Lecture Notes on Combinatory Modal Logic

15-816: Modal Logic
Frank Pfenning

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1 Introduction

The connection between proofs and program so far has been through a proof term assignment for natural deduction. Proof reduction then forms the basis of computation. This might be called the "Howard isomorphism" [How80]. So where does Curry enter into the picture? He had discovered much earlier that one can assign types *combinators* in a way that validates the axioms of intuitionistic propositional logic [Cur34]. Combinators are a variable-free notation for functions, with a form of computation that does not rely on substitution in the way that natural deduction does. In this lecture we first present this connection for intuitionistic logic and then extend it to include necessity and possibility.

2 Hilbert-Style Axiom Systems

Deductive systems in the tradition of Hilbert [HB34] are characterized by a minimal number of inference rules and many axioms. This reflects mathematical tradition where theories are characterized axiomatically, and logic is seen as a particular kind of mathematical theory. Classical and intuitionstic logic, for example, just have a single rule of inference, namely *modus ponens* which we abbreviate as mp:

$$\frac{A \supset B\ \mathit{hil} \quad A\ \mathit{hil}}{B\ \mathit{hil}}\ \mathsf{mp}$$

Here we write A *hil* to indicate that A follows according to the axioms and rules of inference of intuitionistic logic. Where in the judgmental approach we characterize implication via hypothetical judgments, in Hilbert systems *modus ponens* can be seen as a meaning explanation for implication.

Particular logical connectives are now characterized by axioms. For example, we might have the following three axioms for conjunction:

$$
\begin{array}{l}
A \supset B \supset A \wedge B \; \mathit{hil} \\
A \wedge B \supset A \; \mathit{hil} \\
A \wedge B \supset B \; \mathit{hil}
\end{array}
$$

We will not specify right now the axioms for implication, but develop them together with a proof that A *hil* iff A *true*, that is, the axiomatic and the natural deduction approach prove the same theorems. Nevertheless, the two approaches are vastly different because the structure of proofs, of primary importance in the constructive interpretation of logic, is very different, as we will see.

3 From Hilbert Proofs to Natural Deductions

Generally speaking, this direction is easy. Since the rule of *modus ponens* is the same as implication elimination ($\supset E$), all we have to do is to prove the axioms in natural deduction, and any axiomatic proof can be translated into natural deduction form. Once we have completed our axiomatic system, we will show how to prove the axioms at the end of the next section.

4 From Natural Deductions to Hilbert Proofs

The difficult direction is to translate natural deductions to Hilbert proofs. This is difficult because the introduction rule for implication employs hypotheses which are not directly available in Hilbert proofs.

The key to the solution are hypothetical Hilbert derivations. They allow a rather straighforward translation of natural deductions. We prove separately that, on Hilbert proofs, hypotheses can be eliminated using the crucial *deduction theorem*. We will prove this first, because it motivates the particular axioms we need.

We write Γ for hypotheses $A_1\,\mathit{hil}, \ldots, A_n\,\mathit{hil}$.

Theorem 1 (Deduction Theorem) *If* $\Gamma, A\,\mathit{hil} \vdash C\;\mathit{hil}$ *then* $\Gamma \vdash A \supset C\;\mathit{hil}$.

Proof: By induction on the structure of the given proof. We will introduce the necessary axioms for this proof as we go along.

Case:

$$\frac{}{\Gamma, A\ \mathit{hil} \vdash A\ \mathit{hil}}\ \mathsf{hyp}$$

In this case we need to show that $\Gamma \vdash A \supset A\ \mathit{hil}$. We introduce the axiom schema

$$\vdash A \supset A\ \mathit{hil} \qquad (\mathsf{I})$$

from which $\Gamma \vdash A \supset A\ \mathit{hil}$ follows by weakening.

