15-780: Graduate AI

Lecture 2. Proofs & FOL

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Admin

- *Recitations: Fri. 3PM here (GHC 4307)*
- *Vote: useful to have one tomorrow?*
  - *would cover propositional & FO logic*
- *Draft schedule of due dates up on web*
  - *subject to change with notice*

Course email list

- *15780students AT cs.cmu.edu*
- *Everyone's official email should be in the list—we've sent a test message, so if you didn't get it, let us know*

Review

What is AI?

- *Lots of examples: poker, driving robots, flying birds, RoboCup*
- *Things that are easy for humans/animals to do, but no obvious algorithm*
- *Search / optimization / summation*
- *Handling uncertainty*
- *Sequential decisions*

Propositional logic

- *Syntax*
  - *variables, constants, operators*
  - *literals, clauses, sentences*
- *Semantics (model* $\mapsto$ *{T, F})*
- *Truth tables, how to evaluate formulas*
- *Satisfiable, valid, contradiction*
- *Relationship to CSPs*

Propositional logic

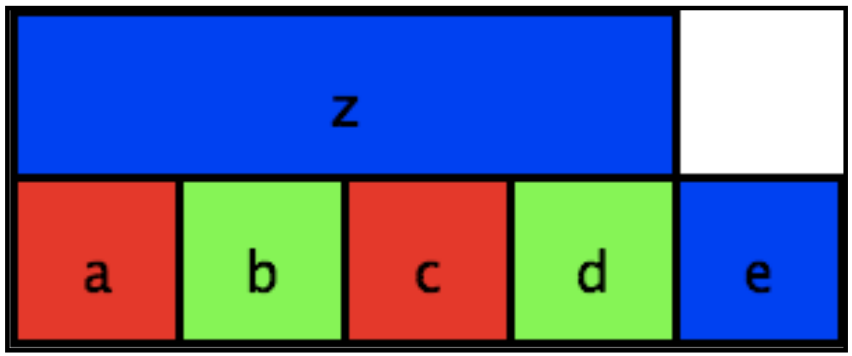
- *Manipulating formulas (e.g., de Morgan)*
- *Normal forms (e.g., CNF)*
- *Tseitin transformation to CNF*
- *Handling uncertainty (independent Nature choices + logical consequences)*
- *Compositional semantics*
- *How to translate informally-specified problems into logic (e.g., 3-coloring)*

NP

Satisfiability

- *SAT: determine whether a propositional logic sentence has a satisfying model*
- *A **decision problem**: instance → yes or no*
- *Fundamental problem in CS*
    - *many decision problems **reduce** to SAT*
    - *informally, if we can solve SAT, we can solve these other problems*
- *A SAT solver is a good AI building block*

Example decision problem

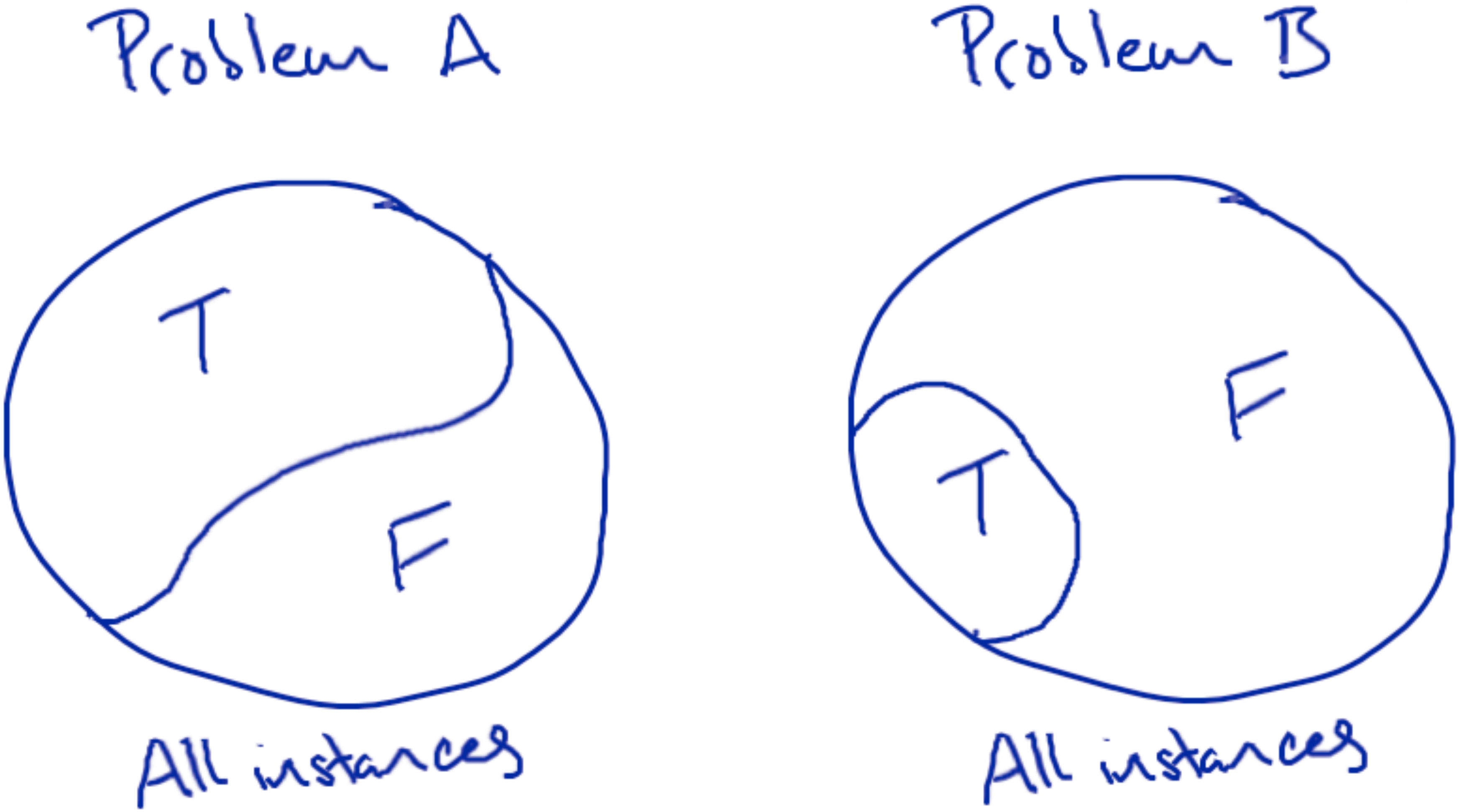


- *k-coloring: can we color a map using only k colors in a way that keeps neighboring regions from being the same color?*

