Laplace-Beltrami: The Swiss Army Knife of Geometry Processing

(SGP 2014 Tutorial—July 7 2014)

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- Expressing tasks in terms of Laplacian/smooth PDEs makes life easier at code/implementation level.
- Lots of existing theory to help understand/interpret algorithms, provide analysis/guarantees.
- Also makes it easy to work with a broad range of geometric data structures (meshes, point clouds, etc.)

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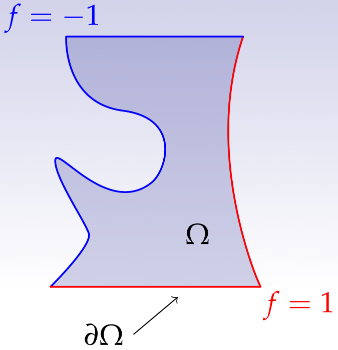
  - Use Laplacian to implement a variety of methods. *(Justin)*

Smooth Theory

The Interpolation Problem



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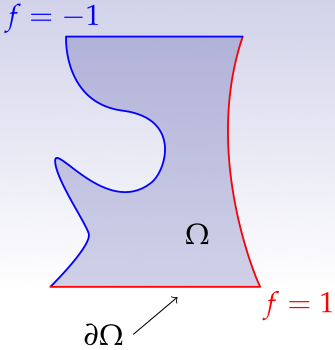

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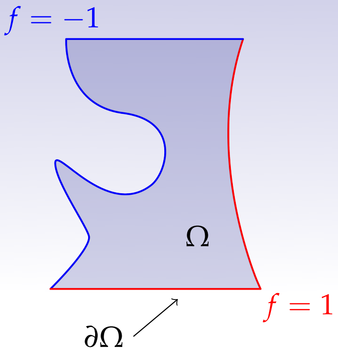

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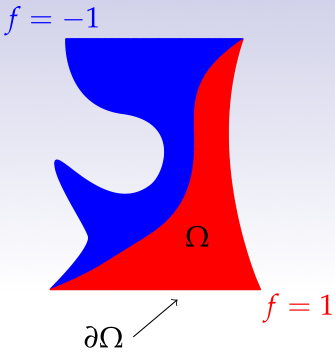

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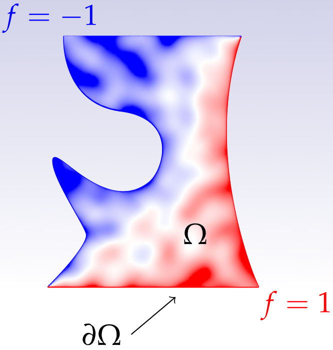

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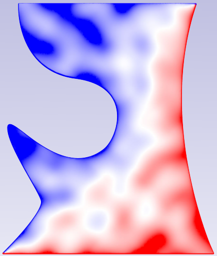
  - large variations over short distances
  - ($\|\nabla f\|$ large)

Dirichlet Energy



non-smooth $f(x)$

$\|\nabla f\|^2$

- $E(f) = \int_\Omega \|\nabla f\|^2 \, dA$
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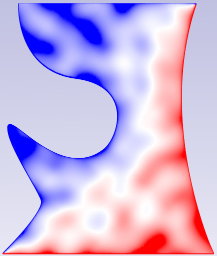

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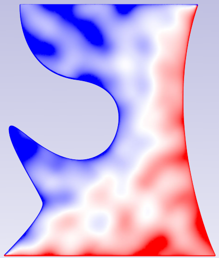

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- how do we find minimum?

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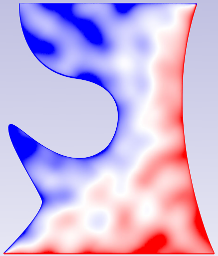

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    - $E(f) = C - \int_\Omega f \Delta f \, dA$

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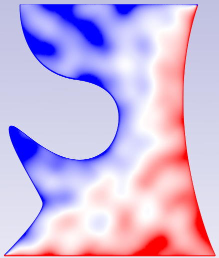

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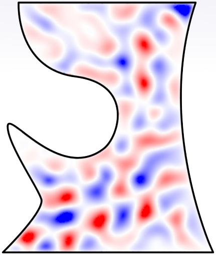

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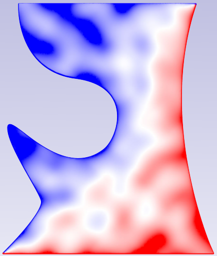

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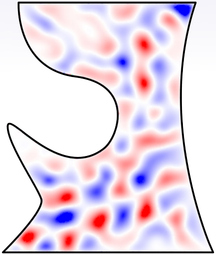

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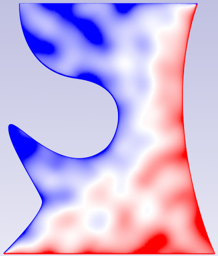

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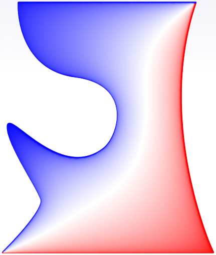

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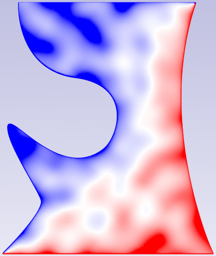

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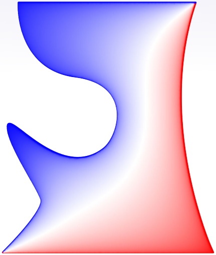

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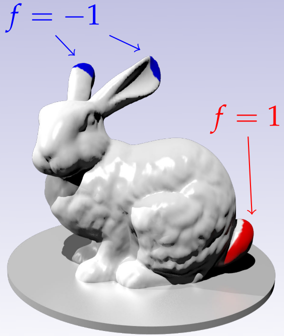
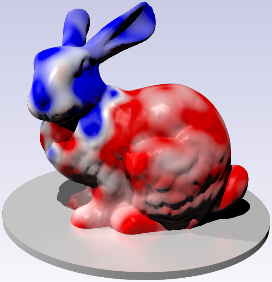
- physical interpretation: temperature at steady state

On a Surface



boundary conditions



nonsmooth $f(x)$

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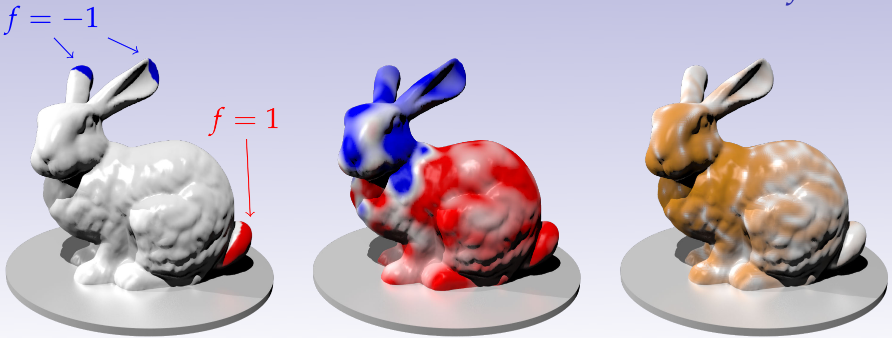

boundary conditions | nonsmooth $f(x)$ | $\|\nabla f\|^2$

- can still define Dirichlet energy $E(f) = \int_M \|\nabla f\|^2$

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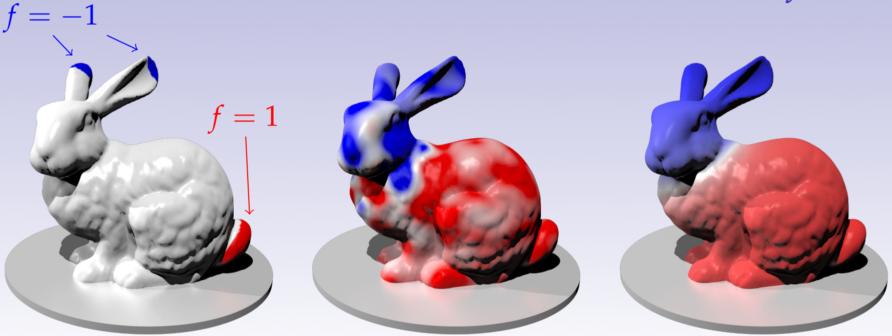

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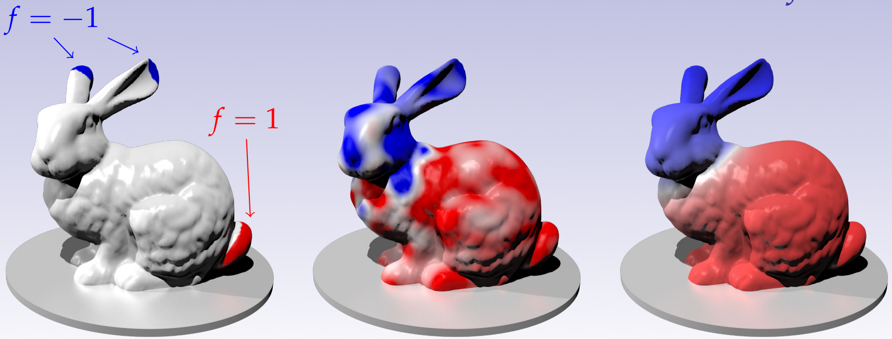

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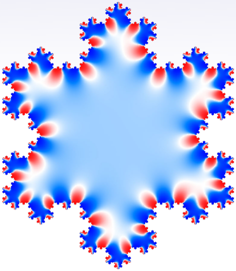
- can still define Dirichlet energy $E(f) = \int_M \|\nabla f\|^2$
- $\nabla E(f) = -\Delta f$, now Δ is the *Laplace-Beltrami operator* of M
- also works in higher dimensions, on discrete graphs/point clouds, …