Case:

$$\frac{B\ \mathit{hil} \in \Gamma}{\Gamma, A\ \mathit{hil} \vdash B\ \mathit{hil}}\ \mathsf{hyp}$$

In this case we need to prove $\Gamma \vdash A \supset B\ \mathit{hil}$. We introduce the axiom schema

$$\vdash B \supset (A \supset B)\ \mathit{hil} \qquad (\mathsf{K})$$

and procceed as follows:

$\Gamma \vdash B\ \mathit{hil}$	By hypothesis $B\ \mathit{hil} \in \Gamma$
$\Gamma \vdash B \supset (A \supset B)\ \mathit{hil}$	By axiom (K) and weakening
$\Gamma \vdash A \supset B\ \mathit{hil}$	By rule (mp)

Case:

$$\frac{\Gamma, A\ \mathit{hil} \vdash B \supset C\ \mathit{hil} \quad \Gamma, A\ \mathit{hil} \vdash B\ \mathit{hil}}{\Gamma, A\ \mathit{hil} \vdash C\ \mathit{hil}}\ \mathsf{mp}$$

$\Gamma \vdash A \supset (B \supset C)\ \mathit{hil}$	By i.h.
$\Gamma \vdash A \supset B\ \mathit{hil}$	By i.h.

At this point we need to introduce the axiom schema

$$\vdash (A \supset (B \supset C)) \supset (A \supset B) \supset (A \supset C)\ \mathit{hil} \qquad (\mathsf{S})$$

Now we can complete the proof.

$\Gamma \vdash (A \supset (B \supset C)) \supset (A \supset B) \supset (A \supset C)\ \mathit{hil}$	By (S) and weakening
$\Gamma \vdash (A \supset B) \supset (A \supset C)\ \mathit{hil}$	By rule (mp)
$\Gamma \vdash A \supset C\ \mathit{hil}$	By rule (mp)

Now we have introduced several axiom schemas X, for each of which we have $\Gamma \vdash X\ \mathit{hil}$. These also have to be covered in the proof.

Case:

$$\frac{}{\Gamma \vdash X\ \mathit{hil}}\ X$$

for an axiom X (either I, K, or S).

$\Gamma \vdash X\ \mathit{hil}$	Axiom
$\Gamma \vdash X \supset (A \supset X)\ \mathit{hil}$	By axiom (K)
$\Gamma \vdash A \supset X\ \mathit{hil}$	By rule (mp)

□

In summary, for the purely implicational fragment we obtain the deduction theorem with the following axioms:

$\vdash A \supset A\ \mathit{hil}$	(I)
$\vdash B \supset (A \supset B)\ \mathit{hil}$	(K)
$\vdash (A \supset (B \supset C)) \supset (A \supset B) \supset (A \supset C)\ \mathit{hil}$	(S)

together with the single rule of modus ponens

$$\frac{\vdash A \supset B\ \mathit{hil} \quad \vdash A\ \mathit{hil}}{\vdash B\ \mathit{hil}}\ \mathsf{mp}$$

From the deduction theorem it follows easily that we can translate natural deduction to Hilbert proofs. We write $\hat{\Gamma}$ for the translation of judgments $A\ \mathit{true}$ to $A\ \mathit{hil}$.

Theorem 2 (From Natural Deduction to Hilbert Proofs) *If* $\Gamma \vdash A\ \mathit{true}$ *then* $\hat{\Gamma} \vdash A\ \mathit{hil}$.

Proof: By induction on the given natural deduction.

Case:

$$\frac{A\ \mathit{true} \in \Gamma}{\Gamma \vdash A\ \mathit{true}}\ \mathsf{hyp}$$

$\hat{\Gamma} \vdash A\ hil$ By hypothesis $A\ hil \in \hat{\Gamma}$.

Case:

$$\frac{\Gamma, A_1\ true \vdash A_2\ true}{\Gamma \vdash A_1 \supset A_2\ true} \supset I$$

$\hat{\Gamma}, A_1\ hil \vdash A_2\ hil$ By i.h.
$\hat{\Gamma} \vdash A_1 \supset A_2\ hil$ By the deduction theorem (Theorem 1)

Case:

$$\frac{\Gamma \vdash B \supset A\ true \quad \Gamma \vdash B\ true}{\Gamma \vdash A\ true} \supset E$$

$\hat{\Gamma} \vdash B \supset A\ hil$ By i.h.
$\hat{\Gamma} \vdash B\ hil$ By i.h.
$\hat{\Gamma} \vdash A\ hil$ By rule (mp)

□

Since deduction in a Hilbert-style system is usually not presented with hypotheses, the following corollary is really the desired result.