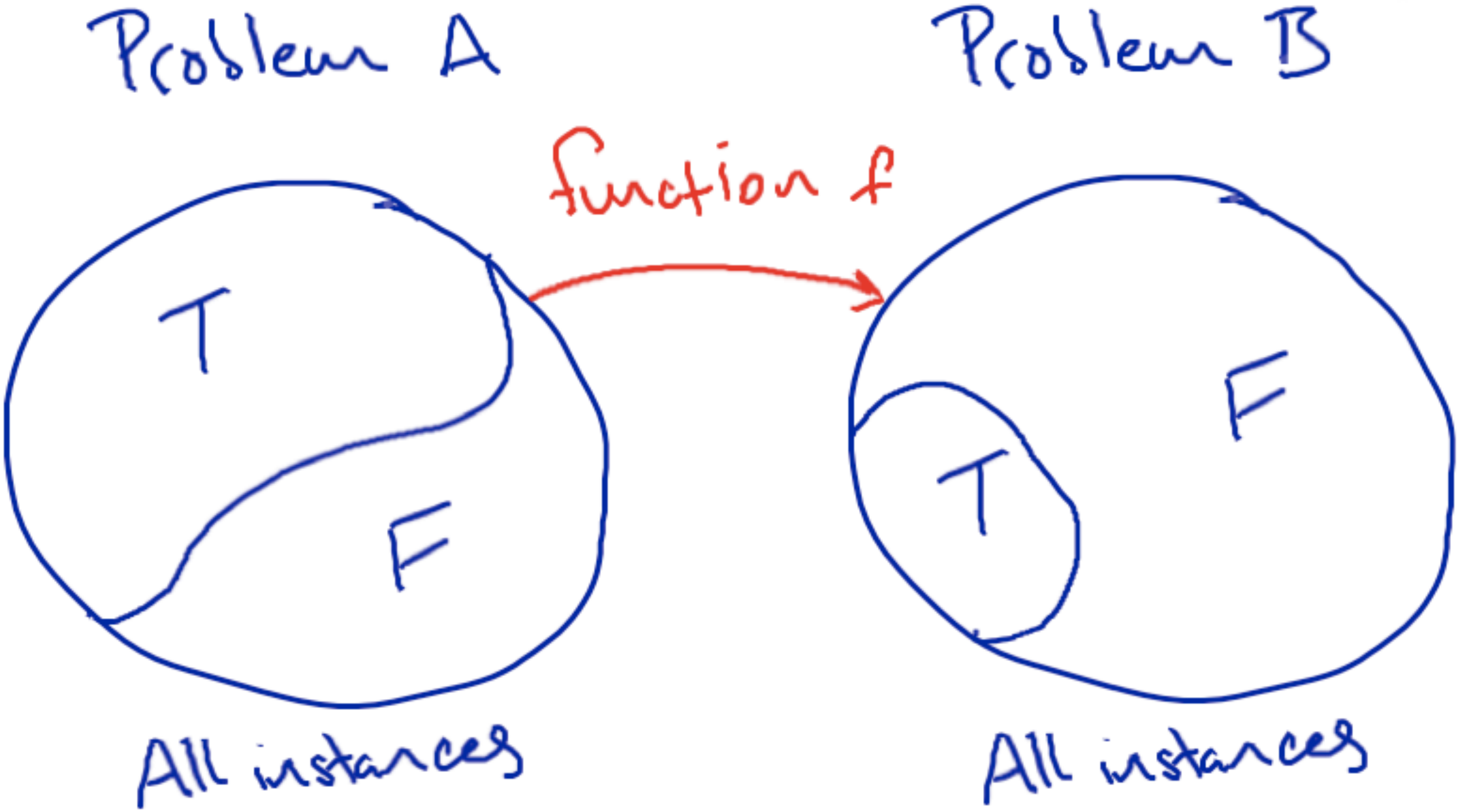
Reduction

- *Loosely, "A reduces to B" means that if we can solve B then we can solve A*
- *Formally, let A, B be decision problems (instances → Y or N)*
- *A reduction is a poly-time function f such that, given an instance a of A*
  - *f(a) is an instance of B, and*
  - $A(a) = B(f(a))$