Existence and Uniqueness

- Laplace's equation

$$\begin{aligned} \Delta f(x) &= 0 & x &\in M \\ f(x) &= f_0(x) & x &\in \partial M \end{aligned}$$

has a unique solution for all reasonable[1] surfaces M



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- gradient descent is exactly the *heat* or *diffusion* equation

$$\frac{df}{dt}(x) = \Delta f(x).$$

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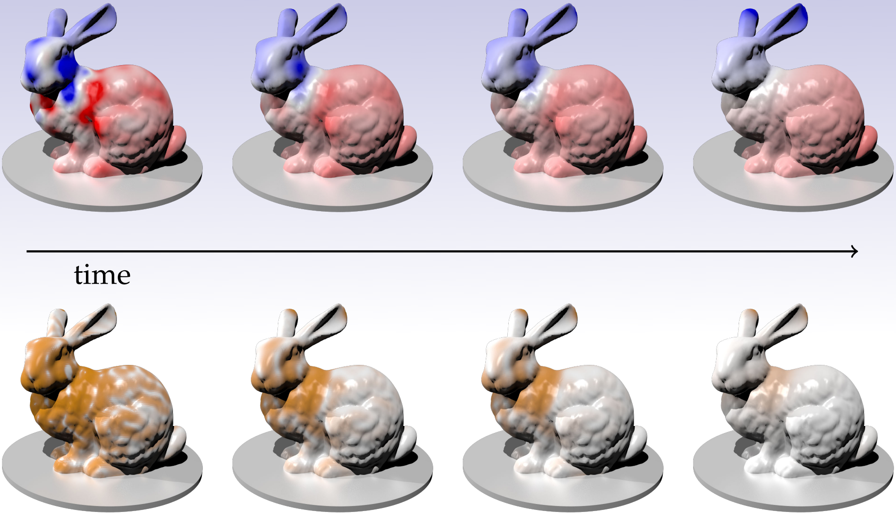
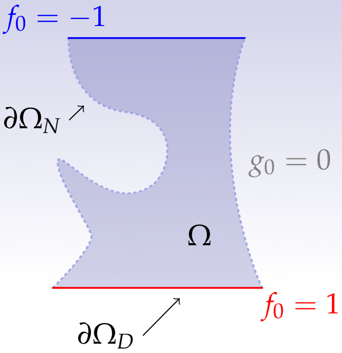
Heat Equation Illustrated



Boundary Conditions



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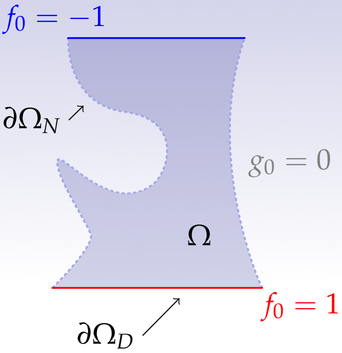

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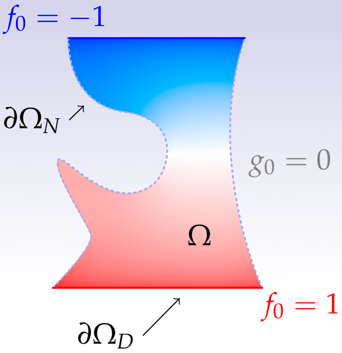

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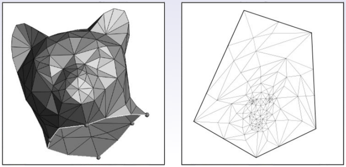
- physical interpretation: free boundary through which heat cannot flow

Interpolation with Δ in Practice

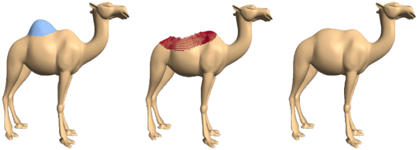
in geometry processing:

- positions
- displacements
- vector fields
- parameterizations
- …you name it

Joshi et al

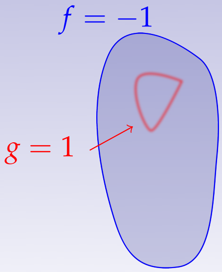


Eck et al



Sorkine and Cohen-Or

Heat Equation with Source



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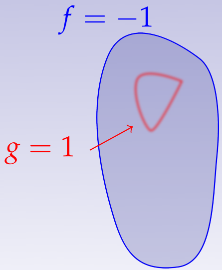

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$$\frac{df}{dt}(x) = g(x) + \Delta f(x)$$

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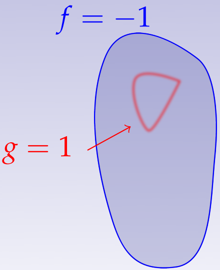

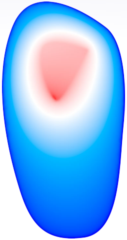

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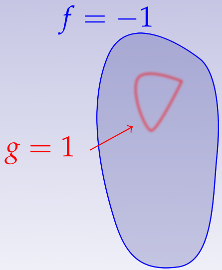

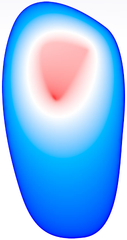

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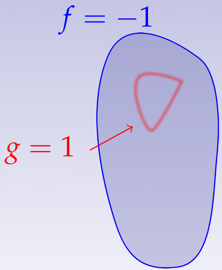

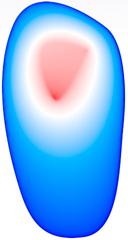

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- becomes Poisson problem, $g = \nabla \cdot \mathbf{v}$

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- *self-adjoint*: $\int_M f\,\Delta g\,dA=-\int_M\langle\nabla f,\nabla g\rangle\,dA=\int_M g\Delta f\,dA$
- *negative*: $\int_M f\,\Delta f\,dA\leq 0$

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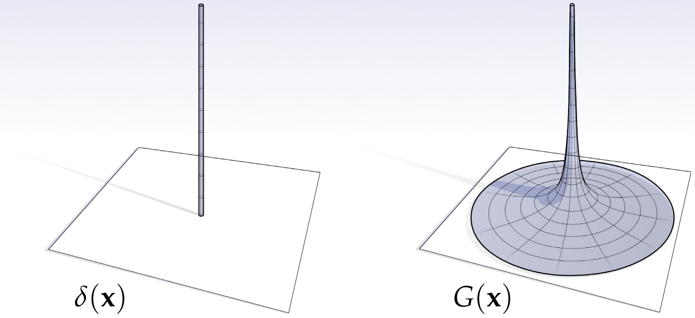
(intuition: $\Delta \approx$ an ∞-*dimensional* negative-semidefinite matrix)

Solving Poisson's Equation with Green's Functions

- the *Green's function* G on $\mathbb{R}^2$ solves $\Delta f = g$ for $g = \delta$



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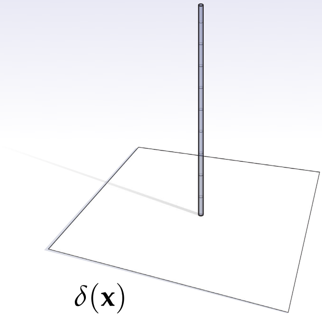

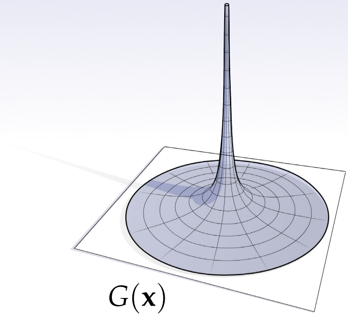

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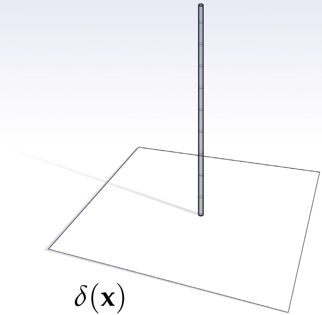
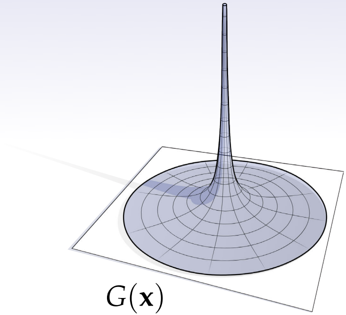
- for any g, $f = G * g$



$\delta(\mathbf{x})$

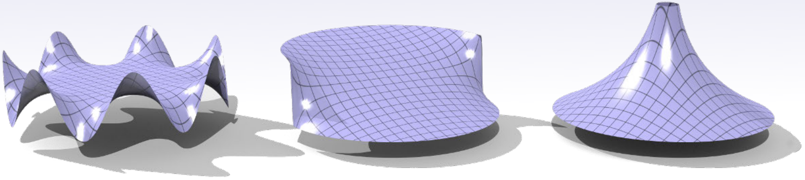


$G(\mathbf{x})$

Essential Algebraic Properties II

a function $f : M \to \mathbb{R}$ with $\Delta f = 0$ is called *harmonic*. Properties:

- f is smooth and analytic



some harmonic $f(x, y)$

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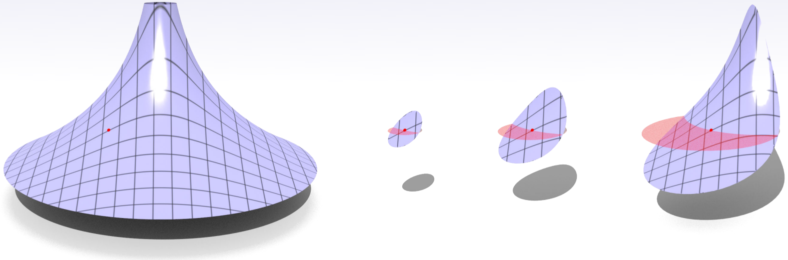

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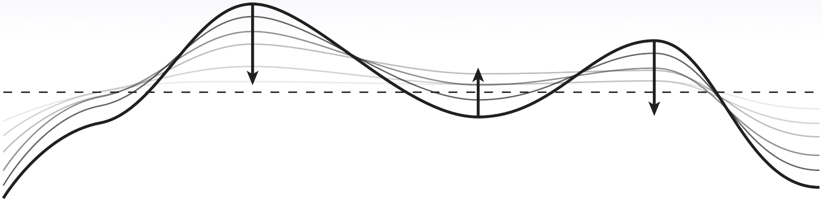
Essential Geometric Properties I

for a curve $\gamma(u) = (x[u], y[u]) : \mathbb{R} \to \mathbb{R}^2$

- total Dirichlet energy $\int \|\nabla x\|^2 + \|\nabla y\|^2$ is arc length



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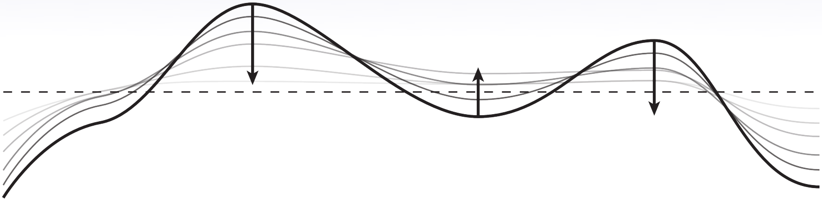

