Strategy-proof Allocation of Multiple Items between Two Agents without Payments or Priors*

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ABSTRACT

We investigate the problem of allocating items (private goods) among competing agents in a setting that is both prior-free and payment-free. Specifically, we focus on allocating multiple heterogeneous items between two agents with additive valuation functions. Our objective is to design strategy-proof mechanisms that are competitive against the most efficient (first-best) allocation. We introduce the family of linear increasing-price (LIP) mechanisms. The LIP mechanisms are strategy-proof, prior-free, and payment-free, and they are exactly the increasing-price mechanisms satisfying a strong responsiveness property. We show how to solve for competitive mechanisms within the LIP family. For the case of two items, we find a LIP mechanism whose competitive ratio is near optimal (the achieved competitive ratio is 0.828, while any strategy-proof mechanism is at most 0.841-competitive). As the number of items goes to infinity, we prove a negative result that any increasing-price mechanism (linear or nonlinear) has a maximal competitive ratio of 0.5. Our results imply that in some cases, it is possible to design good allocation mechanisms without payments and without priors.

Categories and Subject Descriptors

J.4 [**Computer Applications**]: Social and Behavioral Sciences—*Economics*; I.2.11 [**Distributed Artificial Intelligence**]: Multiagent Systems

General Terms

Economics, Theory

Keywords

Mechanism design, prior-free, payment-free

1. INTRODUCTION

We investigate the problem of allocating items (private goods) among competing agents in a setting that is both prior-free and payment-free. That is, we do not assume that we have knowledge about the distribution of the agents' valuations. We also do not allow the mechanism to specify any monetary payments. This is useful in settings where no currency has (yet) been established (as may be the case, for example, in a peer-to-peer network, as well as in many other multiagent systems); or where payments are prohibited by law; or where payments are otherwise inconvenient. Specifically, we focus on allocating multiple heterogeneous items between two agents with additive valuation functions. Our objective is to design strategy-proof mechanisms that are competitive against the efficient (first-best) allocation.

It remains an open question to give an elegant characterization of mechanisms that are strategy-proof, prior-free, and payment-free (for the problem that we study), and we do not know how to solve for the most competitive such mechanism in general. In our attempts to design competitive mechanisms, we introduce the family of *linear increasing-price (LIP) mechanisms*, which are based on a certain artificial currency. The LIP mechanisms are strategy-proof, prior-free, and payment-free. We show how to solve for competitive mechanisms within the LIP family. For the case of two items, we find a LIP mechanism whose competitive ratio is near optimal (the achieved competitive ratio is 0.828, while any strategy-proof mechanism is at most 0.841-competitive). Thus, at least for the case of two items, it does not come at much of a loss to focus only on LIP mechanisms. As the number of items goes to infinity, we prove a negative result that any increasing-price mechanism (linear or nonlinear) has a maximal competitive ratio of 0.5.

By proposing specific competitive strategy-proof mechanisms that do not rely on payments, our paper also helps to answer a question that has recently been drawing the attention of computer scientists: *Are priors and payments necessary for designing good mechanisms?* The idea of designing strategy-proof mechanisms *without* payments that achieve competitive performance was explicitly framed by Procaccia and Tennenholtz [22], in their paper titled *Approximate Mechanism Design Without Money*. That paper carries out a case study on locating a public facility for agents with single-peaked valuations. (The general idea of approximate mechanism design without payments dates back further, at least to work by Dekel *et al.* [10] in a machine learning framework.)

Our paper considers this question in the different context of allocation mechanisms.[1] Unlike the models studied in the above two papers [10, 22], where a consensus agreement may exist, when we are considering the allocation of private goods, the agents are necessarily in conflict.[2] Nevertheless, it turns out that even here, some

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[1]Guo and Conitzer [15] also studied the problem of designing competitive allocation mechanisms without payments, but in a repeated setting. Another difference between [15] and this paper is that the mechanisms proposed in [15] are Bayes-Nash incentive compatible instead of strategy-proof.

[2]For example, both [10] and [22] proposed mechanisms that pick the "median" report from the agents as the final outcome. When the agents' favorite outcomes are identical, the median report is the

positive results can be obtained. Thus, we believe that our results provide additional insights for this line of research. Of course, it is beyond the scope of this paper to answer the above question in its general form; rather, we will be content to focus specifically on designing prior-free, payment-free allocation mechanisms.

Resource allocation mechanisms with payments have been studied extensively in both economics and computer science. Related work that does not require a prior distribution includes the following. For two agents, McAfee [18] analyzes equilibrium behavior under three *simple mechanisms* whose description does not rely on the prior distribution over the agents' valuations. They are the first-price, the second-price, and the cake-cutting mechanisms.[3] For the case of three or more agents, the family of *VCG redistribution mechanisms* are efficient, strategy-proof, and (ex post) individually rational. VCG redistribution mechanisms are Groves mechanisms that allocate resources according to the VCG (Clarke) mechanism, and then redistribute a large portion of the VCG payments back to the agents [2, 8, 21, 14, 13, 19]. The above papers aim to maximize social welfare. Prior-free approaches have also been used for revenue maximization, such as in digital goods auctions [1, 17, 12].

There is also a rich literature on mechanisms without payments. A survey is given in the book chapter by Schummer and Vohra [23]. Barberà [3] gives an introduction to strategy-proof social choice functions. Budish [6] gives a nice survey of existing allocation mechanisms without payments that are designed for practical usage (***e.g.***, the patented Adjusted Winner Procedure [5]). All these mechanisms are manipulable except for the Serial Dictatorship mechanism in Budish and Cantillon [7], in which the authors study user behavior in Harvard Business School course allocation. Several papers suggest that in particular settings, strategy-proof mechanisms without payments, combined with various other restrictions (***e.g.***, efficiency), must come down to mechanisms that are, in a sense, dictatorial [20, 11, 24]. The proposed linear increasing-price mechanisms in our paper are also dictatorial in nature. Mechanism design without payments has also been studied in[16, 9].[4]

2. MODEL DESCRIPTION

We study the problem of allocating m ($m > 1$) heterogeneous items (referred to as items 1 to m) between two agents (referred to as agents 1 and 2). We use $-i$ to denote the agent other than i.

Let O be the set of all possible allocations. An allocation $o \in O$ is denoted by a vector $(p_1, p_2, \ldots, p_m)$ ($0 \leq p_j \leq 1$ for all j), where p_j is the proportion[5] of item j won by agent 1 (so that $1 - p_j$ is the proportion of item j won by agent 2).

We assume that the agents' valuations for the items are additive, and that the agents are risk neutral. We use a vector $(v_1^i, v_2^i, \ldots, v_m^i)$ to denote agent i's type, where v_j^i is agent i's valuation for winning item j ($v_j^i \geq 0$). Additivity and risk neutrality imply that under allocation $(p_1, p_2, \ldots, p_m)$, agent 1's utility equals $\sum_j p_j v_j^1$ and agent 2's utility equals $\sum_j (1 - p_j) v_j^2$.