Corollary 3 (Completeness of IKS) *If* $\vdash A\ true$ *then* $\vdash A\ hil$.

Proof: From Theorem 2 by taking $\Gamma = (\cdot)$. □

We can also return to the translation from Hilbert proofs to natural deductions which we postponed earlier.

Theorem 4 (From Hilbert Proofs to Natural Deductions) *If* $\vdash A\ hil$ *then* $\vdash A\ true$.

Proof: By induction on the structure of the given deduction. We just show the corresponding natural deduction proof term which can be unwound into a proof.

Case: $\vdash A \supset A\ hil$ (I).

$\vdash \lambda x.\, x : A \supset A$

Case: $\vdash B \supset (A \supset B)$ *hil* (K).

$$\vdash \lambda x.\, \lambda y.\, x : B \supset (A \supset B)$$

Case: $\vdash (A \supset (B \supset C)) \supset (A \supset B) \supset (A \supset C)$ (S).

$$\vdash \lambda x.\, \lambda y.\, \lambda z.\, (x\, z)\, (y\, z) : (A \supset (B \supset C)) \supset (A \supset B) \supset (A \supset C)$$

Case:

$$\frac{\vdash B \supset A\ \textit{hil} \quad \vdash B\ \textit{hil}}{\vdash A\ \textit{hil}}\ \mathsf{mp}$$

$\vdash M : B \supset A$	By i.h.
$\vdash N : B$	By i.h.
$\vdash M\, N : A$	By rule $\supset E$

□

5 Combinatory Reduction

In order to understand the computational meaning of Hilbert's system, we assign proof terms to the axioms and inference rules. We write $\vdash^{H} M : A$ if combinator term M represents a Hilbert proof of A.

$$\begin{array}{l}\vdash^{H} S : (A \supset (B \supset C)) \supset (A \supset B) \supset (A \supset C) \\ \vdash^{H} K : A \supset B \supset A \\ \vdash^{H} I : A \supset A\end{array}$$

$$\frac{\vdash^{H} M : A \supset B \quad \vdash^{H} N : A}{\vdash^{H} M\, N : B}\ \mathsf{mp}$$

From the translation of these into proof terms for natural deduction at the end of the last section, we can derive reductions for terms built from combinators. The idea is to supply enough arguments that the resut is no longer a λ-abstraction. Recall that juxtaposition (representing $\supset E$ in natural deduction proof terms and modus ponens in combinator terms) is left-associative. We write $\hat{M}$ for the translation of a combinator into a natural deduction proof term.

$$\begin{array}{lclll}
\hat{I} & = & \lambda x.\, x & I\,x & \longrightarrow & x \\
\hat{K} & = & \lambda x.\, \lambda y.\, x & K\,x\,y & \longrightarrow & x \\
\hat{S} & = & \lambda x.\, \lambda y.\, \lambda z.\, (x\,z)\,(y\,z) & S\,x\,y\,z & \longrightarrow & (x\,z)\,(y\,z)
\end{array}$$

Here, x, y, and z on the right-hand side stand for combinator terms of appropriate type. As an example, we consider $z{:}A \overset{H}{\vdash} S\,K\,K\,z : A$ which reduces to z:

$$\begin{array}{lcl}
S\,K\,K\,z & \longrightarrow & (K\,z)\,(K\,z) \\
 & \longrightarrow & z
\end{array}$$

This means that $S\,K\,K$ acts like I, both in terms of typing and in terms of reduction. When the analogue of local expansion is considered, however, the result is less clear, so I is often retained as a primitive combinator.

Combinatory reduction is interesting from the computational perspective because it does not involve substitution for a bound variable as a primitive operation. It has therefore been considered as the basis for compilation, especially for lazy functional languages, both using a fixed set of combinators (such as S, K, I) or varying set of combinators defined by their own rewrite rules. In either case, we need to compile functional programs to combinator form, which is discussed in the next section.