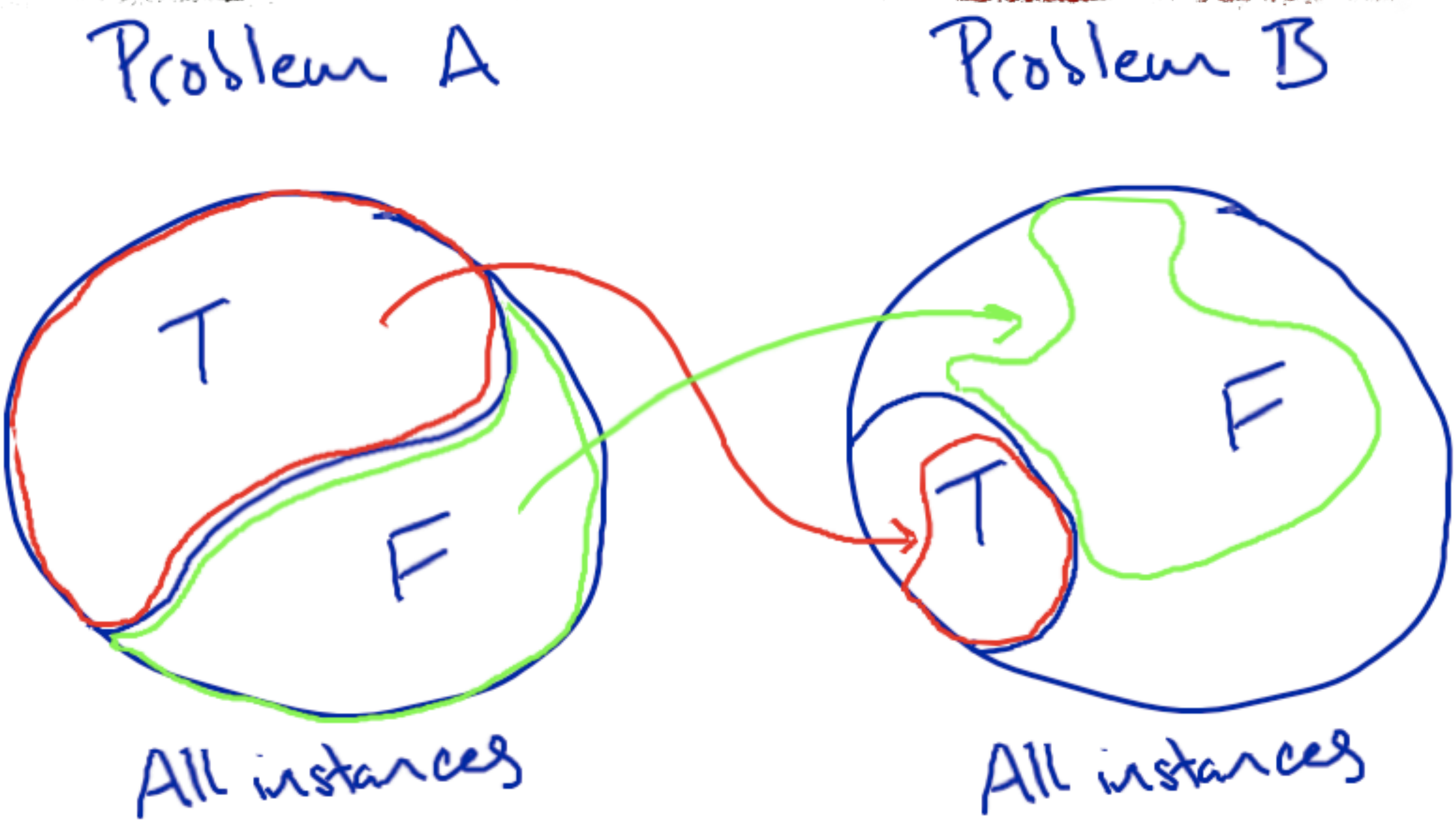
Reduction picture



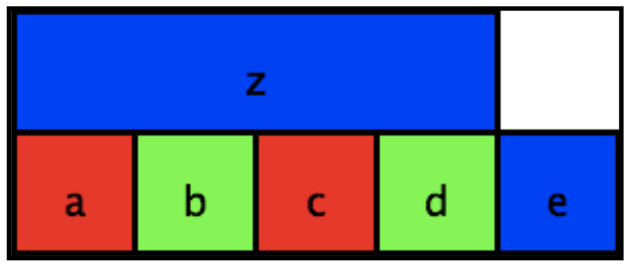
Reduction picture



Reduction picture



Reducing k-coloring → SAT



$$(a_r \vee a_g \vee a_b) \wedge (b_r \vee b_g \vee b_b) \wedge (c_r \vee c_g \vee c_b) \wedge$$

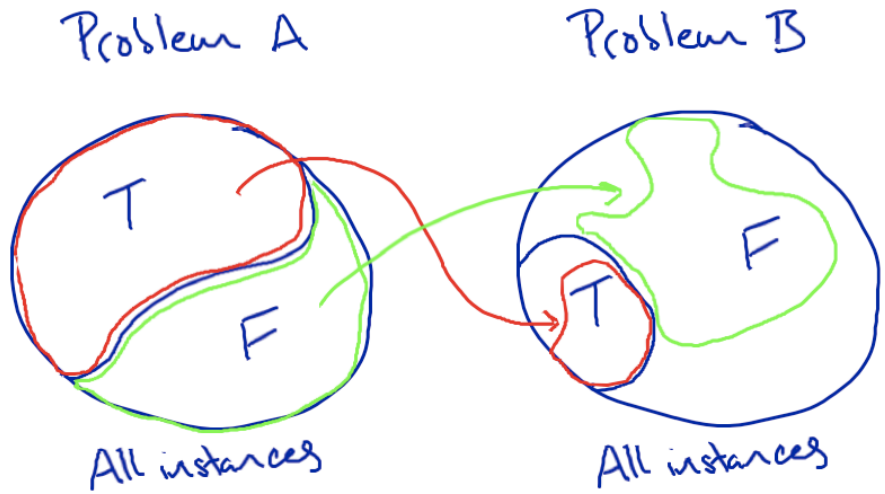
$$(d_r \vee d_g \vee d_b) \wedge (e_r \vee e_g \vee e_b) \wedge (z_r \vee z_g \vee z_b) \wedge$$

$$(\neg a_r \vee \neg b_r) \wedge (\neg a_g \vee \neg b_g) \wedge (\neg a_b \vee \neg b_b) \wedge$$

$$(\neg a_r \vee \neg z_r) \wedge (\neg a_g \vee \neg z_g) \wedge (\neg a_b \vee \neg z_b) \wedge$$

…

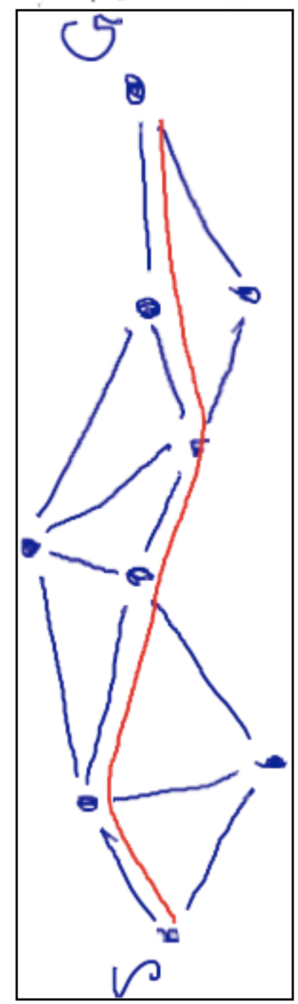
Direction of reduction



- *When A reduces to B:*
  - *if we can solve B, we can solve A*
  - *so B must be at least as hard as A*
- *Trivially, can take an easy problem and reduce it to a hard one*

Not-so-useful reduction

- *Path planning reduces to SAT*
- *Variables: is edge e in path?*
- *Constraints:*
  - *exactly 1 path-edge touches start*
  - *exactly 1 path-edge touches goal*
  - *either 0 or 2 touch each other node*



More useful: SAT $\rightarrow$ CNF-SAT

- *Given any propositional formula, Tseitin transformation produces (in poly time) an equivalent CNF formula*
- *So, given a CNF-SAT solver, we can solve SAT with general formulas*

More useful: CNF-SAT → 3SAT

- *Can reduce even further, to 3SAT*
  - *is 3CNF formula satisfiable?*
  - *3CNF: at most 3 literals per clause*
- *Useful if reducing SAT/3SAT to another problem (to show other problem hard)*