Furthermore, we require that the agents' valuations are normalized. That is, the type space $\mathbb{V}$ consists of vectors $(v_1, v_2, \ldots, v_m)$ with $\sum_j v_j = 1$. As a result, an agent's utility for an allocation can be thought of as her level of satisfaction; if an agent wins all the items, then she is 100% satisfied. The reason that we require this normalization is the following. When payments are available and utility is quasilinear, this provides a way of comparing valuations between agents. However, because payments are unavailable in our context, it is no longer possible to make such a comparison. Hence, the units in which valuations are expressed become meaningless, so that the only meaning that can be derived from an agent's valuations is the *relative* valuations of the items (the ratio of the valuations). If we (say) doubled one agent's valuation for every item, in our payment-free context this would double that agent's utility for every outcome, and as a result her behavior under any mechanism would remain completely unchanged. As a result, there can be no hope of coming anywhere close to maximizing the social welfare without some normalization assumption.

A payment-free mechanism $M : \mathbb{V} \times \mathbb{V} \rightarrow O$ maps the agents' reported type vectors to an allocation. Let $u^i(\vec{v}, o)$ be agent i's utility under allocation o when her true type is $\vec{v}$. Mechanism M is said to be *strategy-proof* if: $\forall i \in \{1, 2\}, \vec{v_i}, \vec{v_i'}$ and $\vec{v_{-i}}$, we have $u^i(\vec{v_i}, M(\vec{v_i}, \vec{v_{-i}})) \geq u^i(\vec{v_i}, M(\vec{v_i'}, \vec{v_{-i}}))$. In words, a mechanism is strategy-proof if no matter what the other agent reports, each agent's best strategy is to report truthfully.

We define the *first-best allocation mechanism* M^* to be the mechanism that always naïvely maximizes the social welfare (without considering incentives). That is, $\forall \vec{v_1}, \vec{v_2}$, $M^*(\vec{v_1}, \vec{v_2}) \in \arg\max_{o \in O} \sum_{i \in \{1,2\}} u^i(\vec{v_i}, o)$.

We will use the first-best mechanism M^* (which is not strategy-proof) as our benchmark when evaluating the performance of strategy-proof mechanisms. (When using M^* as a benchmark, we assume that agents report truthfully, even though they are not incentivized to do so. Hence, M^* always produces the maximal social welfare among all mechanisms, with or without priors, and with or without payments.)

Strategy-proof mechanism M is said to be (at least) α*-competitive* if the social welfare under M is always greater than or equal to α times the social welfare under M^*. Here α is called M's *competitive ratio*. The maximal possible value of α is called M's *maximal competitive ratio*.

Definition 1. Strategy-proof mechanism M is α-competitive against the first-best mechanism M^* if $\forall \vec{v_1}, \vec{v_2}$, we have

$$\sum_{i \in \{1,2\}} u^i(\vec{v_i}, M(\vec{v_1}, \vec{v_2})) \geq \alpha \sum_{i \in \{1,2\}} u^i(\vec{v_i}, M^*(\vec{v_1}, \vec{v_2}))$$

Example 1. The mechanism that always divides every item evenly has maximal competitive ratio 0.5. The mechanism that always gives every item to agent 1 also has maximal competitive ratio 0.5.

Our objective is to design strategy-proof mechanisms with high competitive ratios.

3. UPPER BOUND ON THE COMPETITIVE RATIOS OF STRATEGY-PROOF MECHANISMS

In this section, we derive an upper bound on the competitive ratios of strategy-proof mechanisms. Given our objective, we only

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consensus agreement for all the agents. When allocating private goods (without externalities), consensus agreement never exists—every agent wants every good. Of course, in the worst case (all of these papers are based on worst-case analysis), the agents in the earlier papers are also in conflict.

[3]In fact, the cake-cutting mechanism is payment-free. However, it is not strategy-proof in our sense. In the literature on cake-cutting mechanisms [4], strategy-proofness has another, much weaker meaning: An agent can not *guarantee* a better result by cheating, given that she is ignorant about the other agent's type.

[4]The recently proposed qualitative Vickrey auction[16], a generalization of the traditional Vickrey auction, is another mechanism that does not rely on monetary payments. However, it can not be applied to our problem as it requires that there will be only a single winner, and that the center has preferences over the outcomes.

[5]For indivisible items, p_j is interpreted as the probability that agent 1 wins item j.

need to consider strategy-proof mechanisms that are *symmetric*.[6]

Definition 2. A mechanism M is symmetric if it satisfies

Symmetry over the agents: If we swap the reported type vectors of two of the agents, then the items allocated to these agents are also swapped.

Symmetry over the items: If we swap agent 1's valuations for any two items, and we swap agent 2's valuations for the same two items, then the allocation result for these two items is also swapped.

CLAIM 1. *For any strategy-proof mechanism that is α-competitive, there is a corresponding* symmetric *strategy-proof mechanism that is (at least) α-competitive.*

We omit some of the proofs due to space constraint.

CLAIM 2. *For the case of two agents, any symmetric strategy-proof mechanism is (at least) 0.5-competitive.*

Claim 1 implies that for the purpose of deriving an upper bound on the competitive ratios of strategy-proof mechanisms, we can safely ignore strategy-proof mechanisms that are not symmetric.

Let us recall that a mechanism M is α-competitive if for *all possible* type vectors, the social welfare under M is at least α times the social welfare under the first-best mechanism M^*. If we restrict the type space, then the maximal competitive ratio of M can only stay the same or increase. That is, one way to compute an upper bound on the competitive ratios of strategy-proof mechanisms is to restrict the type space and then solve for the largest possible competitive ratio for any strategy-proof mechanism.

THEOREM 1. *The competitive ratio of any strategy-proof mechanism is at most 0.841. This is true for any number of items and two agents.*

PROOF. We first focus on the case of two items. We consider the following restricted type space: $\{(ih,(N-i)h)|i=0,1,\ldots,N\}$, where $N=50$ and $h=1/N$. Type vector $(ih,(N-i)h)$ can be denoted by the integer i. A mechanism for this restricted type space can be denoted by the p^i_{jk} for $i=1,2$ and $0\le j,k\le N$, where p^i_{jk} is the proportion of item i won by agent 1 when agent 1's report is j and agent 2's report is k.

Strategy-proofness for agent 1 can then be represented by the following set of linear inequalities: $\forall\, 0\le j,j',k\le N$

$$jp^1_{jk}+(N-j)p^2_{jk}\ge jp^1_{j'k}+(N-j)p^2_{j'k}$$

Strategy-proofness for agent 2 can be represented by a similar set of linear inequalities involving the p^i_{jk}.