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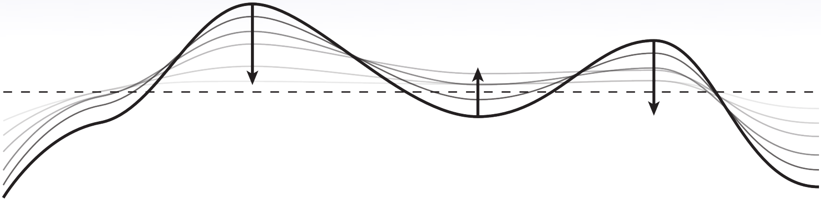

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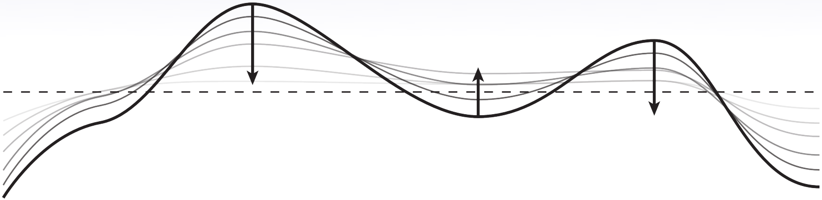

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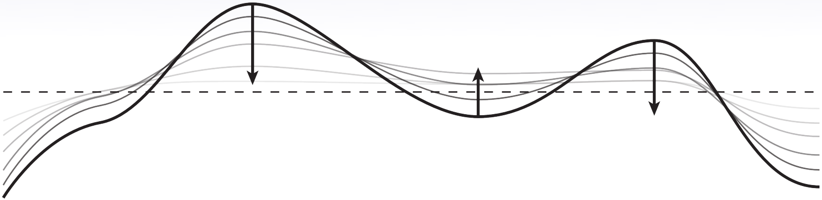

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for a surface $r(u, v) = (x[u, v], y[u, v], z[u, v]) : \mathbb{R} \to \mathbb{R}^3$

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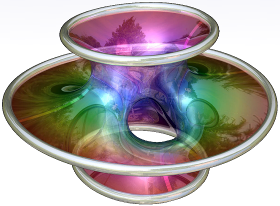

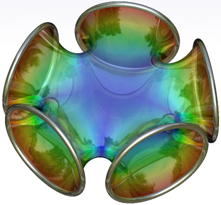

Images: Paul Nylander

Essential Geometric Properties III

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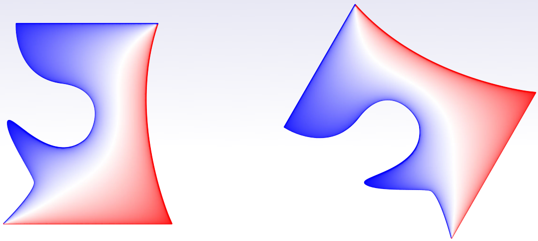

Essential Geometric Properties III

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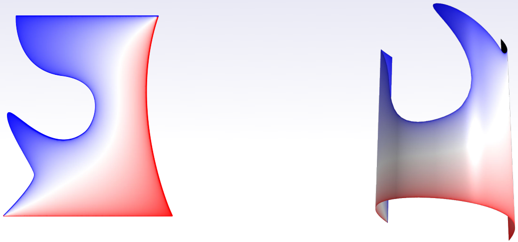
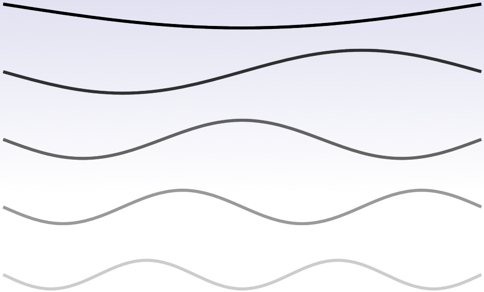
- for a surface Ω, isometric deformations of Ω don't change Δ



Signal Processing on a Line

on line segment $[0, 1]$:

- recall Fourier basis: $\phi_i(x) = \cos(ix)$



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- ϕ is a (Dirichlet) eigenfunction of Δ on M w/ eigenvalue λ:

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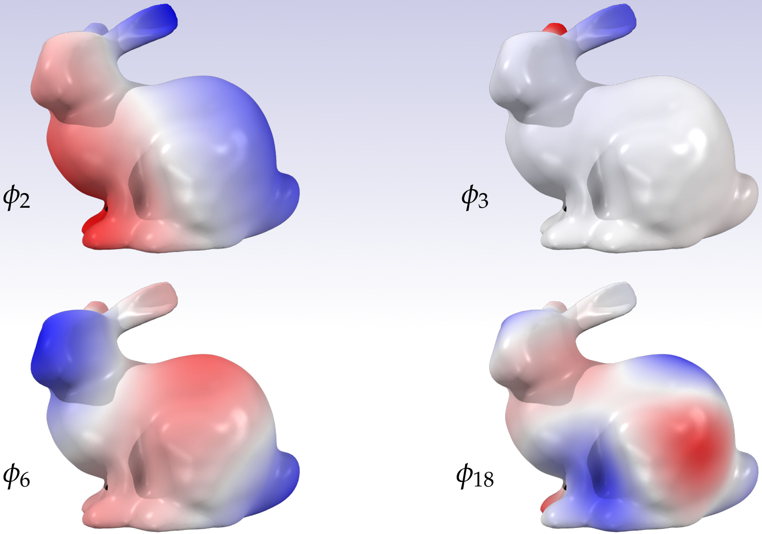
- spectrum is *discrete*: countably many eigenfunctions,

$$0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots$$

Laplacian Spectrum of Bunny



Laplacian Spectrum: Signal Processing

- expand function f in eigenbasis:

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$$E(f) = \int_M \|\nabla f\|^2 \, dA = -\int_M f \Delta f \, dA = \sum_i \alpha_i^2 (-\lambda_i)$$

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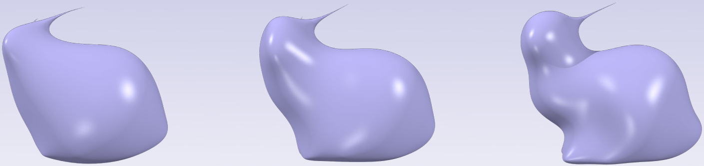

10 modes | 25 modes | 50 modes

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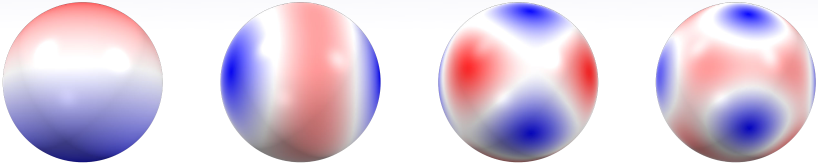
Laplacian Spectrum: Special Cases

perhaps you've heard of

- Fourier basis: $M = \mathbb{R}^n$
- spherical harmonics: $M =$ sphere



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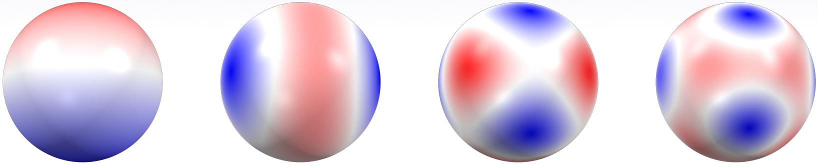
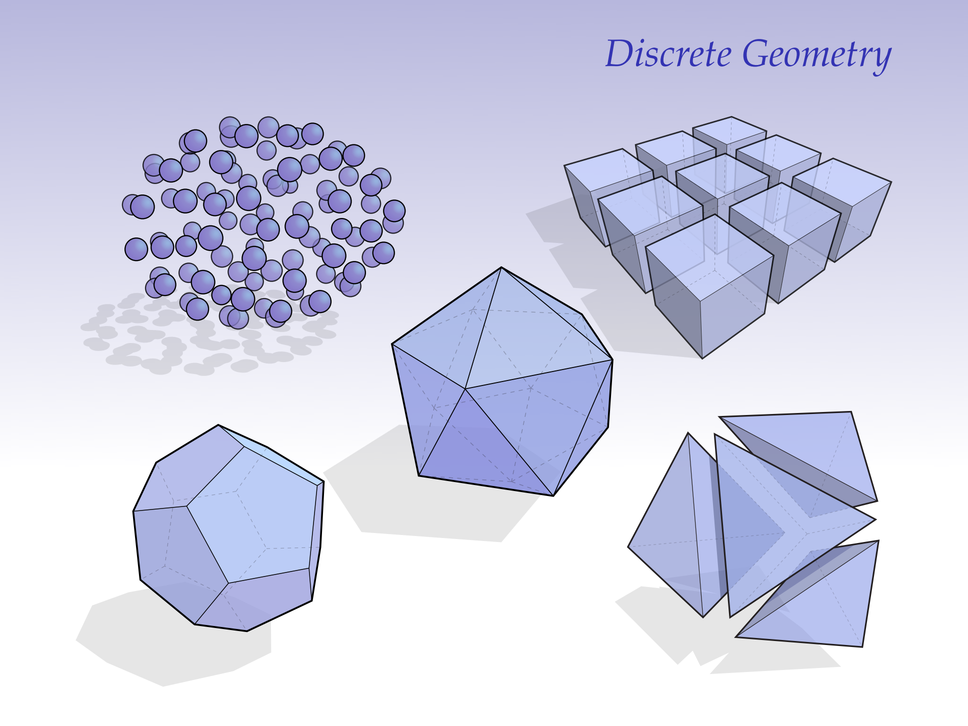
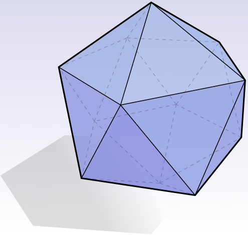
Laplacian spectrum generalizes these to any surface