6 Bracket Abstraction

As with all prior metatheoretic proofs in the constructive section of this course, the proof of the deduction theorem is itself constructive. This means it contains an algorithm for constructing a proof of $\Gamma \vdash A \supset C$ *hil* given a proof of $\Gamma, A\,hil \vdash C$ *hil*. We can extract this algorithm and write is out as an explicit translation on combinators terms. To express hypothetical Hilbert proofs we follow the standard approach for hypothetical judgments, namely labeling assumptions with distinct variables and referring to them as proof terms.

$$\frac{x{:}A \in \Gamma}{\Gamma \overset{H}{\vdash} x : A}\ \mathsf{hyp}$$

Now we can specify $[x]M$ as the combinator term implementing the proof of the deduction theorem:

If $\Gamma, x{:}A \overset{H}{\vdash} M : C$ then $\Gamma \overset{H}{\vdash} [x]M : A \supset C$.

Keeping in mind the cases of the proof of the deduction theorem (Theorem 1), we obtain the following:

$$
\begin{array}{lcll}
[x]x & = & I & \\
[x]y & = & K\,y & \text{for } y \neq x \\
[x](M_1\,M_2) & = & S\,([x]M_1)\,([x]M_2) & \\
[x]X & = & K\,X & \text{where } X = I, K, \text{or } S
\end{array}
$$

We can combine the second and last case as follows:

$$
\begin{array}{lcll}
[x]x & = & I & \\
[x]M & = & K\,M & \text{for } x \text{ not in } M \\
[x](M_1\,M_2) & = & S\,([x]M_1)\,([x]M_2) &
\end{array}
$$

This second version leads to significantly smaller combinator terms, when compared to the first version which inserts applications of S for every subterm when translating $[x]M$, even if x does not occur in M at all. For a more detailed analysis of the complexity of bracket abstraction, see Statman [Sta86].

We can now easily extend bracket abstraction to compilation of functional terms. We write M^* for the compilation of a term $\Gamma \vdash M : A$. We will have that $\hat{\Gamma} \vdash^{H} M^* : A$. In particular, for a closed term with $\vdash M : A$ we obtain $\vdash^{H} M^* : A$, which is a closed combinator term.

$$
\begin{array}{lcl}
(\lambda x.\, M)^* & = & [x]M^* \\
(M_1\,M_2)^* & = & M_1^*\,M_2^* \\
x^* & = & x
\end{array}
$$

7 Hilbert Deduction for Modal Logic

We now revisit the modal logic of necessity and possibility with the goal of developing a complete axiom system. Before that, we have to add one new rule of inference, the rule of necessity.

$$
\frac{\vdash A\ \mathit{hil}}{\vdash \Box A\ \mathit{hil}}\ \mathsf{nec}
$$

For hypothetical Hilbert proofs, we follow the same strategy as for natural deduction: we allow hypotheses *A hil* and *A hvalid*, where the latter means "valid in the sense of Hilbert". We write $\Delta; \Gamma \vdash A\ \mathit{hil}$. The necessity rule then becomes the rule on the left, and we also have an additional hypothesis rule.

$$\frac{\Delta; \bullet \vdash A\ \mathit{hil}}{\Delta; \Gamma \vdash \Box A\ \mathit{hil}}\ \mathsf{nec} \qquad \frac{A\ \mathit{hvalid} \in \Delta}{\Delta; \Gamma \vdash A\ \mathit{hil}}\ \mathsf{vhyp}$$

Anticipating the translation from natural deduction to Hilbert proofs, we prove a generalized version of the deduction theorem. This proof will suggest new axiom schemas to handle necessity and possibility.

Theorem 5 (Modal Deduction Theorem)

(i) If $\Delta; \Gamma, A\ \mathit{hil} \vdash C\ \mathit{hil}$ *then* $\Delta; \Gamma \vdash A \supset C\ \mathit{hil}$.

(ii) If $\Delta, A\ \mathit{hvalid}; \Gamma \vdash C\ \mathit{hil}$ *then* $\Delta; \Gamma \vdash \Box A \supset C\ \mathit{hil}$

Proof: By induction on the structure of the given deduction. Part (i) works exactly as before, in the proof of Theorem 5. We show some cases of part (ii) below; the other cases go as for part (i).