The mechanism characterized by the p^i_{jk} is α-competitive if the following linear inequalities are satisfied: $\forall\, 0\le j,k\le N$

$$jp^1_{jk}+(N-j)p^2_{jk}+k(1-p^1_{jk})+(N-k)(1-p^2_{jk})\ge$$

$$\alpha(\max\{j,k\}+\max\{N-j,N-k\})$$

The largest possible competitive ratio for any mechanism and for the above restricted type space can thus be computed by solving a linear program, which results in 0.841.[7] Any strategy-proof mechanism for the case of $m>2$ items remains strategy-proof when applied to the case of two items (when the agents do not care about the other items). Hence, the upper bound 0.841 still applies. □

[6]This is a frequently used technique in the literature on prior-free mechanism design.

[7]We acknowledge that a computer-assisted proof is not as satisfactory as an easily human-verifiable mathematical proof. Because this is a linear programming problem, in principle, we can give a (nearly) optimal solution to the dual problem to show that it is impossible to better; we do not give such a solution here because it does not seem to shed much light.

4. LINEAR INCREASING-PRICE MECHANISMS

As mentioned earlier, it remains an open question to solve for the most competitive strategy-proof mechanism in general. There are two reasons for this: first, we lack an elegant characterization of all strategy-proof mechanisms for our problem; second, we lack a general approach for evaluating a given mechanism (computing its maximal competitive ratio).

In our attempts to design competitive mechanisms, we start with the family of all strategy-proof mechanisms (SP). We then move on to more and more restricted families of mechanisms: the family of swap-dictatorial mechanisms (SD), the family of increasing-price mechanisms (IP), and finally the family of linear increasing-price mechanisms (LIP). These 4 families are nested as illustrated below:

$$LIP\subsetneq IP\subsetneq SD\subsetneq SP$$

As we move from SP to LIP, we get more and more elegant characterizations of the mechanisms. Finally, the mechanisms in the LIP family can actually be characterized by a single parameter, and we are able to evaluate (the competitiveness of) any given LIP mechanism. That is, we are able to solve for competitive mechanisms within the LIP family.

In a payment-free setting, if we fix agent $-i$'s report, then agent i essentially faces a set of allowable outcomes that she can choose from (each outcome corresponds to an allowable report of i). A necessary condition for a mechanism to be strategy-proof is that the mechanism should always choose i's favorite outcome (among all allowable outcomes). This condition is not sufficient for the mechanism to be strategy-proof for *both* agents, because agent $-i$ may have the power to change the set of allowable outcomes that agent i faces. That is, $-i$ may want to submit a false report to get agent i to a decision $-i$ prefers. However, if we require that the set of allowable outcomes agent i faces is fixed, then the mechanism that picks i's favorite outcome is strategy-proof for both agents. Essentially, in such a mechanism, agent i is the dictator: she chooses her favorite outcome from a set of outcomes predetermined by the mechanism, and agent $-i$ has no choice but to accept this outcome (the decision is solely made by i). This leads to the following family of swap-dictatorial mechanisms (by Claim 1, we only need to consider symmetric mechanisms):

Swap-Dictatorial Mechanisms: With probability 0.5, agent i is the dictator, who chooses her favorite allocation from a predefined set of allowable allocations $\hat{O}^i\subset O$. The $\hat{O}^i$ satisfy the following (symmetry over the agents and the items):

- If $(p_1,p_2,\ldots,p_m)\in\hat{O}^i$, then $(1-p_1,1-p_2,\ldots,1-p_m)\in\hat{O^{-i}}$ for any i.
- If $(p_1,p_2,\ldots,p_m)\in\hat{O}^i$, then $(p_{\sigma(1)},p_{\sigma(2)},\ldots,p_{\sigma(m)})\in\hat{O}^i$ for any permutation σ and i.

Swap-dictatorial mechanisms, as well as other dictatorial mechanisms, have been studied extensively because of their simplicity (*e.g.*, [7]). Many papers in the literature on mechanisms without payments suggest that strategy-proofness, combined with various other properties, can only come down to mechanisms that are dictatorial in nature [20, 11, 24]. However, since we do not assume additional properties, for our problem, there do exist strategy-proof mechanisms that are not dictatorial in nature (that is, $SD\subsetneq SP$).

For purpose of maximizing social welfare, ideally, we want the dictator agent to take only items that she really values, and leave the remaining items to the other agent. This leads to the following family of *increasing-price (IP) mechanisms*.

Increasing-Price (IP) Mechanisms: With probability 0.5, agent i is the dictator, and is endowed with 1 unit of artificial currency. The dictator agent can purchase (proportions of) items (from the mechanism, not from the other agent) with her artificial currency. The (proportions of) items not purchased at the end go to the other agent. Rather than having just a fixed price for each item, there is a price schedule for each item, and the item becomes more expensive as the dictator agent buys more of it. The price schedules are characterized by functions $f_j^i : [0,1] \to \mathbb{R}^+$ for $i = 1, 2$ and $j = 1, 2, \ldots, m$. $f_j^i(x)$ is the instantaneous price per unit charged to agent i (when i is the dictator) if she demands item j, at the point where x units of her artificial currency have already been spent on item j. By Claim 1, we can simply assume $f_j^i = f$ for all i and j. Function f is increasing and positive. We also assume f is differentiable. If, at the end, agent i (when she is the dictator) spent x units of artificial currency on item j, then she is allocated a proportion $\int_0^x \frac{1}{f(t)} dt$ of item j. We will present an example IP mechanism later in this section (which actually belongs to the more restricted class of LIP mechanisms).

The intuition for why increasing-price mechanisms might perform well is as follows. If the dictator agent demands a large proportion of an item, then she will be paying at a high rate, which can only happen when she highly values the item. Because prices are increasing, the optimal strategy for the dictator agent is simply the greedy strategy: purchase (an infinitesimally small amount each time) the best deal (the item with the highest value/price ratio) until the artificial currency runs out. That is, at some point, if the dictator agent's valuation for item j is v_j, and so far x_j units of artificial currency have been spent on item j, then the dictator agent should purchase an infinitesimally small amount of item j^*, where $j^* = \arg\max_j\{\frac{v_j}{f(x_j)}\}$. At the end, for items that have been partly purchased, the final prices must be proportional to the dictator agent's valuations:

LEMMA 1. *Under an IP mechanisms, if the dictator spends $k_1, k_2 (0 < k_i < 1)$ units of artificial currency on items $1, 2$, then the dictator's valuations for these items must be $f_1(k_1) \cdot C$ and $f_2(k_2) \cdot C$ for some C.*

Any increasing and positive function f corresponds to an increasing-price mechanism. Actually, for the purpose of designing competitive mechanisms, we only need to consider functions f that satisfy $\int_0^1 \frac{1}{f(t)} dt = 1$. That is, we only need to consider increasing-price mechanisms in which the dictator agent gets the entirety of an item if and only if she spends all her artificial currency on this item.