Discretization

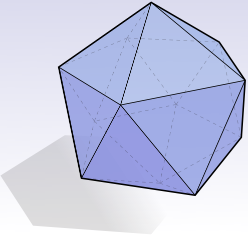

Discrete Geometry

Triangle Meshes



- approximate surface by *triangles*

Triangle Meshes



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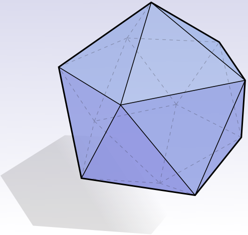

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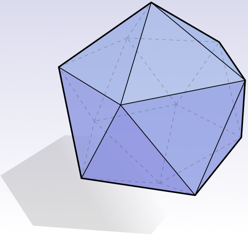

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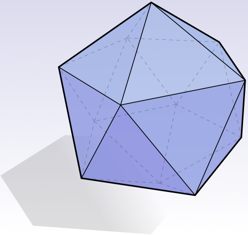

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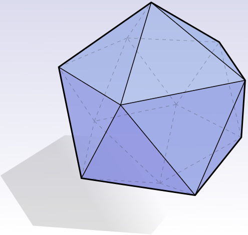

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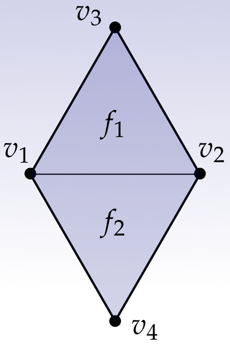
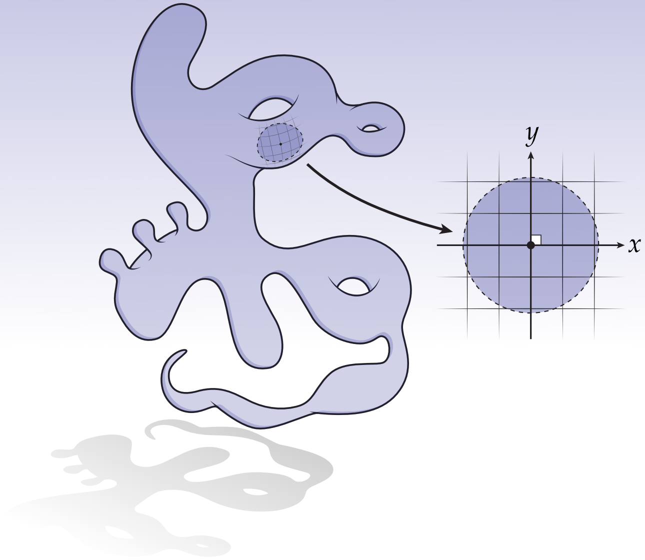
- "glued together" along edges
- many possible data structures
- *half edge, quad edge, corner table, . . .*
- for simplicity: *vertex-face adjacency list*
- (will be enough for our applications!)

Vertex-Face Adjacency List—Example

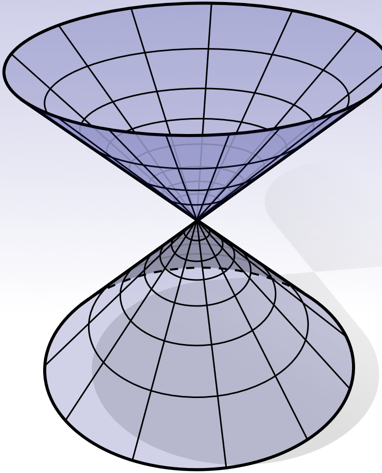
```
# xyz-coordinates of vertices
v 0 0 0
v 1 0 0
v .5 .866 0
v .5 -.866 0

# vertex-face adjacency info
f 1 2 3
f 1 4 2
```



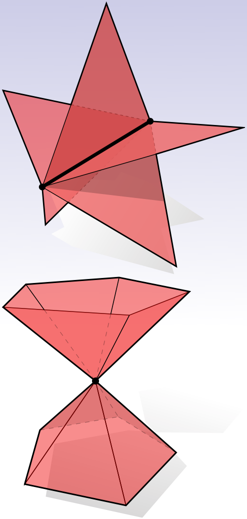
Manifold

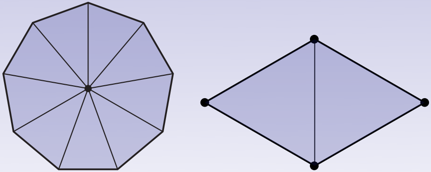


Nonmanifold



Manifold Triangle Mesh





- *manifold* $\Longleftrightarrow$ “locally disk-like”

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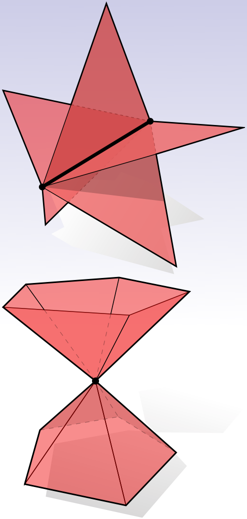

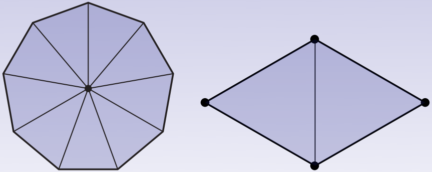

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- Which triangle meshes are manifold?

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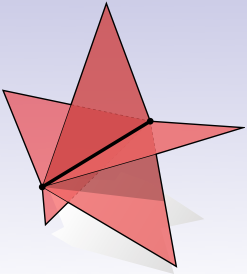

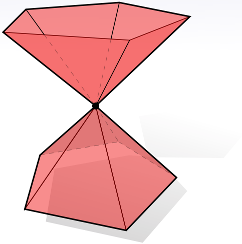

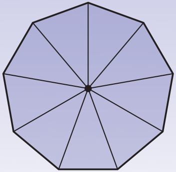

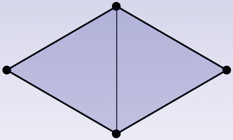

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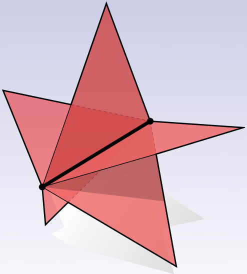
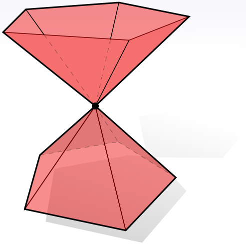
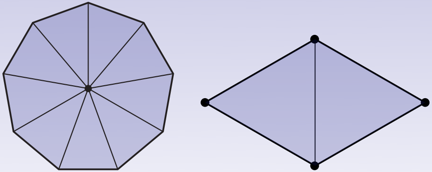
- Which triangle meshes are manifold?
- Two triangles per edge (no "fins")

Manifold Triangle Mesh







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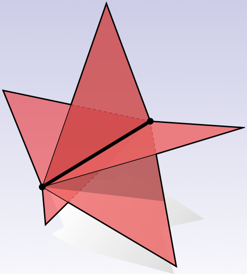
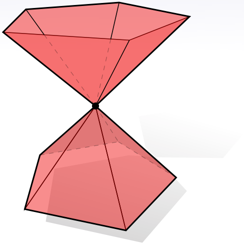
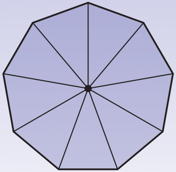
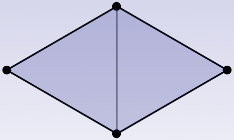
- Which triangle meshes are manifold?
- Two triangles per edge (no "fins")
- Every vertex looks like a "fan"

Manifold Triangle Mesh









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- Two triangles per edge (no "fins")
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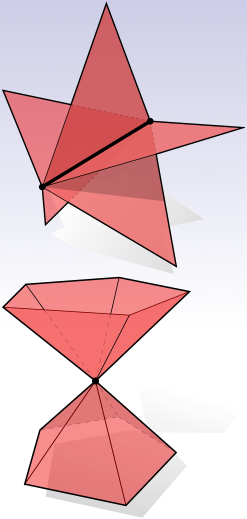
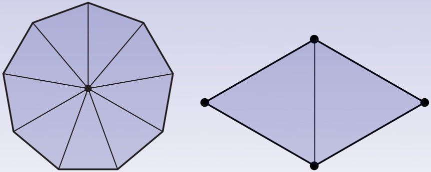
Manifold Triangle Mesh





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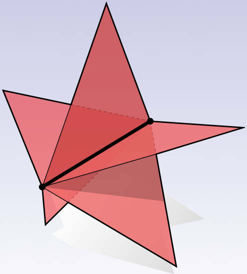

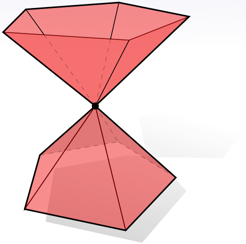

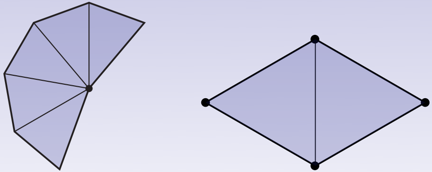

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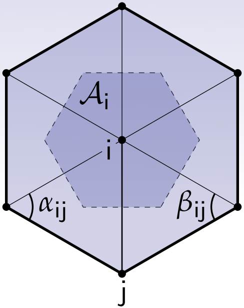

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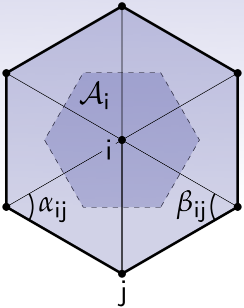

The set $\mathcal{N}(\mathsf{i})$ contains the immediate neighbors of vertex i

The Cotangent Laplacian

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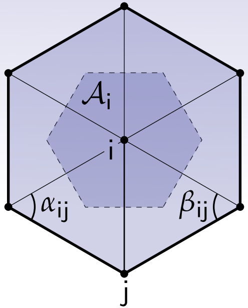

The set $\mathcal{N}(\mathsf{i})$ contains the immediate neighbors of vertex i

The quantity $\mathcal{A}_{\mathsf{i}}$ is *vertex area*—for now: 1/3rd of triangle areas

Origin of the Cotan Formula?