CNF-SAT → 3SAT

- *Must get rid of long clauses*
- *E.g.,* $(a \lor \neg b \lor c \lor d \lor e \lor \neg f)$
- *Replace with*

$$(a \lor \neg b \lor x) \land (\neg x \lor c \lor y) \land$$
$$(\neg y \lor d \lor z) \land (\neg z \lor e \lor \neg f)$$

NP

- *A decision problem is in NP if it reduces to SAT*
- *E.g., TSP, k-coloring, propositional planning, integer programming (decision versions)*
- *E.g., path planning, solving linear equations*

NP-complete

- *Many decision problems reduce back and forth to SAT: they are* ***NP-complete***
    - *Cook showed how to simulate any poly-time nondeterministic computation w/ (very complicated, but still poly-size) SAT problem*
- *Equivalently, SAT is exactly as hard (in theory at least) as these other problems*

S. A. Cook. The complexity of theorem-proving procedures, Proceedings of ACM STOC'71, pp. 151–158, 1971.

Open question: P = NP

- *P = there is a poly-time algorithm to solve*
- *NP = reduces to SAT*
- *We know of no poly-time algorithm for SAT, but we also can't prove that SAT requires more than about linear time!*

Cost of reduction

- *Complexity theorists often ignore little things like constant factors (or even polynomial factors!)*
- *So, is it a good idea to reduce your decision problem to SAT?*
- *Answer: sometimes…*

Cost of reduction

- *SAT is well studied ⇒ fast solvers*
- *So, if there is an efficient reduction, ability to use fast SAT solvers can be a win*
  - *e.g., 3-coloring*
  - *another example later (SATplan)*
- *Other times, cost of reduction is too high*
  - *usu. because instance gets bigger*
  - *will also see example later (MILP)*

Choosing a reduction

- *May be many reductions from problem A to problem B*
- *May have **wildly** different properties*
    - *e.g., solving transformed instance may take seconds vs. days*

Proofs

Entailment

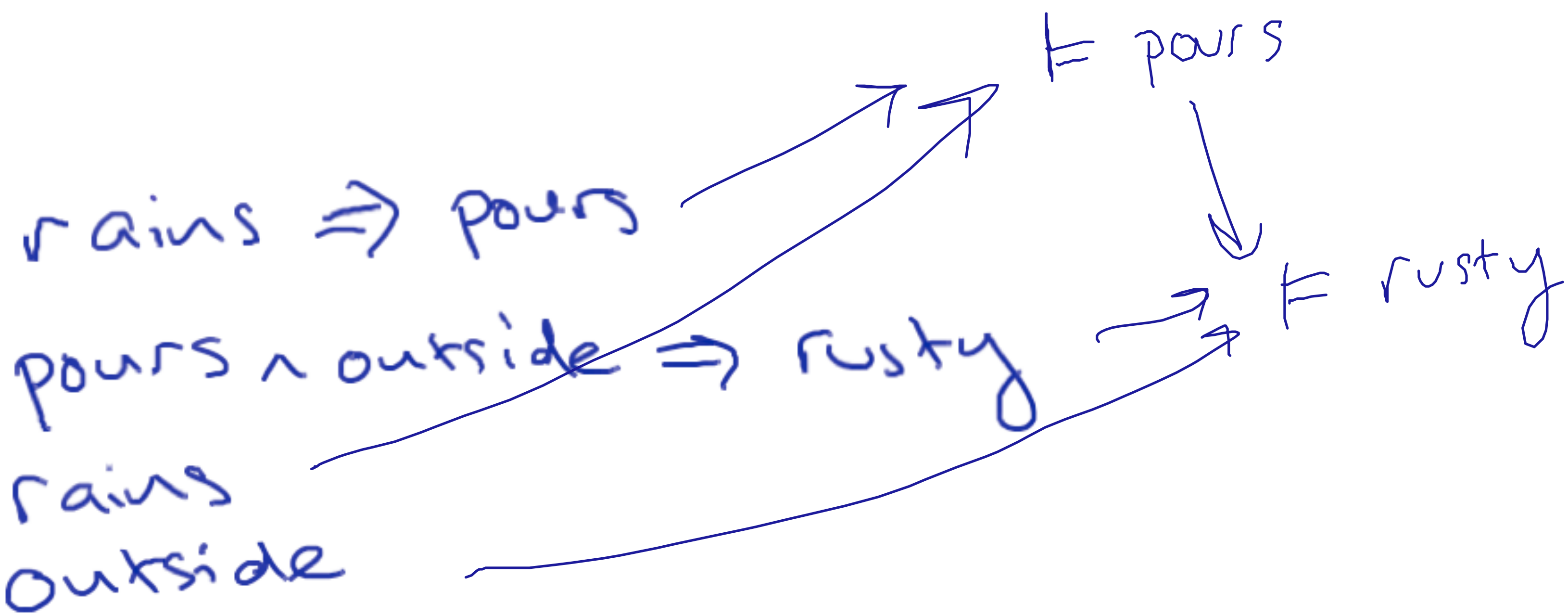
	A	B
T T	F	F
T F	T	T
F T	T	T
F F	F	T

- *Sentence A* ***entails*** *sentence B, A ⊨ B, if B is true in every model where A is*
  - *same as saying that (A ⇒ B) is valid*

Proof tree

- *A tree with a formula at each node*
- *At each internal node, children* $\vDash$ *parent*
- *Leaves:* ***assumptions*** *or* ***premises***
- *Root:* ***consequence***
- *If we believe assumptions, we should also believe consequence*

Proof tree example



Proof by contradiction

- *Assume opposite of what we want to prove, show it leads to a contradiction*
- *Suppose we want to show* $KB \models S$
- *Write KB' for* $(KB \wedge \neg S)$
- *Build a proof tree with*
  - *assumptions drawn from clauses of KB'*
  - *conclusion = F*
  - *so,* $(KB \wedge \neg S) \models F$ *(contradiction)*