CLAIM 3. *For the purpose of designing competitive IP mechanisms, we only need to consider increasing-price mechanisms with f satisfying $\int_0^1 \frac{1}{f(t)} dt = 1$.*

PROOF. If $\int_0^1 \frac{1}{f(t)} dt > 1$, then there exists U ($U < 1$) that satisfies $\int_0^U \frac{1}{f(t)} dt = 1$. $\forall 0 < \epsilon < U$, let $\hat{f}$ be the same as f for $x \le U$, and let $\hat{f}(x)$ take some very high values for $U < x \le 1$ (in a way that makes $\hat{f}$ increasing), so that $\int_0^1 \frac{1}{\hat{f}(t)} dt \le 1+\epsilon$. Since the dictator agent will never spend more than U units of artificial currency on any item (it is pointless for the dictator agent to continue purchasing an item when she has already obtained the entirety of this item), on the region that matters to the mechanism ($0 \le x \le U$), f and $\hat{f}$ are identical. Thus, we only need to consider functions f satisfying $\int_0^1 \frac{1}{f(t)} dt \le 1 + \epsilon$ for arbitrary small value ϵ. That is, we only need to consider cases where $\int_0^1 \frac{1}{f(t)} \le 1$.

If $\int_0^1 \frac{1}{f(t)} dt = p < 1$, then let $\hat{f} = pf$, so that we have $\int_0^1 \frac{1}{\hat{f}(t)} dt = 1$. We denote the proportion of item j won by agent i under f when i is the dictator by q_j^i. The proportion of item j won by agent i under f when i is not the dictator is then $1 - q_j^{-i}$. The proportion of item j won by agent i under $\hat{f}$ when i is the dictator is $\frac{q_j^i}{p}$ (under $\hat{f}$, a dictator gets $\frac{1}{p}$ times as much item per unit of artificial currency at every amount of currency spent), and the proportion of item j won by agent i under $\hat{f}$ when i is not the dictator is $1 - \frac{q_j^{-i}}{p}$. The social welfare under f equals $\sum_{i,j} \frac{q_j^i v_j^i + (1-q_j^{-i}) v_j^i}{2}$. The social welfare under $\hat{f}$ equals $\sum_{i,j} \frac{q_j^i v_j^i / p + (1 - q_j^{-i}/p) v_j^i}{2}$, which is at least 1 (as in the proof of Claim 2). It turns out that the social welfare under f is always less than or equal to the social welfare under $\hat{f}$, as proved below. $\sum_{i,j} \frac{q_j^i v_j^i + (1-q_j^{-i}) v_j^i}{2} =$
$\sum_{i,j} \frac{q_j^i v_j^i + (p-q_j^{-i}) v_j^i + (1-p) v_j^i}{2} = \sum_{i,j} \frac{q_j^i v_j^i + (p-q_j^{-i}) v_j^i}{2} +$
$\sum_{i,j} \frac{(1-p) v_j^i}{2} = \sum_{i,j} \frac{q_j^i v_j^i + (p-q_j^{-i}) v_j^i}{2} + (1-p) =$
$p \sum_{i,j} \frac{q_j^i v_j^i / p + (1-q_j^{-i}/p) v_j^i}{2} + (1-p) \le \sum_{i,j} \frac{q_j^i v_j^i / p + (1-q_j^{-i}/p) v_j^i}{2}$.

Hence, we only need to consider f satisfying $\int_0^1 \frac{1}{f(t)} = 1$. □

Finally, the family of linear increasing-price mechanisms is described below:

Linear Increasing-Price (LIP) Mechanisms: Linear increasing-price mechanisms are increasing-price mechanisms characterized by a linear function $f(x) = ax + b$, where a and b are positive constants. (a has to be positive for f to be increasing. b has to be positive to avoid negative prices or division-by-zero.) Since we only consider f satisfying $\int_0^1 \frac{1}{f(t)} dt = 1$, we have $b = \frac{a}{e^a - 1}$. That is, a LIP mechanism is characterized by a single parameter a. From now on, we use $LIP(a)$ to denote the LIP mechanism with parameter a. We use b to denote the value $\frac{a}{e^a - 1}$.

Example 2. Let $a = 2$ ($b = \frac{2}{e^2 - 1}$) and $m = 2$. Let the agents' type vectors be $(1, 0)$ and $(0.5, 0.5)$, respectively. Under $LIP(a)$, with 0.5 probability, agent 1 is the dictator. Since agent 1's type vector is $(1, 0)$, she will spend all her artificial currency on item 1. The resulting allocation is $(1, 0)$: agent 1 wins the entirety of item 1, while agent 2 gets what is left (the entirety of item 2). With 0.5 probability, agent 2 is the dictator. Since agent 2's type vector is $(0.5, 0.5)$, she will divide her artificial currency evenly on items 1 and 2. The resulting allocation is $(1 - \int_0^{0.5} \frac{1}{at+b} dt, 1 - \int_0^{0.5} \frac{1}{at+b} dt) = (0.283, 0.283)$: agent 2 wins $\int_0^{0.5} \frac{1}{at+b} dt = 0.717$ proportion of both item 1 and 2, while agent 1 gets what is left ($1 - \int_0^{0.5} \frac{1}{at+b} dt = 0.283$ proportion of both items). In total, the resulting allocation under $LIP(a)$ is $(1 - \frac{1}{2} \int_0^{0.5} \frac{1}{at+b} dt, \frac{1}{2} - \frac{1}{2} \int_0^{0.5} \frac{1}{at+b} dt) = (0.642, 0.642)$.

Besides simplicity, the linear increasing-price mechanisms possess a nice property that is not shared by other (non-linear) increasing-price mechanisms. Before defining this property, we need the following definitions. Suppose we are considering an IP mechanism characterized by function f.

Definition 3. A type vector $\vec{v} \in \mathbb{V}$ is *strictly full ranked* for f if a dictator agent with true type $\vec{v}$ will purchase positive proportions of every item under f.

Every strictly full ranked type vector $\vec{v} = (v_1, v_2, \ldots, v_m)$ corresponds to a vector $(t_1, t_2, \ldots, t_m)$ with $\sum_{j=1}^m t_j = 1$, where t_j (> 0) denotes the amount of artificial currency that an agent with type vector $\vec{v}$ will spend on item j (when she is the dictator). The final value/price ratio $\frac{v_j}{f(t_j)}$ should be the same for all j (Lemma 1).

Definition 4. A type vector $\vec{v} \in \mathbb{V}$ is *full ranked* if $\vec{v} \in \mathbb{W}$, where $\mathbb{W}$ is the closure of the set of all strictly full ranked type vectors.