Case:

$$\frac{}{\Delta, A\ \mathit{hvalid}; \Gamma \vdash A\ \mathit{hil}}\ \mathsf{vhyp}$$

In this case we need to show $\Delta; \Gamma \vdash \Box A \supset A\ \mathit{hil}$. We introduce the new axiom schema

$$\vdash \Box A \supset A\ \mathit{hil} \qquad (\mathsf{T}^{\Box})$$

from which the desired conclusion follows by weakening.

Case:

$$\frac{C\ \mathit{hvalid} \in \Delta}{\Delta, A\ \mathit{hvalid}; \Gamma \vdash C\ \mathit{hil}}\ \mathsf{vhyp}$$

$\Delta; \Gamma \vdash C\ \mathit{hil}$	By rule vhyp
$\Delta; \Gamma \vdash C \supset (\Box A \supset C)\ \mathit{hil}$	By axiom K
$\Delta; \Gamma \vdash \Box A \supset C\ \mathit{hil}$	By modus ponens

Case:

$$\frac{\Delta, A\ \mathit{hvalid}; \bullet \vdash C_1\ \mathit{hil}}{\Delta, A\ \mathit{hvalid}; \Gamma \vdash \Box C_1\ \mathit{hil}}\ \mathsf{nec}$$

$\Delta; \bullet \vdash \Box A \supset C_1\ \mathit{hil}$ By i.h.
$\Delta; \Gamma \vdash \Box(\Box A \supset C_1)\ \mathit{hil}$ By rule nec

At this point we need to introduce a new axiom schema

$$\vdash \Box(\Box A \supset C) \supset (\Box A \supset \Box C)\ \mathit{hil} \qquad (\mathsf{K4}^{\Box})$$

and continue with the proof

$\Delta; \Gamma \vdash \Box A \supset \Box C_1\ \mathit{hil}$ By modus ponens

$\Box$

Because there are no special rules regarding possibility, we do not need any special cases in the modal deduction theorem. However, we will need appropriate axioms in order to translate natural deductions to Hilbert proofs. We let this proof give us the remaining axioms. We extend the notation $\hat{\Gamma}$ to valid hypotheses as before.

Theorem 6 (From Modal Natural Deduction to Hilbert Proofs)

1. *If* $\Delta; \Gamma \vdash A\ \mathit{true}$ *then* $\hat{\Delta}; \hat{\Gamma} \vdash A\ \mathit{hil}$.
2. *If* $\Delta; \Gamma \vdash A\ \mathit{poss}$ *then* $\hat{\Delta}; \hat{\Gamma} \vdash \Diamond A\ \mathit{hil}$.

Proof: By induction on the structure of the given deduction.

Case:

$$\frac{A\ \mathit{true} \in \Gamma}{\Delta; \Gamma \vdash A\ \mathit{true}}\ \mathsf{hyp}$$

$\hat{\Delta}; \hat{\Gamma} \vdash A\ \mathit{hil}$ By hypothesis $A\ \mathit{hil} \in \hat{\Gamma}$

Case:

$$\frac{A\ \mathit{valid} \in \Delta}{\Delta; \Gamma \vdash A\ \mathit{true}}\ \mathsf{vhyp}$$

$\hat{\Delta}; \hat{\Gamma} \vdash A\ \mathit{hil}$ By hypothesis $A\ \mathit{hvalid} \in \hat{\Delta}$.

Case:

$$\frac{\Delta; \bullet \vdash A_1 \; true}{\Delta; \Gamma \vdash \Box A_1 \; true} \; \Box I$$

$\hat{\Delta}; \bullet \vdash A_1 \; hil$ — By i.h.
$\hat{\Delta}; \hat{\Gamma} \vdash \Box A_1 \; hil$ — By rule nec

Case:

$$\frac{\Delta; \Gamma \vdash \Box B \; true \quad \Delta, B \; hvalid; \Gamma \vdash A \; true}{\Delta; \Gamma \vdash A \; true} \; \Box E$$

$\hat{\Delta}, B \; hvalid; \hat{\Gamma} \vdash A \; hil$ — By i.h.
$\hat{\Delta}; \hat{\Gamma} \vdash \Box B \supset A \; hil$ — By the modal deduction theorem
$\hat{\Delta}; \hat{\Gamma} \vdash \Box B \; hil$ — By i.h.
$\hat{\Delta}; \hat{\Gamma} \vdash A \; hil$ — By modus ponens

Case:

$$\frac{\Delta; \Gamma \vdash \Box B \; true \quad \Delta, B \; hvalid; \Gamma \vdash A \; poss}{\Delta; \Gamma \vdash A \; poss} \; \Box E$$

As in the previous case, with $\Diamond A$ instead of A in the conclusion.