Proof by contradiction

KB

rains ⇒ pours

pours ∧ outside ⇒ rusty

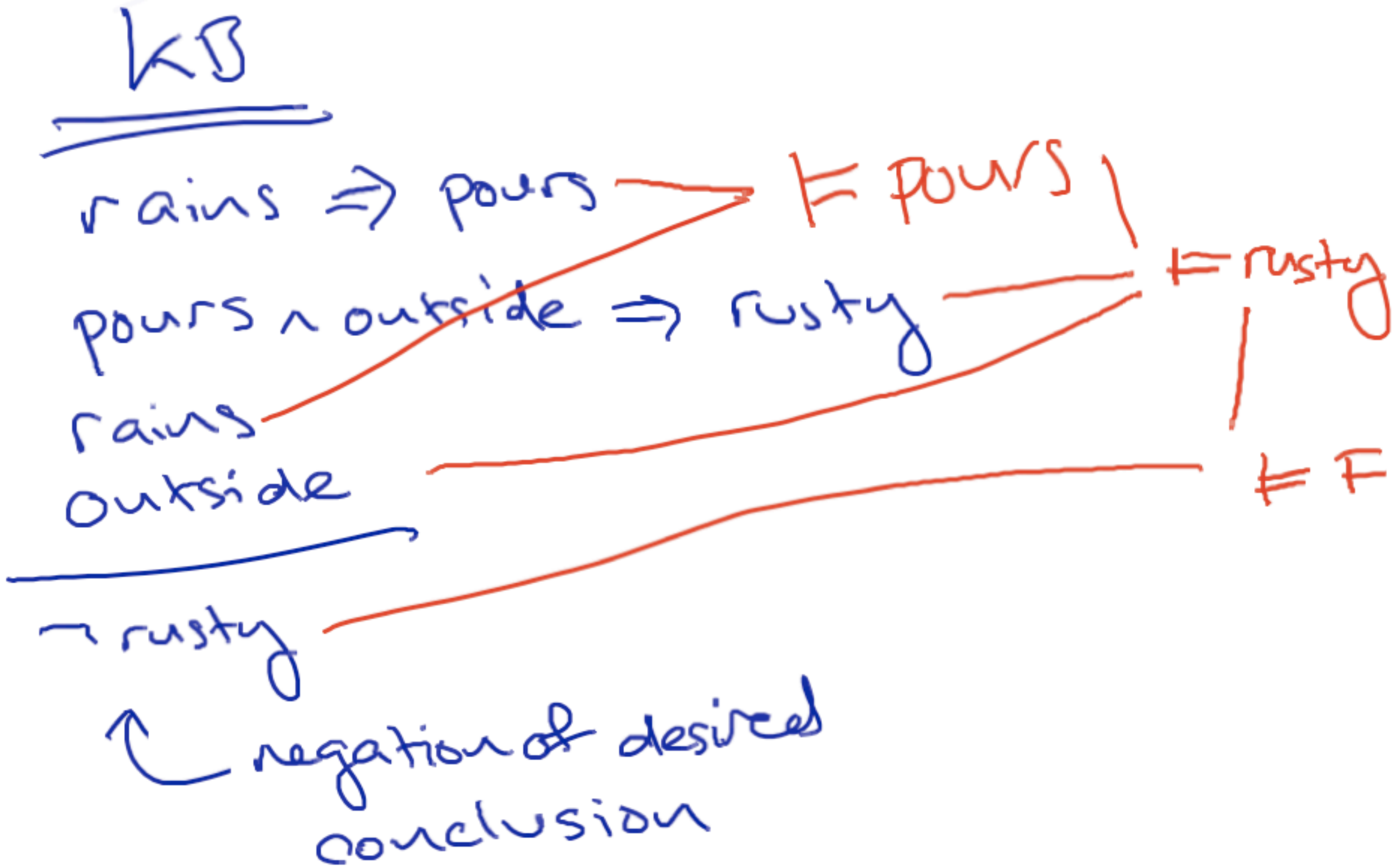
rains

outside

¬ rusty

← negation of desired conclusion

Proof by contradiction



Inference rules

Inference rule

- *To make a proof tree, we need to be able to figure out new formulas entailed by KB*
- *Method for finding entailed formulas =* ***inference rule***
- *We've implicitly been using one already*

Modus ponens

$$\frac{(a \wedge b \wedge c \Rightarrow d) \quad a \quad b \quad c}{d}$$

- *Probably most famous inference rule: all men are mortal, Socrates is a man, therefore Socrates is mortal*
- *Quantifier-free version:*

  $man(Socrates) \wedge$

  $(man(Socrates) \Rightarrow mortal(Socrates))$

Another inference rule

$$\frac{(a \Rightarrow b) \quad \neg b}{\neg a}$$

- ***Modus tollens***
- *If it's raining the grass is wet; the grass is not wet, so it's not raining*

One more…

$$\frac{(\alpha \vee c) \quad (\neg c \vee \beta)}{\alpha \vee \beta}$$

- ***Resolution***
  - *α, β are arbitrary subformulas*
- *Combines two formulas that contain a literal and its negation*
- *Not as commonly known as modus ponens / tollens*

Resolution example

- *Modus ponens / tollens are special cases*
- *Modus tollens:*

  $(\neg raining \lor grass\text{-}wet) \land \neg grass\text{-}wet \models \neg raining$

Resolution example

- $rains \Rightarrow pours$
- $pours \wedge outside \Rightarrow rusty$
- *Can we conclude* $rains \wedge outside \Rightarrow rusty$*?*

Resolution example

- $rains \Rightarrow pours$
- $pours \wedge outside \Rightarrow rusty$
- *Can we conclude* $rains \wedge outside \Rightarrow rusty$*?*

$$\neg rains \vee pours$$
$$\neg pours \vee \neg outside \vee rusty$$

Resolution example

- $rains \Rightarrow pours$
- $pours \wedge outside \Rightarrow rusty$
- *Can we conclude* $rains \wedge outside \Rightarrow rusty$?

$$\frac{\begin{array}{c}\neg rains \vee pours \\ \neg pours \vee \neg outside \vee rusty\end{array}}{\neg rains \vee \neg outside \vee rusty}$$

Resolution

$$\frac{(\alpha \lor c) \quad (\neg c \lor \beta)}{\alpha \lor \beta}$$

- *Simple proof by case analysis*
- *Consider separately cases where we assign c = True and c = False*

Resolution case analysis

$$(\alpha \vee c) \wedge (\neg c \vee \beta)$$

$c = T$: $\quad T \wedge \beta = \beta$

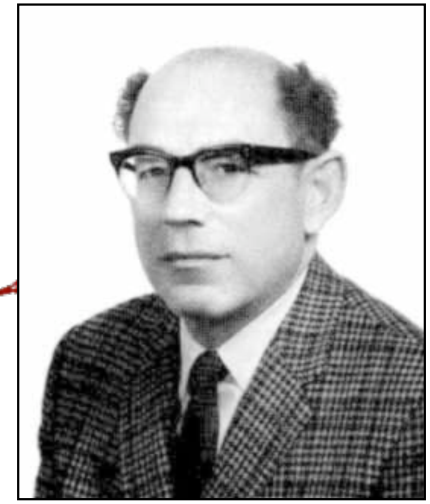
$c = F$: $\quad \alpha \wedge T = \alpha$

$$\alpha \vee \beta$$

Soundness and completeness

- *An inference procedure is* ***sound*** *if it can only conclude things entailed by KB*
  - *common sense; haven't discussed anything unsound*
- *A procedure is* ***complete*** *if it can conclude everything entailed by KB*