For a full ranked vector $\vec{v}$, we also have that the final value/price ratio $\frac{v_j}{f(t_j)}$ should be the same for all j.

Not all type vectors are full ranked type vectors. If an agent has very low valuations for some items, then she will not spend any artificial currency on those items if $f(0)$ is sufficiently high. For small $f(0)$, most type vectors are full ranked. In the rest of this paper (when solving for the competitive ratios of LIP mechanisms), we focus on full ranked type vectors, and treat vectors that are not full ranked as exceptions.

CLAIM 4. *For cases of at least three items, LIP mechanisms are the only IP mechanisms satisfying the following condition:*

Strong responsiveness: *For two agents with full ranked type vectors, if one agent values an item more than the other agent, then she should win a greater proportion of this item than the other agent.*

We first prove the following lemma, which will be also used later in the paper.

LEMMA 2. *Let $\vec{v} = (v_1, v_2, \ldots, v_m)$ be a full ranked vector under $LIP(a)$. Let $\vec{v}$'s payment vector $(t_1, t_2, \ldots, t_m)$ ($\sum_{j=1}^{m} t_j = 1$) be such that an agent with true type $\vec{v}$ will spend t_j units of artificial currency on item j under $LIP(a)$ (when she is the dictator). Then, the v_j and the t_j satisfy $v_j = \frac{at_j+b}{a+mb}$ for all j.*

PROOF. The final value/price ratio $\frac{v_j}{at_j+b}$ should be the same for all j, by Lemma 1. Since $\sum v_j = 1$, we have $v_j = \frac{at_j+b}{a+mb}$ for all j. $\square$

Now we are ready to prove the above claim.

PROOF OF CLAIM 4. We first prove that LIP mechanisms satisfy the strong responsiveness condition.

Lemma 2 says that under a LIP mechanism, an agent's value for an item is linear in the amount of artificial currency this agent would spend on the item as a dictator. Therefore, if one agent values an item more than the other agent, then, as the dictator, she would spend more on this item than the other agent, which means she wins more of the item at the end.

We now prove that LIP mechanisms are the only IP mechanisms satisfying the strong responsiveness condition, for cases of at least three items.

Let us consider an IP mechanism characterized by an increasing positive function f. If $\exists$ nonnegative t_a, t_b, t'_a, t'_b, so that $0 \le t_a + t_b = t'_a + t'_b = t \le 1$ and $f(t_a) + f(t_b) > f(t'_a) + f(t'_b)$ are both satisfied, then we can construct the following full ranked type vectors:

$$\left(\frac{f(1-t)}{f(1-t)+f(t_a)+f(t_b)+(m-3)f(0)}, \frac{f(t_a)}{f(1-t)+f(t_a)+f(t_b)+(m-3)f(0)}, \frac{f(t_b)}{f(1-t)+f(t_a)+f(t_b)+(m-3)f(0)}, \frac{f(0)}{f(1-t)+f(t_a)+f(t_b)+(m-3)f(0)}, \ldots, \frac{f(0)}{f(1-t)+f(t_a)+f(t_b)+(m-3)f(0)}\right)$$ and
$$\left(\frac{f(1-t)}{f(1-t)+f(t'_a)+f(t'_b)+(m-3)f(0)}, \frac{f(t'_a)}{f(1-t)+f(t'_a)+f(t'_b)+(m-3)f(0)}, \frac{f(t'_b)}{f(1-t)+f(t'_a)+f(t'_b)+(m-3)f(0)}, \frac{f(0)}{f(1-t)+f(t'_a)+f(t'_b)+(m-3)f(0)}, \ldots, \frac{f(0)}{f(1-t)+f(t'_a)+f(t'_b)+(m-3)f(0)}\right).$$

The two vectors are constructed in such a way that agent 1 will spend $1-t$ units of artificial currency on item 1, t_a units on item 2, t_b units on item 3, and 0 units on the other items, while agent 2 will spend $1-t$ units of artificial currency on item 1, t'_a units on item 2, t'_b units on item 3, and 0 units on the other items. Agent 1 values item 1 less than agent 2 (the denominator is larger), but they will spend the same amount of artificial currency on item 1. So, they win the same proportion of item 1 at the end. Now if we increase the value of agent 1 for item 1 by a tiny amount (still keeping it less than the value of agent 2), then we have a situation where agent 1 values item 1 less, but wins a greater proportion of it at the end (agent 1 now spends more on item 1). That is, to satisfy the strong responsiveness condition, whenever $0 \le t_a + t_b = t'_a + t'_b = t \le 1$ for nonnegative t_a, t_b, t'_a, t'_b, we must have $f(t_a) + f(t_b) = f(t'_a) + f(t'_b)$. That is, $\forall 0 \le c \le t \le 1$, we have $f(t) + f(0) = f(t-c) + f(c)$. Since we assume f is differentiable, by taking the derivative over t on both sides of the equality, we have that $f'(t) = f'(t-c)$. The values of t and c can be arbitrary. That is, f' is a constant. f must be linear. $\square$

The above claim provides another justification (other than simplicity) why, among all IP mechanisms, we focus on LIP mechanisms. In the next section, we solve for competitive mechanisms within the LIP family.

5. COMPETITIVE LINEAR INCREASING-PRICE MECHANISMS

Since a linear increasing-price mechanism is characterized by a single parameter, if, for a given value of a, we are able to evaluate the competitiveness of $LIP(a)$, then the task of solving for competitive LIP mechanisms can be done simply by searching for the optimal value of a.

In what follows, we discuss how to evaluate the competitiveness of $LIP(a)$, for a given value of a and a given number of items.

5.1 Two Items

We first focus on the case of two items.

We denote the type vectors of agent 1 and 2 by $(x, 1-x)$ and $(y, 1-y)$, respectively ($1 \ge x \ge y \ge 0$). We abuse notation by using x to refer to both the value x and the type vector whose first element is x. We do the same for y.