- Many different ways to derive it

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- For three different derivations, see [Crane et al., 2013a]

MacNeal, 1949

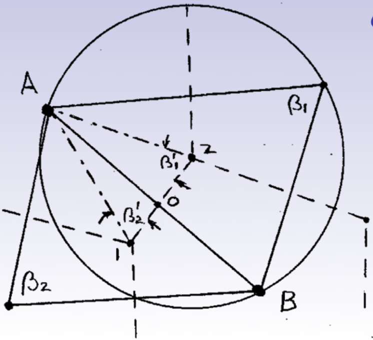
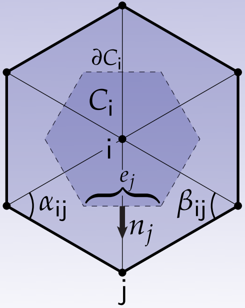


Fig. 25.

"If the network is first laid out on a large sheet of drawing paper, the angles can be measured with a protractor and the distances scaled off with sufficient accuracy in a short time."

"If the mesh is sufficiently fine, this will not lead to a large error. It indicates, however, that an attempt should be made to keep the triangles as nearly regular as possible."

Cotan-Laplacian via Finite Volumes



- Integrate over each dual cell C_i

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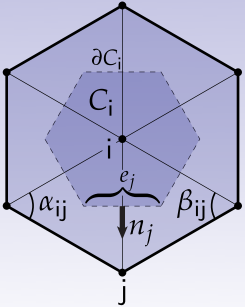

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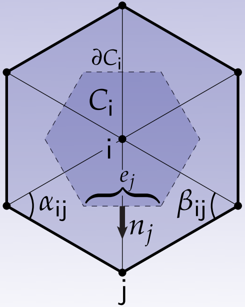

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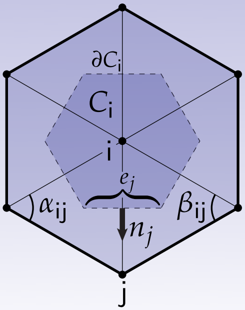

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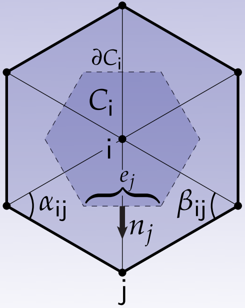

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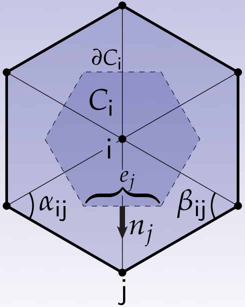

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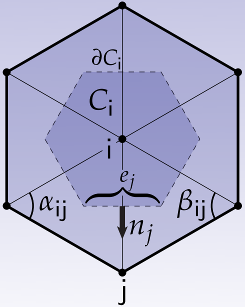

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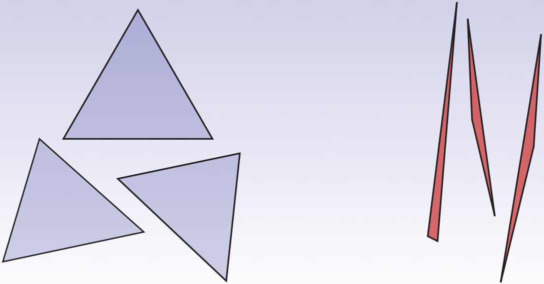
- (Can divide by $\mathcal{A}_i$ to approximate *pointwise* value)

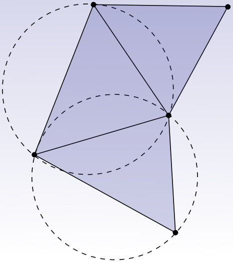
Triangle Quality—Rule of Thumb



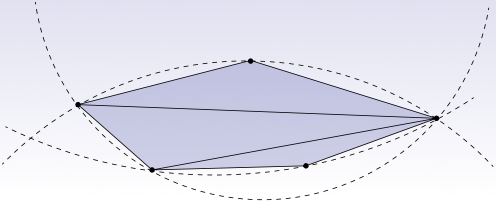
good triangles bad triangles

(For further discussion see Shewchuk, *"What Is a Good Linear Finite Element?"*)

Triangle Quality—Delaunay Property

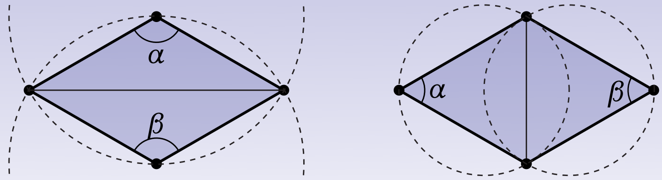


Delaunay



Not Delaunay

Local Mesh Improvement



- Some simple ways to improve quality of Laplacian

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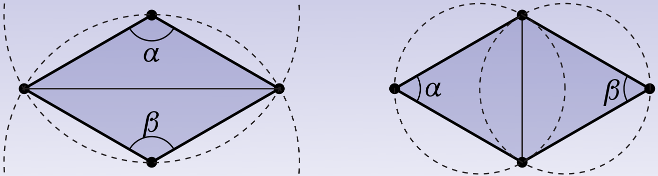

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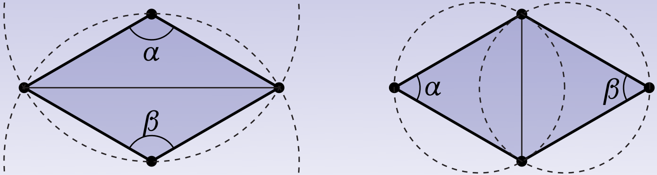

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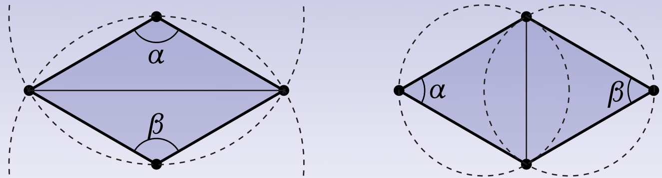

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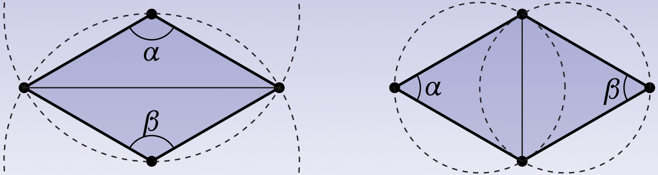

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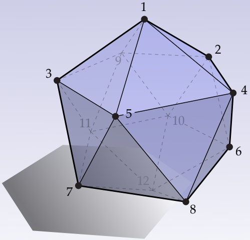
- For more, see [Dunyach et al., 2013, Wojtan et al., 2011].

Meshes and Matrices



$$\begin{bmatrix}
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1 & -5 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & -5 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
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1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -5 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -5 & 1 & 1 \\
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0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & -5
\end{bmatrix}$$

(Laplace matrix, ***ignoring weights!***)

- So far, Laplacian expressed as a sum:

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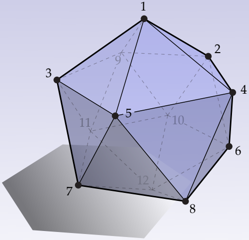

$$\begin{bmatrix} -5 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -5 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & -5 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -5 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & -5 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -5 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & -5 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & -5 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -5 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -5 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & -5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & -5 \end{bmatrix}$$

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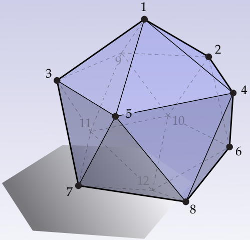

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-5 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
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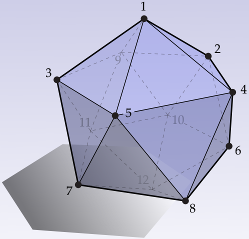

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$$\begin{bmatrix}
-5 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & -5 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & -5 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
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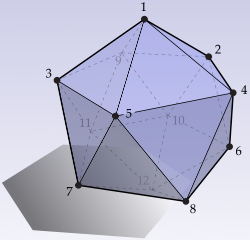
Meshes and Matrices



$$\begin{bmatrix} -5 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -5 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & -5 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -5 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & -5 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -5 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & -5 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & -5 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -5 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -5 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & -5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & -5 \end{bmatrix}$$

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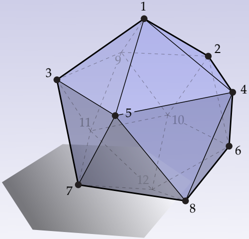

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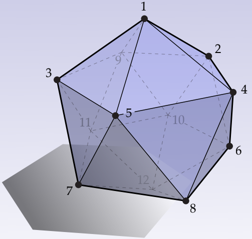

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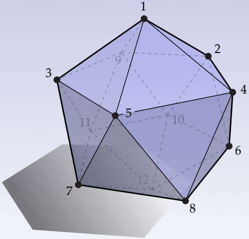

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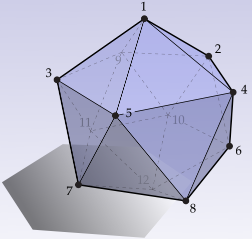

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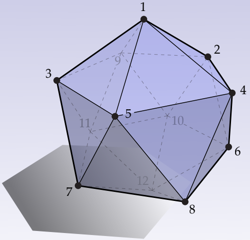

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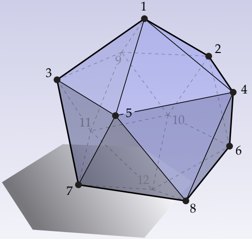

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- (MATLAB: `sparse`, SuiteSparse: `cholmod_sparse`, Eigen: `SparseMatrix`)

Mass Matrix

- Matrix C encodes only *part* of Laplacian—recall that

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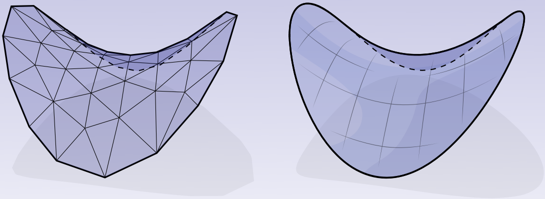
- Applying L to a column vector $u \in \mathbb{R}^{|V|}$ "implements" the cotan formula shown above

Discrete Poisson / Laplace Equation



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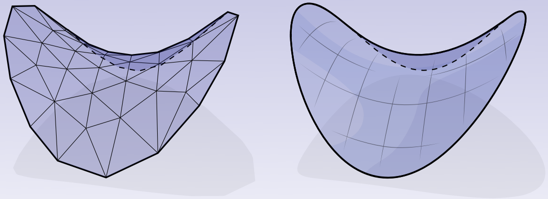

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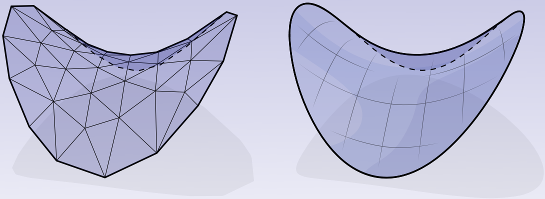

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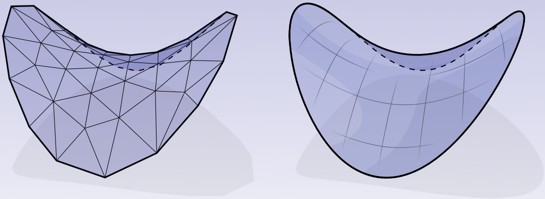

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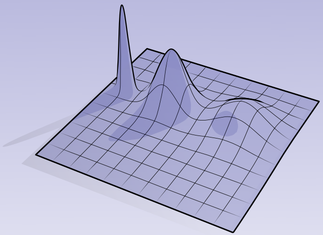
- Laplace is just Poisson with "zero" on right hand side!