Case:

$$\frac{\Delta; \Gamma \vdash A \; poss}{\Delta; \Gamma \vdash \Diamond A \; true} \; \Diamond I$$

$\hat{\Delta}; \hat{\Gamma} \vdash \Diamond A \; hil$ — By i.h.

Case:

$$\frac{\Delta; \Gamma \vdash A \; true}{\Delta; \Gamma \vdash A \; poss} \; \mathsf{poss}$$

Here we need a new axiom schema

$$\vdash A \supset \Diamond A \; hil \qquad (\mathsf{T}^{\Diamond})$$

$\hat{\Delta}; \hat{\Gamma} \vdash A \; hil$ — By i.h.
$\hat{\Delta}; \hat{\Gamma} \vdash \Diamond A \; hil$ — By modus ponens

Case:

$$\frac{\Delta; \Gamma \vdash \Diamond B\ \mathit{true} \quad \Delta; \bullet, B\ \mathit{true} \vdash A\ \mathit{poss}}{\Delta; \Gamma \vdash A\ \mathit{poss}}\ \Diamond E$$

$\hat{\Delta}; \hat{\Gamma} \vdash \Diamond B\ \mathit{hil}$ By i.h.(i)
$\hat{\Delta}; B\ \mathit{hil} \vdash \Diamond A\ \mathit{hil}$ By i.h.(ii)
$\hat{\Delta}; \bullet \vdash B \supset \Diamond A\ \mathit{hil}$ By the modal deduction theorem
$\hat{\Delta}; \hat{\Gamma} \vdash \Box(B \supset \Diamond A)\ \mathit{hil}$ By rule nec

At this point we need a new axiom schema

$$\vdash \Box(B \supset \Diamond A) \supset (\Diamond B \supset \Diamond A)\ \mathit{hil} \qquad (\mathsf{K4}^{\Diamond})$$

and we can proceed with the proof

$\hat{\Delta}; \hat{\Gamma} \vdash \Diamond B \supset \Diamond A\ \mathit{hil}$ By modus ponens
$\hat{\Delta}; \hat{\Gamma} \vdash \Diamond A\ \mathit{hil}$ By modus ponens

□

The Hilbert system that can be read off from these proofs is the following.

Inference Rules.

$$\frac{\vdash A \supset B\ \mathit{hil} \quad \vdash A\ \mathit{hil}}{\vdash B\ \mathit{hil}}\ \mathsf{mp} \qquad \frac{\vdash A\ \mathit{hil}}{\vdash \Box A\ \mathit{hil}}\ \mathsf{nec}$$

Axioms. The names for the modal axioms below are not standard, although derived from standard notation.

$\vdash A \supset A\ \mathit{hil}$ (I)
$\vdash A \supset B \supset A\ \mathit{hil}$ (K)
$\vdash (A \supset B \supset C) \supset (A \supset B) \supset (A \supset C)\ \mathit{hil}$ (S)

$\vdash \Box A \supset A\ \mathit{hil}$ ($\mathsf{T}^{\Box}$)
$\vdash \Box(\Box A \supset B) \supset (\Box A \supset \Box B)\ \mathit{hil}$ ($\mathsf{K4}^{\Box}$)

$\vdash A \supset \Diamond A\ \mathit{hil}$ ($\mathsf{T}^{\Diamond}$)
$\vdash \Box(A \supset \Diamond B) \supset (\Diamond A \supset \Diamond B)\ \mathit{hil}$ ($\mathsf{K4}^{\Diamond}$)

Theorem 7 (From Hilbert Proofs to Modal Natural Deduction) *If* $\vdash A$ *hil then* $\vdash A$ *true*