CLAIM 5. *Under $LIP(a)$, with probability 0.5, agent 1 is the dictator, whose optimal strategy (when she is the dictator) is as follows.*

- *If $\frac{x}{a+b} \ge \frac{1-x}{b}$, then agent 1 will spend all her artificial currency on item 1. At the end, agent 1 gets item 1 in its entirety while agent 2 gets what 1 does not take (item 2 in its entirety). It should be noted that this is the resulting allocation* when agent 1 is the dictator. *When agent 2 is the dictator, we may get a different allocation.*
- *If $\frac{1-x}{a+b} \ge \frac{x}{b}$, then agent 1 will spend all her artificial currency on item 2. At the end, agent 1 gets item 2 in its entirety while agent 2 gets item 1 in its entirety.*
- *Otherwise, agent 1 will spend $t = \frac{x(a+2b)-b}{a}$ units of artificial currency on item 1, and $1-t = \frac{(1-x)(a+2b)-b}{a}$ units of artificial currency on item 2. At the end, the instantaneous prices of items 1 and 2 will be $at+b = x(a+2b)$ and $a(1-t)+b = (1-x)(a+2b)$, respectively. (We note that the prices are proportional to agent 1's type vector $(x, 1-x)$, as they should be.) At the end, agent 1 gets a proportion $\frac{\ln(at+b)}{a} - \frac{\ln(b)}{a}$ of item 1 and a proportion $\frac{\ln(a(1-t)+b)}{a} - \frac{\ln(b)}{a}$ of item 2, while agent 2 gets the remainder.*

For $j = 1, 2$, we use $p_j(x, y)$ to denote the proportion of item j won by agent 1 at the end, when agent 1's reported type vector is x and agent 2's reported type vector is y. (This proportion takes the randomization over who is the dictator into account.) The value of $p_j(x, y)$ can be computed as shown above. $p_1(x, y)$ is increasing in x and decreasing in y. $p_2(x, y)$ is decreasing in x and increasing in y. We use $S(x, y)$ to denote the social welfare under $LIP(a)$. That is, $S(x, y) = xp_1(x, y) + (1-x)p_2(x, y) + y(1-p_1(x, y)) + (1-y)(1-p_2(x, y))$. The social welfare under the first-best mechanism M^* equals $x + 1 - y$.

By definition, the maximal competitive ratio of $LIP(a)$ can be computed as

$$\min_{1\geq x\geq y\geq 0} \frac{S(x,y)}{x+1-y}$$

We now show how to bound the above expression from both below and above.

Let N be a large positive integer. Let $h = \frac{1}{N}$ be the step size. Let the x_i be defined as $x_i = ih$ for $i = 0, 1, \ldots, N$. Similarly, let the y_i be defined as $y_i = ih$ for $i = 0, 1, \ldots, N$.

We have that

$$\min_{1\geq x\geq y\geq 0} \frac{S(x,y)}{x+1-y} \geq \min_{N>i\geq j\geq 0}\{\min_{\substack{x_i+h\geq x\geq x_i\\ y_j+h\geq y\geq y_j}} \frac{S(x,y)}{x_i+h+1-y_j}\}$$

$$\geq \min_{N>i\geq j\geq 0} \frac{\begin{array}{c}x_ip_1(x_i,y_j+h)+(1-x_i-h)p_2(x_i+h,y_j)\\ +y_j(1-p_1(x_i+h,y_j))\\ +(1-y_j-h)(1-p_2(x_i,y_j+h))\end{array}}{x_i+h+1-y_j}$$

We also have that

$$\min_{1\geq x\geq y\geq 0} \frac{S(x,y)}{x+1-y} \leq \min_{N\geq i\geq j\geq 0} \frac{S(x_i,y_j)}{x_i+1-y_j}$$

$$= \min_{N\geq i\geq j\geq 0} \frac{\begin{array}{c}x_ip_1(x_i,y_j)+(1-x_i)p_2(x_i,y_j)\\ +y_j(1-p_1(x_i,y_j))\\ +(1-y_j)(1-p_2(x_i,y_j))\end{array}}{x_i+1-y_j}$$

We note that the x_i and the y_i are constants. The values of the $p_k(x_i, y_j)$ are also constants (for fixed a). That is, based on the above two inequalities, we are able to compute a constant upper bound and a constant lower bound on the maximal competitive ratio of $LIP(a)$. When $a = 2$, the lower bound is 0.828. Since any lower bound on the maximal competitive ratio is also a competitive ratio, $LIP(2)$ is (at least) 0.828-competitive. That is, the obtained $LIP(2)$ mechanism is near optimal for the case of two items (we recall that Theorem 1 says that any strategy-proof mechanism is at most 0.841-competitive).

THEOREM 2. *For the case of two items and two agents, the competitive ratio of $LIP(2)$ is at least 0.828, and at most 0.829.*

5.2 Three or More Items

With more than two items, we need a different technique to bound the maximal competitive ratio of a given LIP mechanism.

Let α be the maximal competitive ratio of $LIP(a)$ (for some given a and m). Let $\mathbb{W}$ be the set of full ranked type vectors under $LIP(a)$. Let $\alpha^{\mathbb{W}}$ be the maximal competitive ratio of $LIP(a)$ if we restrict the type space to $\mathbb{W}$. The following claim says that a lower bound on α can be obtained based on $\alpha^{\mathbb{W}}$.

CLAIM 6. *Let α be the maximal competitive ratio of $LIP(a)$. Let $\alpha^{\mathbb{W}}$ be the maximal competitive ratio of $LIP(a)$ if we restrict the type space to the set of full ranked type vectors $\mathbb{W}$. We have*

$$\frac{a+b}{a+2mb}\alpha^{\mathbb{W}} \leq \alpha$$

Before proving this claim, let us introduce the following definition and lemma.

Definition 5. Let $\vec{v} = (v_1, v_2, \ldots, v_m)$, which may or may not be full ranked. Let $\vec{v}$'s payment vector $(t_1, t_2, \ldots, t_m)$ be such that an agent with true type $\vec{v}$ will spend t_j units of artificial currency on item j (when she is the dictator). We define $\phi(\vec{v}) = (v'_1, v'_2, \ldots, v'_m)$, where $v'_j = \frac{at_j+b}{a+mb}$ for all j. That is, $\phi(\vec{v})$ is the (unique) full ranked type vector corresponding to the payment vector of $\vec{v}$.

If $\vec{v}$ is already full ranked, then $\phi(\vec{v}) = \vec{v}$. In any case, an agent with true type $\phi(\vec{v})$ will act in the same way as an agent with true type $\vec{v}$, since their corresponding payment vectors are the same.

LEMMA 3. *$\forall\vec{v} = (v_1, v_2, \ldots, v_m), \forall j$, let $\phi(\vec{v}) = (v'_1, \ldots, v'_m)$. Then, we have $v_j + \frac{b}{a+mb} \geq v'_j$ and $v_j\frac{a+b}{a+mb} \leq v'_j$. That is, if we change $\vec{v}$ into $\phi(\vec{v})$, the value of an element increases at most by $\frac{b}{a+mb}$, and the value of an element decreases at most by a factor of $\frac{a+b}{a+mb}$.*

PROOF. Let $(t_1, t_2, \ldots, t_m)$ be the payment vector of $\vec{v}$ and $\phi(\vec{v})$. Let $S = \{j|t_j > 0, j = 1, 2, \ldots, m\}$ and $T = \{j|t_j = 0, j = 1, 2, \ldots, m\}$. We have that for all $j \in S$, $\frac{v_j}{at_j+b} = C$ for a common constant C. We also have that for all $j \in T$, $C \geq \frac{v_j}{at_j+b} = \frac{v_j}{b}$.