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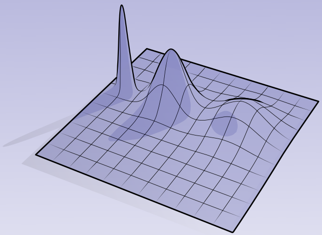

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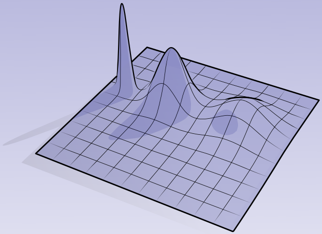

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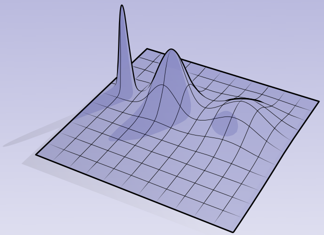

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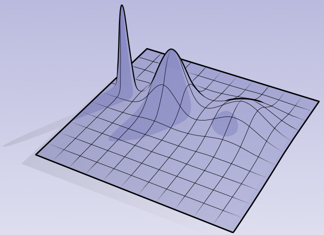

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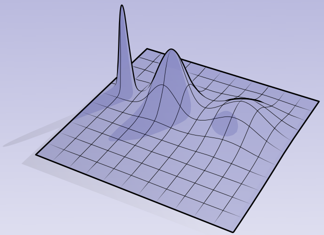

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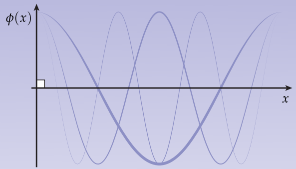
- Implicit update becomes linear system $(\mathsf{I} - \mathsf{h}\mathsf{L})\mathsf{u}_{k+1} = \mathsf{u}_k$



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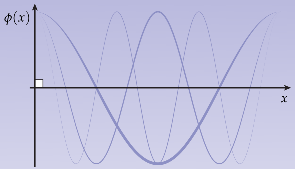

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- Can be solved using *(inverse) power method*:
  - Pick random u_0
  - Until convergence:
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    - Remove mean value from u_{k+1}
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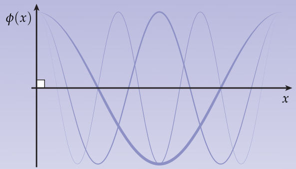

Discrete Eigenvalue Problem

- Smallest eigenvalue problem $\Delta u = \lambda u$ becomes

$$\mathsf{L}\mathsf{u} = \lambda \mathsf{u}$$

  for smallest nonzero eigenvalue λ.
- Can be solved using *(inverse) power method*:
  - Pick random u_0
  - Until convergence:
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- By ***prefactoring*** L, overall cost is nearly identical to solving a single Poisson equation!

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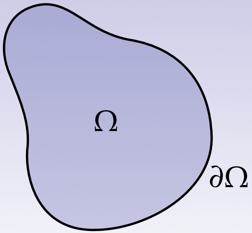

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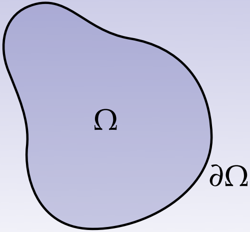

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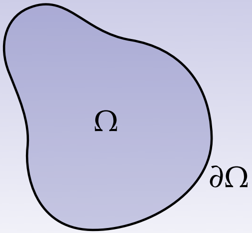

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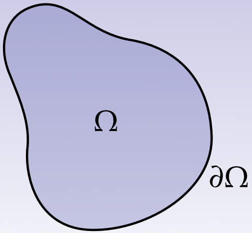

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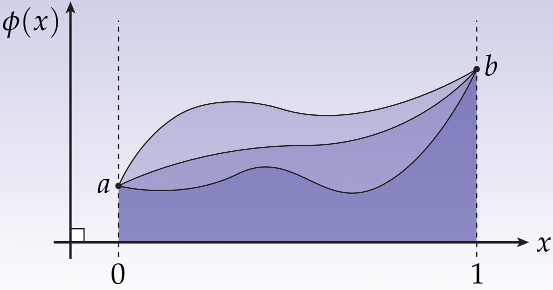

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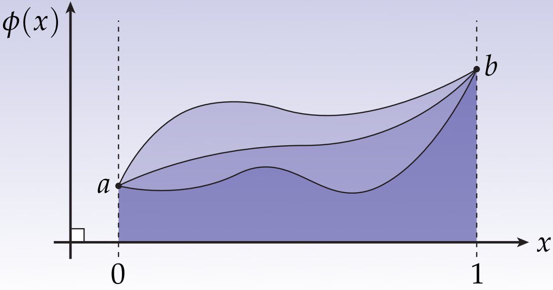

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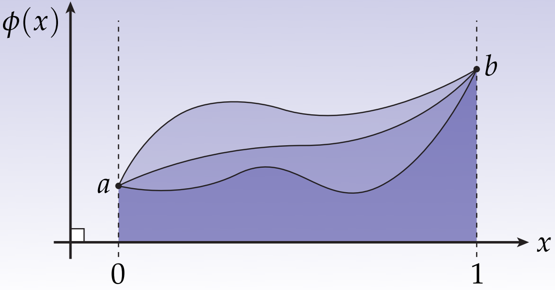

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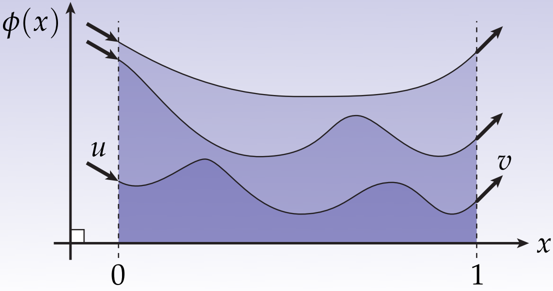

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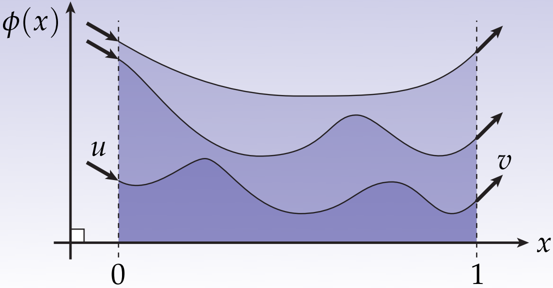

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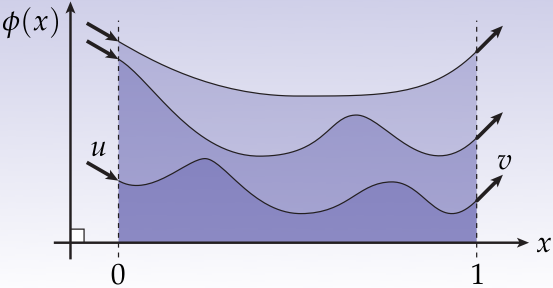

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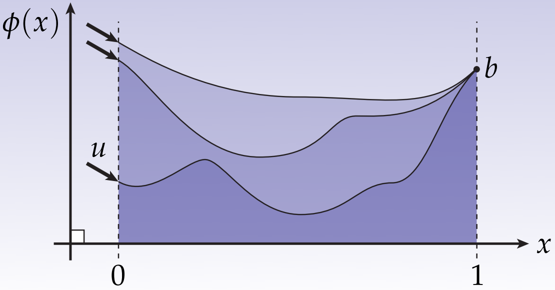

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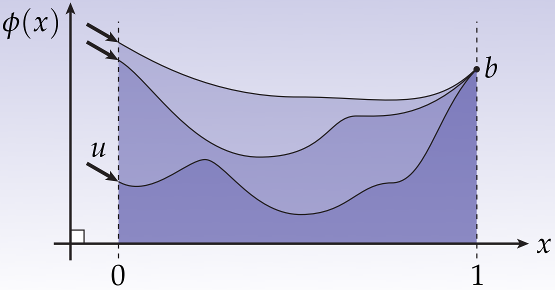

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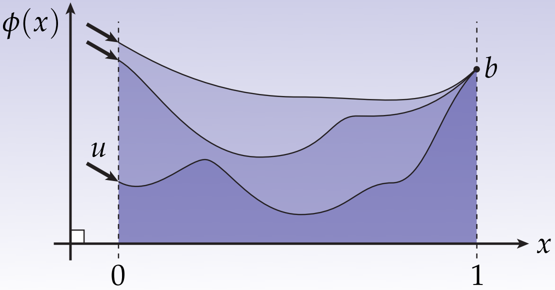

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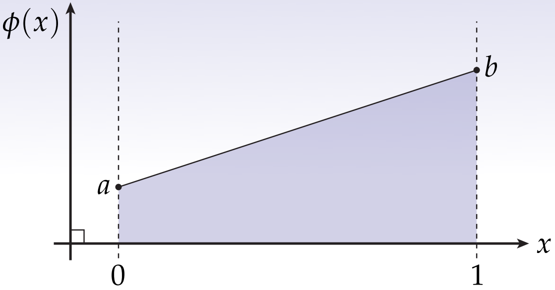