Proof: By straightforward induction, proving all the axiom schemas directly via natural deduction. □

The usual presentation uses the same inference rules, but a slightly different set of axioms. Specifically, we replace

$$\vdash \Box(\Box A \supset B) \supset (\Box A \supset \Box B)\ hil \quad (\mathsf{K4}^{\Box})$$
$$\vdash \Box(A \supset \Diamond B) \supset (\Diamond A \supset \Diamond B)\ hil \quad (\mathsf{K4}^{\Diamond})$$

by

$$\vdash \Box(A \supset B) \supset (\Box A \supset \Box B)\ hil \quad (\mathsf{K}^{\Box})$$
$$\vdash \Box A \supset \Box\Box A\ hil \quad (\mathsf{4}^{\Box})$$

$$\vdash \Box(A \supset B) \supset (\Diamond A \supset \Diamond B)\ hil \quad (\mathsf{K}^{\Diamond})$$
$$\vdash \Diamond\Diamond A \supset \Diamond A\ hil \quad (\mathsf{4}^{\Diamond})$$

It is easy to see that $K^{\Box}$ and $4^{\Box}$ implies $K4^{\Box}$, and that $K^{\Diamond}$ and $4^{\Diamond}$ implies $K4^{\Diamond}$, which means these for axioms are also complete with respect to modal natural deduction. We have proven them in an earlier lecture, so they are also sound. See also Exercise 5.

8 Modal Combinators

This section is is speculative in the sense that no theorems relating combinatory reduction and λ-calculus reduction have been checked.

We now extend the proof term assignment from earlier, using the more familiar second set of axioms.

Inference Rules.

$$\frac{\vdash^{H} M : A \supset B \quad \vdash^{H} N : A}{\vdash^{H} M\, N : B}\ \mathsf{mp} \qquad \frac{\vdash^{H} M : A}{\vdash^{H} `M' : \Box A}\ \mathsf{nec}$$

Axioms. The names for the modal axioms below are not standard, but derived from standard names. For example, Simpson [Sim94] calls $(\Box A \supset A) \wedge (A \supset \Diamond A)$ the axiom T; we name the first conjunct $T^{\Box}$ and the second conjunct $T^{\Diamond}$. Note that $K^{\Box}$ and $K^{\Diamond}$ do not have anything to do with the usual K combinator of intuitionistic combinatory logic, but with the axiom K of modal logic.

$$
\begin{array}{l}
\overset{H}{\vdash} I : A \supset A \\
\overset{H}{\vdash} K : A \supset B \supset A \\
\overset{H}{\vdash} S : (A \supset B \supset C) \supset (A \supset B) \supset (A \supset C) \\
\\
\overset{H}{\vdash} T^{\Box} : \Box A \supset A \\
\overset{H}{\vdash} K^{\Box} : \Box(A \supset B) \supset (\Box A \supset \Box B) \\
\overset{H}{\vdash} 4^{\Box} : \Box A \supset \Box\Box A \\
\\
\overset{H}{\vdash} T^{\Diamond} : A \supset \Diamond A \\
\overset{H}{\vdash} K^{\Diamond} : \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) \\
\overset{H}{\vdash} 4^{\Diamond} : \Diamond\Diamond A \supset \Diamond A
\end{array}
$$

The reductions:

$$
\begin{array}{lcl}
I\,x & \longrightarrow & x \\
K\,x\,y & \longrightarrow & x \\
S\,x\,y\,z & \longrightarrow & (x\,z)\,(y\,z) \\
\\
T^{\Box}\,`u' & \longrightarrow & u \\
K^{\Box}\,`u'\,`w' & \longrightarrow & `u\,w' \\
4^{\Box}\,`u' & \longrightarrow & ``u'' \\
\\
K^{\Diamond}\,`u'\,(T^{\Diamond}\,x) & \longrightarrow & T^{\Diamond}\,(u\,x) \\
4^{\Diamond}\,(T^{\Diamond}\,(T^{\Diamond}\,x)) & \longrightarrow & T^{\Diamond}\,x
\end{array}
$$

Bracket abstraction:

(i) If $\Delta; \Gamma, x{:}A \overset{H}{\vdash} M : C$ then $\Delta; \Gamma \overset{H}{\vdash} [x]M : A \supset C$.