We get $\sum_{j\in S} v_j = C(a + |S|b)$. We also get $\sum_{j\in T} v_j \leq C(|T|b)$. Since $\sum_{j\in S\cup T} v_j = 1$, we have $C(a + mb) \geq 1$. That is, for $j \in S$, $v_j \geq \frac{at_j+b}{a+mb} = v'_j$. For $j \in T$, $v'_j = \frac{b}{a+mb}$. Therefore, for any j, $v_j + \frac{b}{a+mb} \geq v'_j$.

Since $\sum_{j\in S\cup T} v_j = 1$ and $v_j \geq 0$ for all j, we have $\sum_{j\in S} v_j \leq 1$. That is, $C(a+b) \leq C(a+|S|b) \leq 1$. That is, $C \leq \frac{1}{a+b}$. Hence, for any j, $v_j \leq \frac{at_j+b}{a+b}$. Let us recall that $v'_j = \frac{at_j+b}{a+mb}$. Therefore, for any j, $v_j\frac{a+b}{a+mb} \leq v'_j$. □

Now we are ready to prove Claim 6.

PROOF OF CLAIM 6. Let $\vec{v_1}, \vec{v_2} \in \mathbb{V}$ be any two type vectors.

Let S be the obtained social welfare (under $LIP(a)$) when the agents report $\vec{v_1}$ and $\vec{v_2}$, respectively. Let M be the first-best social welfare when the agents report $\vec{v_1}$ and $\vec{v_2}$, respectively. Let S^ϕ be the obtained social welfare (under $LIP(a)$) when the agents report $\phi(\vec{v_1})$ and $\phi(\vec{v_2})$, respectively. Let M^ϕ be the first-best social welfare when the agents report $\phi(\vec{v_1})$ and $\phi(\vec{v_2})$, respectively.

We consider what happens when agents report $\phi(\vec{v_1})$ and $\phi(\vec{v_2})$ instead of $\vec{v_1}$ and $\vec{v_2}$. The allocation does not change. Since there are m items and by Lemma 3 the valuation of an item goes up by at most $\frac{b}{a+mb}$, we have $S^\phi \leq m\frac{b}{a+mb} + S$. Since by Lemma 3 the valuation of an item goes down by at most a factor of $\frac{a+b}{a+mb}$, we have $M^\phi \geq \frac{a+b}{a+mb}M$. Therefore $\frac{S+m\frac{b}{a+mb}}{\frac{a+b}{a+mb}M} \geq \frac{S^\phi}{M^\phi}$. Since $S \geq 1$ (as in the proof of Claim 2), we have $\frac{S+m\frac{b}{a+mb}S}{\frac{a+b}{a+mb}M} \geq \frac{S^\phi}{M^\phi}$. That is, $\frac{S}{M} \geq \frac{a+b}{a+2mb}\frac{S^\phi}{M^\phi} \geq \frac{a+b}{a+2mb}\alpha^{\mathbb{W}}$. □

Claim 6 implies that if we can get a lower bound on $\alpha^{\mathbb{W}}$, then by multiplying it by $\frac{a+b}{a+2mb}$, we get a lower bound on α. So, we now focus on deriving a lower bound on the maximal competitive ratio of $LIP(a)$ considering only full ranked type vectors.

Let x,y be the agents' valuations for item 1 (or any other item). Without loss of generality, we assume $x \geq y$. Since we are only dealing with full ranked type vectors, we have $x = \frac{at_x+b}{a+mb}$ for some $0 \leq t_x \leq 1$, where t_x is the amount of artificial currency agent 1 spends on item 1 when she is the dictator. Similar observations hold for y. That is, $y = \frac{at_y+b}{a+mb}$ for some $0 \leq t_y \leq 1$, where t_y is the amount of artificial currency agent 2 spends on item 1 when she is the dictator. Let $u = \frac{y}{x}$. We have $\frac{b}{a+b} \leq u \leq 1$.

Under $LIP(a)$, the proportion of item 1 won by agent 1 when 1 is the dictator is $\frac{\ln(at_x+b)}{a} - \frac{\ln(b)}{a}$. The proportion of item 1 won by agent 1 when 1 is not the dictator is $1 - \frac{\ln(at_y+b)}{a} + \frac{\ln(b)}{a}$. In total, the proportion of item 1 won by agent 1 is $\frac{1}{2} + \frac{\ln(\frac{at_x+b}{at_y+b})}{2a} = \frac{1}{2} + \frac{\ln(\frac{x}{y})}{2a} = \frac{-\ln(u)}{2a} + \frac{1}{2}$. Similarly, the proportion of item 1 won by agent 2 is $\frac{\ln(u)}{2a} + \frac{1}{2}$.

We use $R(x,y)$ to denote the sum of the agents' utilities derived from item 1 when the agents' valuations for item 1 are x and y, respectively ($x \geq y$). Let $\theta(a)$ be defined as the minimum ratio between $R(x,y)$ and x over all x, y. That is, $\theta(a)$ is the minimum ratio of achieved utility over optimal utility for item 1 under $LIP(a)$, when we only consider full ranked vectors. $\theta(a)$ only depends on a (not on m). We call it the *intrinsic value* of a.

CLAIM 7. *The intrinsic value $\theta(a)$ is less than or equal to the maximal competitive ratio of $LIP(a)$ considering only full ranked type vectors.*

PROOF. By symmetry over the items, the achieved utility over optimal utility for any item is at least $\theta(a)$. Hence, the maximal competitive ratio is at least $\theta(a)$. $\square$

Let N be a large positive integer. Let $h = \frac{a}{N(a+b)}$ be the step size. Let the u_i be defined as $u_i = \frac{b}{a+b} + ih$ for $i = 0, 1, \ldots, N$.