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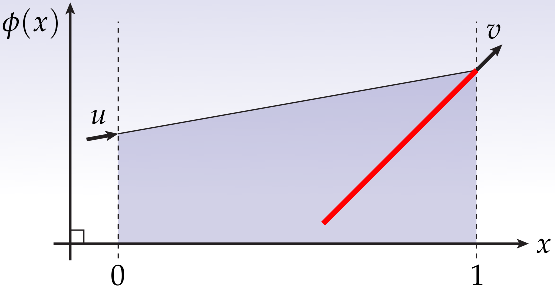

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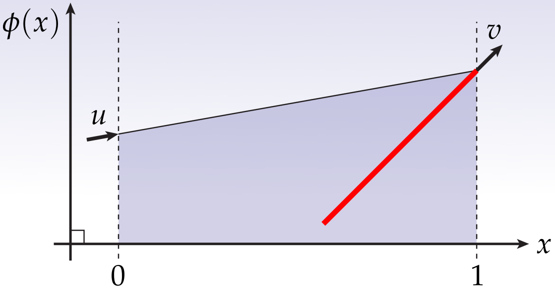

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- In general: solutions to PDE may *not* exist for given BCs

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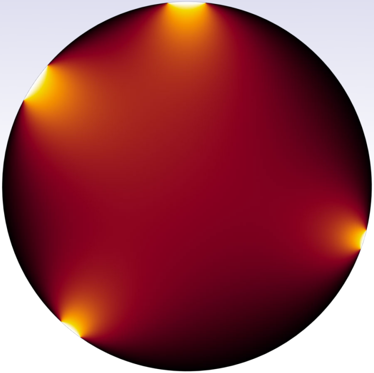
- Yes: Laplace is steady-state solution to heat flow $\frac{d}{dt}\phi = \Delta\phi$



- Dirichlet data is just “heat” along boundary

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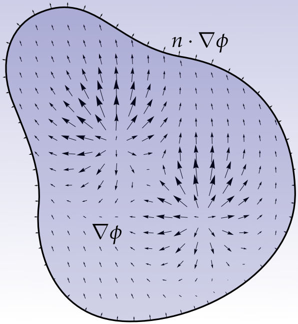

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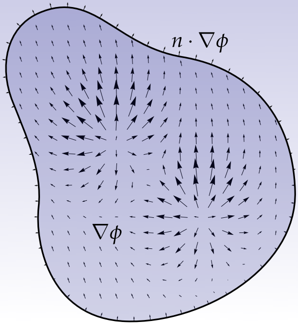

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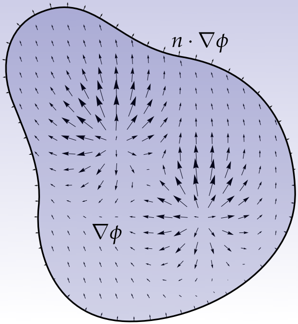

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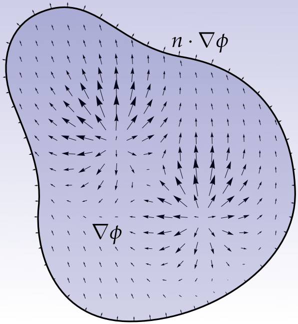

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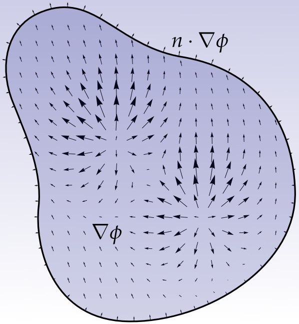

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- Conclusion: can only solve $\Delta \phi = 0$ if Neumann BCs have *zero mean!*

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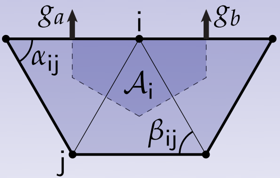
- Can skip matrix multiply and compute entries of RHS directly: $\mathcal{A}_\mathsf{i} f_\mathsf{i} - \sum_{\mathsf{j}\in\mathcal{N}_\partial(\mathsf{i})}(\cot\alpha_{ij} + \cot\beta_{ij})u_j$
- Here $\mathcal{N}_\partial(\mathsf{i})$ denotes neighbors of i *on the boundary*

Discrete Boundary Conditions - Neumann



- Integrate both sides of $\Delta u = f$ over cell C_i ("finite volume")

$$\int_{C_i} f \overset{!}{=} \int_{C_i} \Delta u = \int_{C_i} \nabla \cdot \nabla u = \int_{\partial C_i} n \cdot \nabla u$$

- Gives usual cotangent formula for interior vertices; for boundary vertex i, yields

$$\mathcal{A}_{ii} \overset{!}{=} \tfrac{1}{2}(g_a + g_b) + \tfrac{1}{2} \sum_{j \in \mathcal{N}_{\text{int}}} (\cot \alpha_{ij} + \cot \beta_{ij})(u_j - u_i)$$

- Here g_a, g_b are prescribed normal derivatives; just subtract from RHS and solve $Cu = Mf$ as usual

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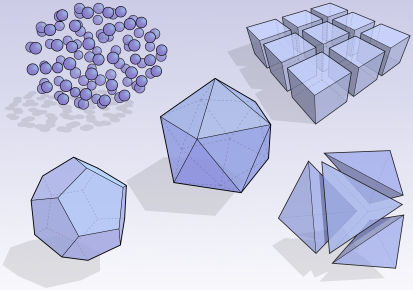

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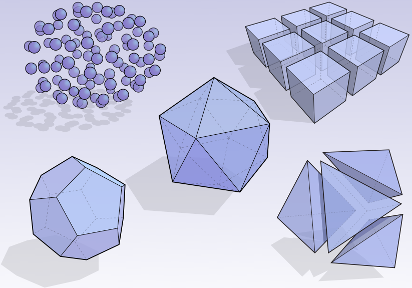

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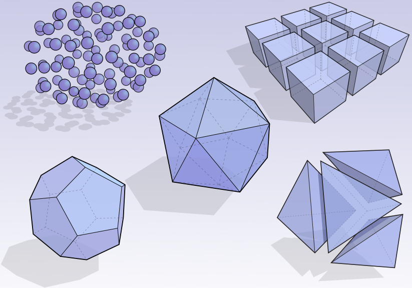

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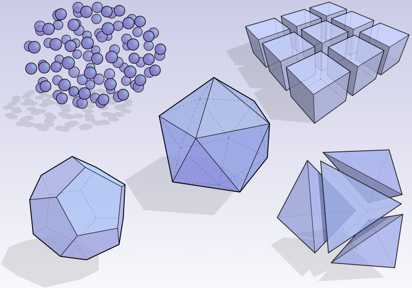

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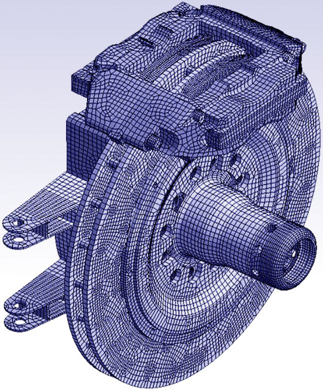
- Also: more accurate discretization on triangle meshes

Quad, Polygon Meshes



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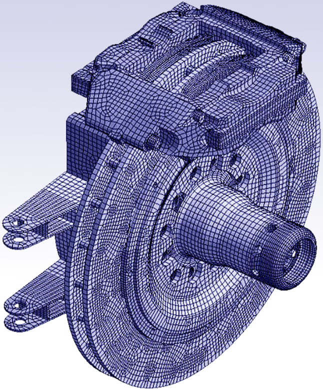

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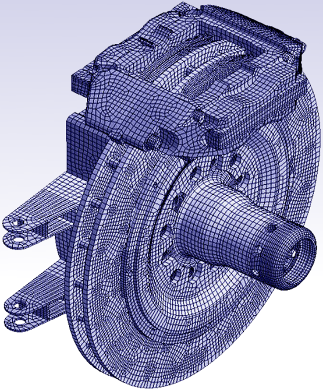

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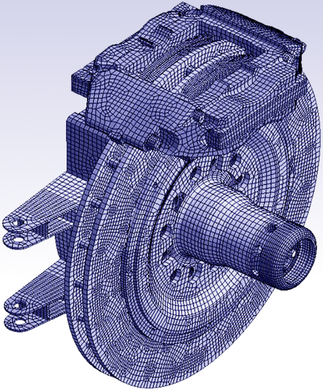

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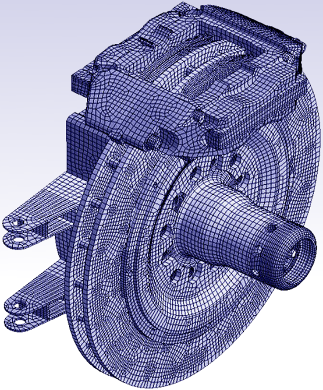

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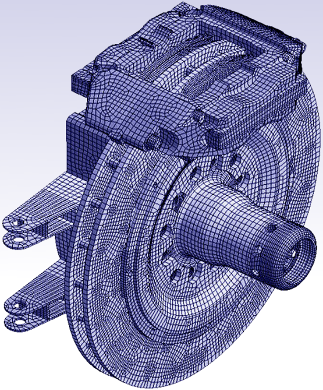

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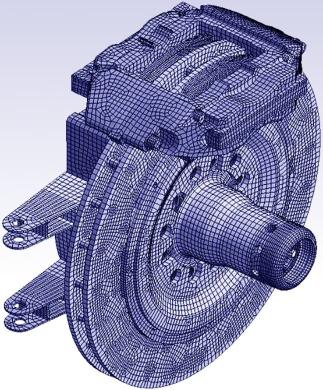

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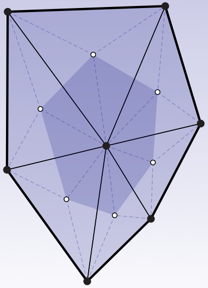
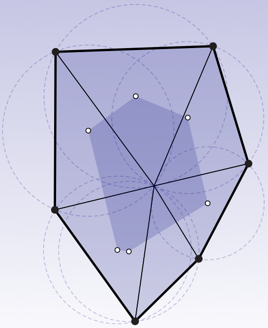


barycentric



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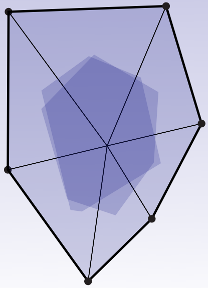