Completeness



J. A. Robinson
1918–1974

- *Modus ponens by itself is* ***incomplete***
- *Resolution + proof by contradiction is* ***complete*** *for propositional formulas represented as sets of clauses*
  - *famous theorem due to Robinson*
  - *if KB ⊨ F, we'll derive empty clause*
- *Caveat: also need* ***factoring****, removal of redundant literals (a ∨ b ∨ a) ⊨ (a ∨ b)*

Algorithms

- *We now have our first* algorithm for SAT*
  - *remove redundant literals (factor) wherever possible*
  - *pick an application of resolution according to some fair rule*
  - *add its consequence to KB*
  - *repeat*
- *Not a great algorithm, but works*

Variations

- *Horn clause inference*
- *MAXSAT*
- *Nonmonotonic logic*

Horn clauses

- *Horn clause:* $(a \wedge b \wedge c \Rightarrow d)$
- *Equivalently,* $(\neg a \vee \neg b \vee \neg c \vee d)$
- *Disjunction of literals, **at most one** of which is positive*
- *Positive literal =* ***head****, rest =* ***body***

Use of Horn clauses

- *People find it easy to write Horn clauses (listing out conditions under which we can conclude head)*

  $$happy(John) \wedge happy(Mary) \Rightarrow happy(Sue)$$

- *No negative literals in above formula; again, easier to think about*

Why are Horn clauses important

- *Modus ponens alone is complete*
- *So is modus tollens alone*
- *Inference in a KB of propositional Horn clauses is linear*
  - *e.g., by forward chaining*

Forward chaining

- *Look for a clause with all body literals satisfied*
- *Add its head to KB (modus ponens)*
- *Repeat*
- *See RN for more details*

MAXSAT

- *Given a CNF formula* $C_1 \wedge C_2 \wedge \ldots \wedge C_n$
- *Clause weights* $w_1, w_2, \ldots w_n$ *(weighted version) or* $w_i = 1$ *(unweighted)*
- *Find model which satisfies clauses of* ***maximum total weight***
  - *decision version: max weight* $\geq w$*?*
- *More generally, weights on variables (bonus for setting to T): MAXVARSAT*

Nonmonotonic logic

- *Suppose we believe all birds can fly*
- *Might add a set of sentences to KB*

$$bird(Polly) \Rightarrow flies(Polly)$$

$$bird(Tweety) \Rightarrow flies(Tweety)$$

$$bird(Tux) \Rightarrow flies(Tux)$$

$$bird(John) \Rightarrow flies(John)$$

…

Nonmonotonic logic

- *Fails if there are penguins in the KB*
- *Fix: instead, add*

$$bird(Polly) \land \neg ab(Polly) \Rightarrow flies(Polly)$$

$$bird(Tux) \land \neg ab(Tux) \Rightarrow flies(Tux)$$

…

- *ab(Tux) is an "abnormality predicate"*
- *Need separate $ab_i(x)$ for each type of rule*

Nonmonotonic logic

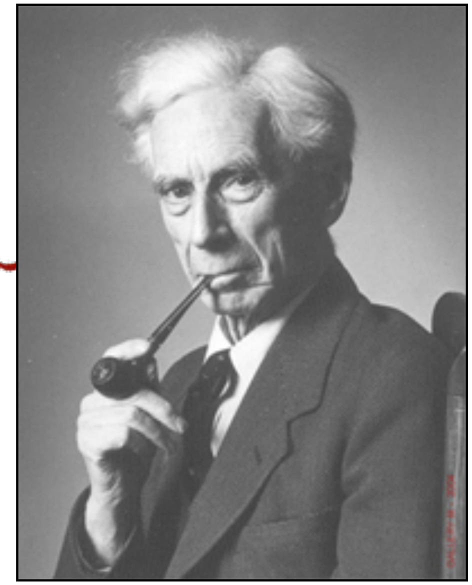
- *Now set as few abnormality predicates as possible (a MAXVARSAT problem)*
- *Can prove flies(Polly) or flies(Tux) with no ab(x) assumptions*
- *If we assert ¬flies(Tux), must now assume ab(Tux) to maintain consistency*
- *Can't prove flies(Tux) any more, but can still prove flies(Polly)*

Nonmonotonic logic

- *Works well as long as we don't have to choose between big sets of abnormalities*
  - *is it better to have 3 flightless birds or 5 professors that don't wear jackets with elbow-patches?*
  - *even worse with nested abnormalities: birds fly, but penguins don't, but superhero penguins do, but …*

First-order logic

First-order logic



Bertrand Russell
1872-1970

- *So far we've been using opaque vars like **rains** or **happy(John)***
- *Limits us to statements like "it's raining" or "if John is happy then Mary is happy"*
- *Can't say "all men are mortal" or "if John is happy then someone else is happy too"*

Predicates and objects

- *Interpret happy(John) or likes(Joe, pizza) as a **predicate** applied to some **objects***
- *Object = an object in the world*
- *Predicate = boolean-valued function of objects*
- *Zero-argument predicate x() plays same role that Boolean variable x did before*

Distinguished predicates

- *We will assume three distinguished predicates with fixed meanings:*
  - *True / T, False / F*
  - *Equal(x, y)*
- *We will also write* $(x = y)$ *and* $(x \neq y)$

Equality satisfies usual axioms

- *Reflexive, transitive, symmetric*
- *Substituting equal objects doesn't change value of expression*

  $(John = Jonathan) \wedge loves(Mary, John)$
  $\Rightarrow loves(Mary, Jonathan)$

Functions

- ***Functions*** *map zero or more objects to another object*
  - *e.g., professor(15-780), last-common-ancestor(John, Mary)*
- *Zero-argument function is the same as an object—John v. John()*

The **nil** object

- *Functions are untyped: must have a value for **any** set of arguments*
- *Typically add a **nil** object to use as value when other answers don't make sense*

Types of values

- *Expressions in propositional logic could only have Boolean (T/F) values*
- *Now we have two types of expressions: object-valued and Boolean-valued*
  - *done(slides(15-780))* $\Rightarrow$ *happy(professor(15-780))*
- *Functions map objects to objects; predicates map objects to Booleans; connectives map Booleans to Booleans*