We observe that

$$\theta(a) = \min_{\frac{b}{a+mb} \leq y \leq x \leq \frac{a+b}{a+mb}} \frac{x(\frac{-\ln(u)}{2a} + \frac{1}{2}) + y(\frac{\ln(u)}{2a} + \frac{1}{2})}{x}$$

$$= \min_{\frac{b}{a+b} \leq u \leq 1} \frac{-\ln(u)}{2a} + \frac{1}{2} + \frac{u\ln(u)}{2a} + \frac{u}{2}$$

$$\geq \min_{0 \leq i < N} \min_{u_i \leq u \leq u_i+h} \frac{(u-1)\ln(u)}{2a} + \frac{1}{2} + \frac{u}{2}$$

$$\geq \min_{0 \leq i < N} \frac{(u_i+h-1)\ln(u_i+h)}{2a} + \frac{1}{2} + \frac{u_i}{2}$$

Given a, the u_i are constants. The above expression is the minimum of N constants. It gives a lower bound on $\theta(a)$. We denote it by $\underline{\theta(a)}$. The following expression gives an upper bound on $\theta(a)$ (denoted by $\overline{\theta(a)}$).

$$\theta(a) = \min_{\frac{b}{a+b} \leq u \leq 1} \frac{-\ln(u)}{2a} + \frac{1}{2} + \frac{u\ln(u)}{2a} + \frac{u}{2}$$

$$\leq \min_{0 < i \leq N} \frac{(u_i-1)\ln(u_i)}{2a} + \frac{1}{2} + \frac{u_i}{2}$$

$$\leq \min_{0 \leq i < N} \frac{(u_i+h-1)\ln(u_i+h)}{2a} + \frac{1}{2} + \frac{u_i}{2} + \frac{h}{2}$$

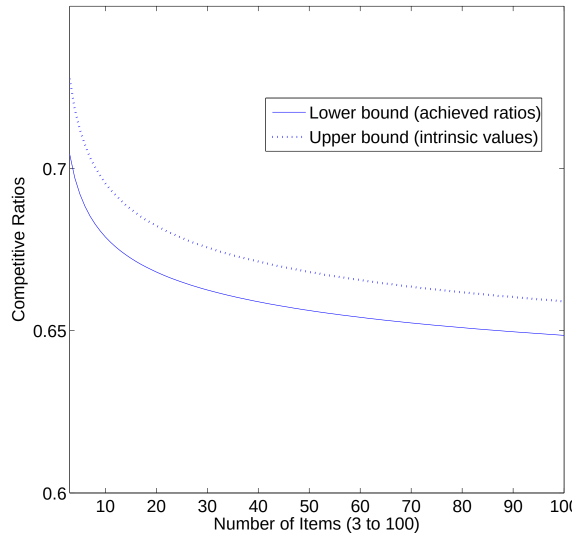

Figure 1: Obtained Competitive Ratios

That is, the obtained lower bound $\underline{\theta(a)}$ and upper bound $\overline{\theta(a)}$ differ only by at most $\frac{h}{2}$, which can be made arbitrarily small.

Since $\theta(a) \leq \alpha^{\mathbb{W}}$, we have that α is bounded below by $\frac{a+b}{a+2mb}\theta(a)$.[8]

Next, we prove that $\theta(a)$ serves as an upper bound on α.[9]

CLAIM 8. $\theta(a) \geq \alpha$.

PROOF. Let $\bar{\alpha}$ be the maximal competitive ratio of $LIP(a)$ when there are only two items. We have $\bar{\alpha} \geq \alpha$. Hence we only need to show $\theta(a) \geq \bar{\alpha}$.

For the case of two items, let us consider the case where agent 1's type vector is $(\frac{u}{u+1}, \frac{1}{u+1})$, and agent 2's type vector is $(\frac{1}{u+1}, \frac{u}{u+1})$. Here, $\frac{b}{a+b} \leq u \leq 1$. It is easy to see that these two type vectors are full ranked. The utility of agent 1 under $LIP(a)$ equals $\frac{u}{u+1}(\frac{1}{2} + \frac{\ln(u)}{2a}) + \frac{1}{u+1}(\frac{1}{2} + \frac{-\ln(u)}{2a})$. The utility of agent 2 is the same. The first-best social welfare is $\frac{2}{u+1}$. So, $\bar{\alpha}$ is at most $2\frac{\frac{1}{2} + \frac{u}{u+1}\frac{\ln(u)}{2a} + \frac{1}{u+1}\frac{-\ln(u)}{2a}}{\frac{2}{u+1}} = \frac{u+1}{2} + u\frac{\ln(u)}{2a} + \frac{-\ln(u)}{2a}$.

Since u can take any value from $\frac{b}{a+b}$ to 1, $\bar{\alpha} \leq \min_{\frac{b}{a+b} \leq u \leq 1} \frac{1}{2} + \frac{u}{2} + \frac{u\ln(u)}{2a} - \frac{\ln(u)}{2a}$. The expression on the right side of the inequality is exactly $\theta(a)$. $\square$

Theorem 3 summarizes the development in this section.

THEOREM 3. *For the case of m items and two agents, $LIP(a)$ is at least $\frac{a+b}{a+2mb}\theta(a)$-competitive, and at most $\theta(a)$-competitive.*

We illustrate the results in this section with Figure 1. For three to one hundred items, we searched for the LIP mechanism (from $\{LIP(a)|a = 0.01, 0.02, 0.03, \ldots, 20\}$) that maximizes $\frac{a+b}{a+2mb}\theta(a)$ (the corresponding upper bounds $\theta(a)$ are also presented).

[8]When we compute this lower bound, we actually compute $\frac{a+b}{a+2mb}\underline{\theta(a)}$.

[9]When we compute this upper bound, we actually compute $\overline{\theta(a)}$.

6. LARGE NUMBERS OF ITEMS

We now show a negative result: as the number of items goes to infinity, any increasing-price mechanism (whether it is linear or nonlinear) has maximal competitive ratio 0.5. That is, in the limit, they are no more competitive than the mechanism that simply divides the items evenly.

THEOREM 4. *For the case of two agents, as the number of items m goes to infinity, the maximal competitive ratio of any increasing-price mechanism is 0.5.*

PROOF. Let M be any increasing-price mechanism, characterized by the price function f. Let the type vectors of the agents be $(\frac{f(1)}{f(1)+(m-1)f(0)}, \frac{f(0)}{f(1)+(m-1)f(0)}, \ldots, \frac{f(0)}{f(1)+(m-1)f(0)})$ and $(1, 0, \ldots, 0)$, respectively. Either agent, when she is the dictator, will choose to spend all her artificial currency on item 1.

When agent 1 is the dictator, the social welfare under M equals $\frac{f(1)}{f(1)+(m-1)f(0)}$. When agent 2 is the dictator, the social welfare under M equals $1 + \frac{(m-1)f(0)}{f(1)+(m-1)f(0)}$. The social welfare under the first-best mechanism equals $1 + \frac{(m-1)f(0)}{f(1)+(m-1)f(0)}$. The competitive ratio of M is then at most $\frac{1}{1+\frac{(m-1)f(0)}{f(1)+(m-1)f(0)}} = \frac{f(1)+(m-1)f(0)}{f(1)+2(m-1)f(0)}$. As $m \to \infty$, this ratio goes to 0.5. That is, the maximal competitive ratio of any increasing-price mechanism is at most 0.5 as $m \to \infty$. On the other hand, 0.5 is a lower bound on the competitive ratios of strategy-proof mechanisms by Claim 2. $\square$

7. FUTURE RESEARCH

One direction for future research is to find out whether higher competitive ratios can be achieved by focusing on other families of strategy-proof mechanisms. We could also consider more general settings in which the agents may express complementary/substitutable preferences over the items.

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