(ii) If $\Delta, u{:}A; \Gamma \overset{H}{\vdash} M : C$ then $\Delta; \Gamma \overset{H}{\vdash} [\![u]\!]M : \Box A \supset C$.

$$
\begin{array}{lcll}
[x]x & = & I & \\
[x]M & = & K\,M & \text{if } x \notin M \\
[x](M\,N) & = & S\,([x]M)\,([x]N) & \\
[x]`M' & = & K\,`M' & \\
\\
[\![u]\!]u & = & T^{\Box} & \\
[\![u]\!]M & = & K\,M & \text{for } u \notin M \\
[\![u]\!](M\,N) & = & S\,([\![u]\!]M)\,([\![u]\!]N) & \\
[\![u]\!]`M' & = & K4^{\Box}\,(`[\![u]\!]M') &
\end{array}
$$

Observe that the case for $[x]`M'$ is actually redundant with the second case, because x may not occur in M.

Translation from natural deduction:

(i) If $\Delta; \Gamma \vdash M : A$ then $\Delta; \Gamma \vdash^{H} M^* : A$.

(ii) If $\Delta; \Gamma \vdash E \div A$ then $\Delta; \Gamma \vdash^{H} E^+ : \Diamond A$.

$$
\begin{array}{lcl}
(x)^* & = & x \\
(M\,N)^* & = & (M^*)\,(N^*) \\
(\lambda x.\,M)^* & = & [x]M^* \\
\\
(u)^* & = & u \\
(\mathbf{box}\,M)^* & = & `(M^*)' \\
(\mathbf{let\,box}\,u = M\ \mathbf{in}\ N)^* & = & ([\![u]\!]N^*)\,M^* \\
(\mathbf{dia}\,E)^* & = & E^+ \\
\\
(\mathbf{let\,dia}\,x = M\ \mathbf{in}\ F)^+ & = & K4^{\Diamond}\,`[x]F^{+}'\,M^* \\
(\mathbf{let\,box}\,u = M\ \mathbf{in}\ F)^+ & = & ([\![u]\!]F^+)\,M^* \\
(M)^+ & = & T^{\Diamond}\,M^*
\end{array}
$$

Exercises

Exercise 1 *Add conjunction $A \wedge B$ and truth $\top$ to the development of this lecture.*

(i) Add appropriate axioms to intuitionistic modal logic.

(ii) Extend the deduction theorem (Theorem 5), showing the new cases.

(iii) Extend the translation from natural deduction to Hilbert proofs (Theorem 6), showing the new cases.

(iv) Extend the translation from Hilbert proofs to natural deductions by giving proof terms for the new axioms.

(v) Name the new axioms schemas as combinators and give appropriate reductions.

(vi) Show the new cases in bracket abstraction $[x]M$ and $[\![x]\!]M$.

(vii) Show the new cases in the definition of M^* and E^+.

Exercise 2 *Add disjunction $A \vee B$ and falsehood $\bot$ to the development of this lecture as in Exercise 1.*

Exercise 3 *Give reduction rules directly for $\mathsf{K4}^{\Box}$ and $\mathsf{K4}^{\Diamond}$.*

Exercise 4 *Assume we have a closed term $\vdash M : A$ so that $\stackrel{H}{\vdash} M^* : A$. How do reductions in M relate to combinatory reductions in M^*?*

(i) State and proof some form of correspondence for the non-modal intuitionistic case.

(ii) Generalize the correspondence to the modal case. You may use the reductions you devise in Exercise 3 if appropriate.

Exercise 5 *You may carry out this exercise using Hilbert proofs or combinators. Please note any other axioms or combinators you need besides those mentioned.*

(i) Prove that $\mathsf{K}^{\Box}$ and $\mathsf{4}^{\Box}$ together can prove $\mathsf{K4}^{\Box}$ and vice versa.

(ii) Prove that $\mathsf{K}^{\Diamond}$ and $\mathsf{4}^{\Diamond}$ together can prove to $\mathsf{K4}^{\Diamond}$ and vice versa.

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