Definitions

- ***Term*** *= expression referring to an object*
  - *John*
  - *left-leg-of(father-of(president-of(USA)))*
- ***Atom*** *= predicate applied to objects*
  - *happy(John)*
  - *raining*
  - *at(robot, Wean-5409, 11AM-Wed)*

Definitions

- ***Literal*** *= possibly-negated atom*
  - *happy(John), ¬happy(John)*
- ***Sentence*** *or* ***formula*** *= literals joined by connectives like* ∧∨¬⇒
  - *raining*
  - *done(slides(780)) ⇒ happy(professor)*
- ***Expression*** *= term or formula*

Semantics

- *Models are now much more complicated*
  - *List of objects (nonempty, may be infinite)*
  - *Lookup table for each function mentioned*
  - *Lookup table for each predicate mentioned*
- *Meaning of sentence: model → {T, F}*
- *Meaning of term: model → object*

For example

KB describing example

- $alive(cat)$
- $ear\text{-}of(cat) = ear$
- $in(cat, box) \wedge in(ear, box)$
- $\neg in(box, cat) \wedge \neg in(cat, nil) \ldots$
- $ear\text{-}of(box) = ear\text{-}of(ear) = ear\text{-}of(nil) = nil$
- $cat \neq box \wedge cat \neq ear \wedge cat \neq nil \ldots$

Aside: avoiding verbosity

- ***Closed-world assumption****: literals not assigned a value in KB are false*
  - *avoid stating ¬in(box, cat), etc.*
- ***Unique names assumption****: objects with separate names are separate*
  - *avoid box ≠ cat, cat ≠ ear, …*

Aside: typed variables

- *KB also illustrates need for* ***data types***
- *Don't want to have to specify ear-of(box) or ¬in(cat, nil)*
- *Could design a type system*
  - *argument of happy() is of type* animate
- *Include rules saying function instances which disobey type rules have value nil*

Model of example

- *Objects: C, B, E, N*
- *Function values:*
  - *cat: C, box: B, ear: E, nil: N*
  - *ear-of(C): E, ear-of(B): N, ear-of(E): N, ear-of(N): N*
- *Predicate values:*
  - *in(C, B), ¬ in(C, C), ¬ in(C, N), …*

Failed model

- *Objects: C, E, N*
- *Fails because there's no way to satisfy inequality constraints with only 3 objects*

Another possible model

- *Objects: C, B, E, N, X*
- *Extra object X could have arbitrary properties since it's not mentioned in KB*
- *E.g., X could be its own ear*

An embarrassment of models

- *In general, can be infinitely many models*
  - *unless KB limits number somehow*
- *Job of KB is to rule out models that don't match our idea of the world*
- *Saw how to rule out CEN model*
- *Can we rule out CBENX model?*

Getting rid of extra objects

- *Can use quantifiers to rule out CBENX model:*

  $\forall x.\ x = cat \lor x = box \lor x = ear \lor x = nil$

- *Called a* ***domain closure assumption***

Quantifiers, informally

- *Add* ***quantifiers*** *and* ***object variables***
  - $\forall x.\ man(x) \Rightarrow mortal(x)$
  - $\neg \exists x.\ lunch(x) \wedge free(x)$
- $\forall$*: no matter how we replace object variables with objects, formula is still true*
- $\exists$*: there is some way to fill in object variables to make formula true*

New syntax

- *Object variables are **terms***
- *Build atoms from variables x, y, … as well as constants John, Fred, …*
  - *man(x), loves(John, z), mortal(brother(y))*
- *Build formulas from these atoms*
  - *man(x) ⇒ mortal(brother(x))*
- *New syntactic construct: term or formula w/ **free variables***

New syntax ⇒ new semantics

- ***Variable assignment*** *for a model M maps syntactic variables to model objects*
  - *x: C, y: N*
- *Meaning of expression w/ free vars: look up in assignment, then continue as before*
  - *term: (model, var asst) → object*
  - *formula: (model, var asst) → truth value*

Example

- *Model: CEBN model from above*
- *Assignment: (x: C, y: N)*
- $alive(ear(x)) \mapsto alive(ear(C)) \mapsto alive(E) \mapsto T$

Working with assignments

- *Write ε for an arbitrary assignment (e.g., all variables map to nil)*
- *Write (V / x: obj) for the assignment which is just like V except that variable x maps to object obj*

More new syntax: Quantifiers, binding

- *For any variable x and formula F,* $(\forall x.\ F)$ *and* $(\exists x.\ F)$ *are formulas*
- *Adding quantifier for x is called **binding** x*
  - *In* $(\forall x.\ likes(x, y))$*, x is bound, y is free*
- *Can add quantifiers and apply logical operations like* $\wedge \vee \neg$ *in any order*
- *But must eventually wind up with **ground** formula (no free variables)*

Semantics of ∀

- *Sentence (∀x. S) is T in (M, V) if S is T in (M, V / x: obj) for all objects obj in M*

Example

- *M has objects (A, B, C) and predicate happy(x) which is true for A, B, C*
- *Sentence* $\forall x.$ *happy(x) is satisfied in (M, ε)*
  - *since happy(A), happy(B), happy(C) are all satisfied in M*
  - *more precisely, happy(x) is satisfied in (M, ε/x:A), (M, ε/x:B), (M, ε/x:C)*

Semantics of ∃

- *Sentence (∃x. S) is true in (M, V) if there is some object obj in M such that S is true in (M, V / x: obj)*

Example

- *M has objects (A, B, C) and predicate*
  - *happy(A) = happy(B) = True*
  - *happy(C) = False*
- *Sentence* $\exists x.$ *happy(x) is satisfied in (M, ε)*
- *Since happy(x) is satisfied in (M, ε/x:B)*

Scoping rules (so we don't have to write a gazillion parens)

- *In* $(\forall x.\, F)$ *and* $(\exists x.\, F)$, F = ***scope*** *= part of formula where quantifier applies*
- *Variable x is bound by* ***innermost*** *possible quantifier (matching name, in scope)*
- *Two variables in different scopes can have same name—they are still different vars*
- *Quantification has lowest precedence*

Scoping examples

- $(\forall x.\ happy(x)) \vee (\exists x.\ \neg happy(x))$
  - *Either everyone's happy, or someone's unhappy*
- $\forall x.\ (raining \wedge outside(x) \Rightarrow (\exists x.\ wet(x)))$
  - *The x who is outside may not be the one who is wet*

Scoping examples

- *English sentence "everybody loves somebody" is ambiguous*
- *Translates to logical sentences*
  - $\forall x.\ \exists y.\ loves(x, y)$
  - $\exists y.\ \forall x.\ loves(x, y)$

Equivalence in FOL

Entailment, etc.

- *As before, entailment, satisfiability, validity, equivalence, etc. refer to all possible models*
  - *these words only apply to ground sentences, so variable assignment doesn't matter*
- *But now, can't determine by enumerating models, since there could be infinitely many*
- *So, must do reasoning via equivalences or entailments*

Equivalences

- *All transformation rules for propositional logic still hold*
- *In addition, there is a "De Morgan's Law" for moving negations through quantifiers*

$$\neg \forall x.\, S \equiv \exists x.\, \neg S$$

$$\neg \exists x.\, S \equiv \forall x.\, \neg S$$

- *And, rules for getting rid of quantifiers*

Generalizing CNF

- *Eliminate ⇒, move ¬ in w/ De Morgan*
  - *but ¬ moves through quantifiers too*
- *Get rid of quantifiers (see below)*
- *Distribute ∧∨, or use Tseitin*

Do we really need ∃?

- $\exists x.\ happy(x)$
- $happy(happy\_person())$

- $\forall y.\ \exists x.\ loves(y, x)$
- $\forall y.\ loves(y, loved\_one(y))$

Skolemization

Thoraf Albert Skolem
1887–1963

- *Called **Skolemization** (after Thoraf Albert Skolem)*
- *Eliminate ∃ by substituting a function of arguments of all enclosing ∀ quantifiers*
- *Make sure to use a new name!*

Do we really need ∀?

- *Positions of quantifiers irrelevant (as long as variable names are distinct)*
  - $\forall x.\ happy(x) \wedge \forall y.\ takes(y, CS780)$
  - $\forall x.\ \forall y.\ happy(x) \wedge takes(y, CS780)$
- *So, might as well drop them*
  - $happy(x) \wedge takes(y, CS780)$

Getting rid of quantifiers

- ***Standardize apart*** *(avoid name collisions)*
- *Skolemize*
- *Drop* ∀ *(free variables implicitly universally quantified)*
- *Terminology: still called "free" even though quantification is implicit*

For example

- $\forall x.\ man(x) \Rightarrow mortal(x)$
  - $\neg man(x) \vee mortal(x)$
- $\forall y.\ \exists x.\ loves(y, x)$
  - $loves(y, f(y))$
- $\forall x.\ honest(x) \Rightarrow happy(Diogenes)$
  - $\neg honest(x) \vee happy(Diogenes)$
- $(\forall x.\ honest(x)) \Rightarrow happy(Diogenes)$

Exercise

- $(\forall x.\ honest(x)) \Rightarrow happy(Diogenes)$

$\neg(\forall x.\ honest(x)) \vee happy(D)$

$(\exists x.\ \neg honest(x)) \vee happy(D))$

$\neg honest(foo()) \vee happy(D)$

Proofs in FOL

FOL is special

- *Despite being much more powerful than propositional logic, there is still a* ***sound*** *and* ***complete*** *inference procedure for FOL w/ equality*
- *Almost any significant extension breaks this property*
- *This is why FOL is popular: very powerful language with a sound & complete inference procedure*

Proofs

- *Proofs by contradiction work as before:*
  - *add ¬S to KB*
  - *put in CNF*
  - *run resolution*
  - *if we get an empty clause, we've proven S by contradiction*
- *But, CNF and resolution have changed*

Generalizing resolution

- *Propositional:* $(\neg a \lor b) \land a \models b$
- *FOL:*

  $(\neg man(x) \lor mortal(x)) \land man(Socrates)$

  $\models (\neg man(Socrates) \lor mortal(Socrates)) \land man(Socrates)$

  $\models mortal(Socrates)$
- *Difference: had to substitute* $x \rightarrow Socrates$

Universal instantiation

- *What we just did is UI:*

  $(\neg man(x) \vee mortal(x))$

  $\models (\neg man(Socrates) \vee mortal(Socrates))$

- *Works for $x \rightarrow$ any term not containing x*

  $\ldots \models (\neg man(uncle(y)) \vee mortal(uncle(y)))$

- *For proofs, need a good way to find useful instantiations*

Substitution lists

- *List of variable → term pairs*
- *Values may contain variables (leaving flexibility about final instantiation)*
- *But, no LHS may be contained in any RHS*
  - *i.e., applying substitution twice is the same as doing it once*
- *E.g., L = (x → Socrates, y → uncle(z))*

Substitution lists

- *Apply a substitution to an expression: syntactically substitute vars → terms*
- *E.g., L = (x → Socrates, y → uncle(z))*
  - *mortal(x) ∧ man(y): L →*
    mortal(Socrates) ∧ man(uncle(z))
- *Substitution list ≠ variable assignment*

Unification

- *Two FOL terms* ***unify*** *with each other if there is a substitution list that makes them syntactically identical*
- *man(x), man(Socrates) unify using the substitution* $x \rightarrow$ *Socrates*
- *Importance: purely syntactic criterion for identifying useful substitutions*

Unification examples

- *loves(x, x), loves(John, y) unify using* $x \rightarrow John, y \rightarrow John$
- *loves(x, x), loves(John, Mary) can't unify*
- *loves(uncle(x), y), loves(z, aunt(z)):*

Unification examples

- *loves(x, x), loves(John, y) unify using* $x \rightarrow John, y \rightarrow John$
- *loves(x, x), loves(John, Mary) can't unify*
- *loves(uncle(x), y), loves(z, aunt(z)):*
  - $z \rightarrow uncle(x), y \rightarrow aunt(uncle(x))$
  - *loves(uncle(x), aunt(uncle(x)))*

Quiz

- *Can we unify*

  knows(John, x)   knows(x, Mary)

- *What about*

  knows(John, x)   knows(y, Mary)

Quiz

- *Can we unify*

  $$knows(John, x) \quad knows(x, Mary)$$

  No!

- *What about*

  $$knows(John, x) \quad knows(y, Mary)$$

  $$x \rightarrow Mary, y \rightarrow John$$

Standardize apart

- *But knows(x, Mary) is logically equivalent to knows(y, Mary)!*
- *Moral: standardize apart before unifying*

Most general unifier

- *May be many substitutions that unify two formulas*
- *MGU is unique (up to renaming)*
- *Simple, moderately fast algorithm for finding MGU (see RN); more complex, linear-time algorithm*

Linear unification. MS Paterson, MN Wegman. Proceedings of the eighth annual ACM symposium on Theory of Computing, 1976.