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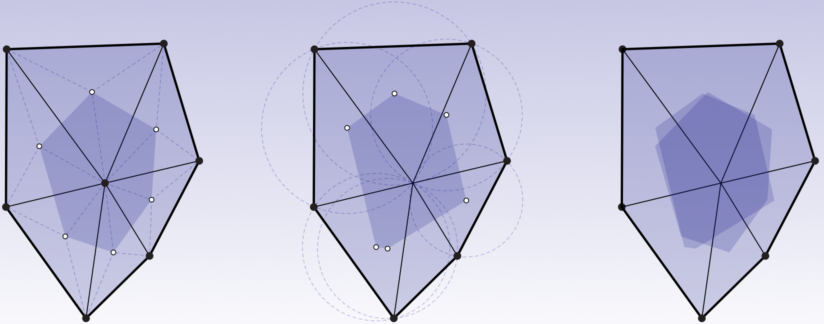

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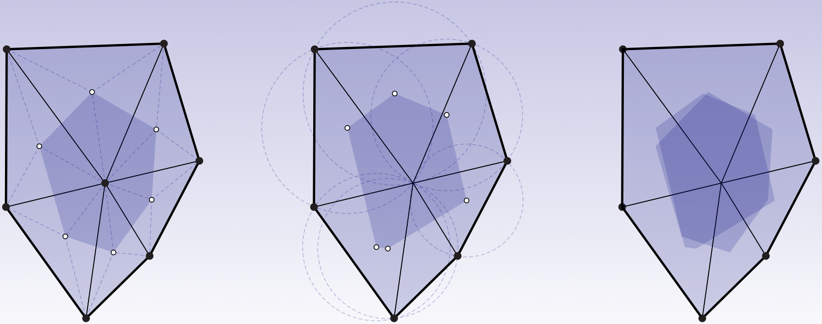

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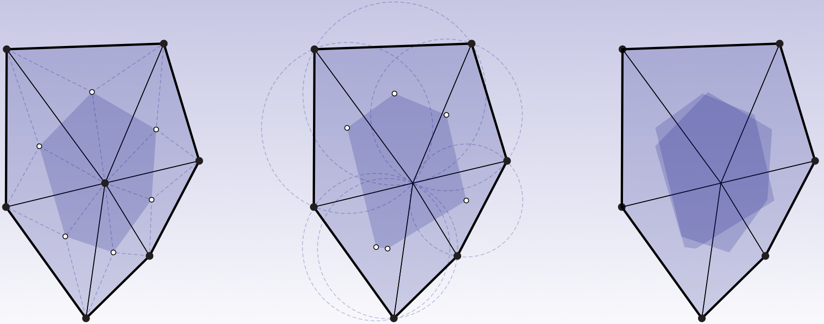

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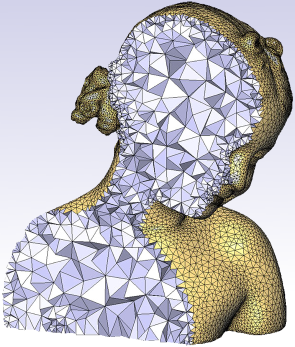
  [de Goes et al., 2012, de Goes et al., 2013, de Goes et al., 2014]

Volumes / Tetrahedral Meshes



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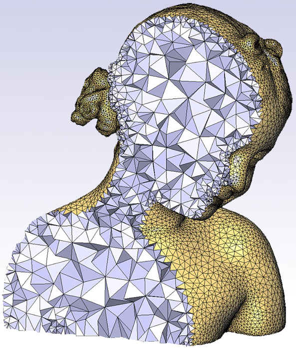

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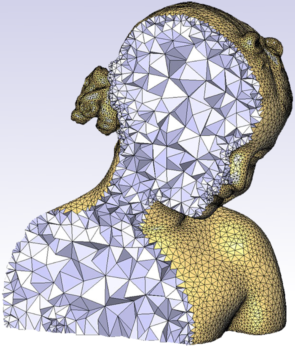

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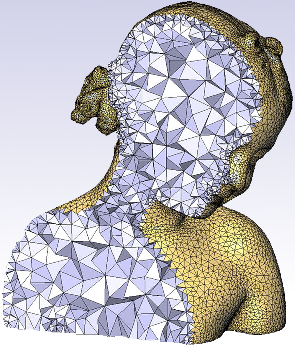

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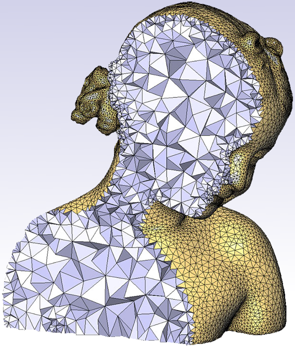

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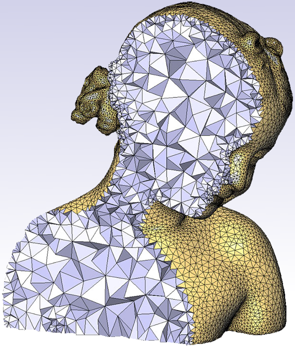

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- Added bonus: play with definition of dual to improve accuracy [Mullen et al., 2011].

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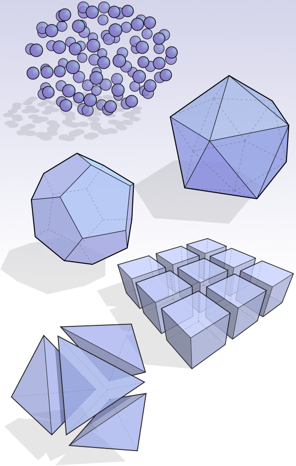
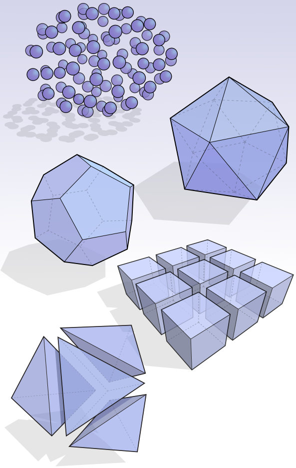


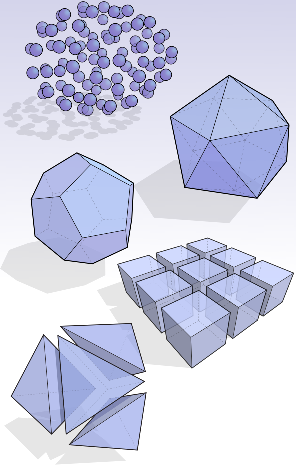
- Covered some standard discretizations

...and More!



- Covered some standard discretizations
- Many possibilities (level sets, hex meshes…)

...and More!



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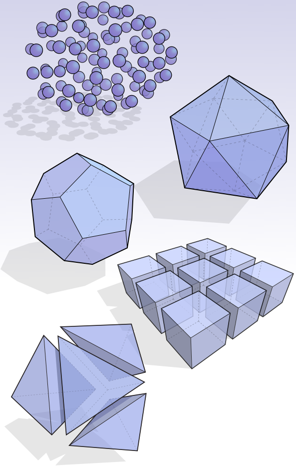
...and More!



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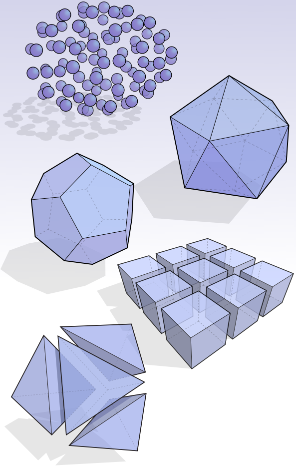
...and More!



- Covered some standard discretizations
- Many possibilities (level sets, hex meshes…)
- Often enough to have *gradient* G and inner product W.
- (weak!) Laplacian is then $\mathsf{C} = \mathsf{G}^\mathsf{T}\mathsf{W}\mathsf{G}$ (think Dirichlet energy)
- Key message:
  build Laplace; do lots of cool stuff.

Applications

$$\{\text{simple pre-processing}\} \longrightarrow \Delta^{(-1)} \longrightarrow \{\text{simple post-processing}\}$$

"Our method boils down to 'backslash' in Matlab!"

Reminder: Model Equations

$$\boxed{\Delta f = 0}$$

Laplace equation

Linear solve

$$\boxed{\Delta f = g}$$

Poisson equation

Linear solve

$$\boxed{f_t = \Delta f}$$

Heat equation

ODE time-step

$$\boxed{\Delta \phi_i = \lambda_i \phi_i}$$

Vibration modes

Eigenproblem

Look here!

Reminder: Model Equations

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Eigenproblem

$\boxed{\Delta f = 0}$

Reminder: Variational Interpretation

$$\min_{f(x)} \int_{\Sigma} \|\nabla f(x)\|^2 \, dA$$

$\updownarrow$ `<calculus>`

$$\boxed{\Delta f(x) = 0}$$

Reminder: Variational Interpretation

$$\min_{f(x)} \int_\Sigma \|\nabla f(x)\|^2 \, dA$$

$\updownarrow$ `<calculus>`

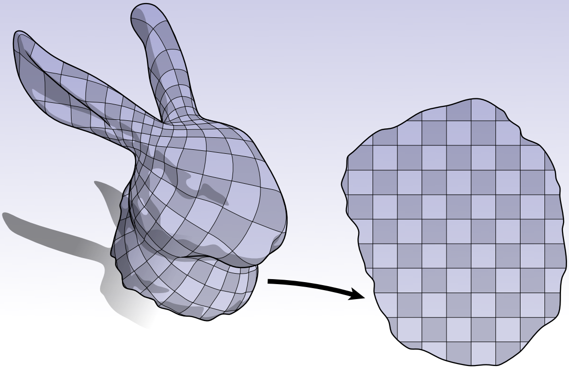
$$\boxed{\Delta f(x) = 0}$$

The (inverse) Laplacian **wants** to make functions smooth.

"Elliptic regularity"

$\boxed{\Delta f = 0}$

Application: Mesh Parameterization



Want **smooth** $f : M \to \mathbb{R}^2$.

$\boxed{\Delta f = 0}$

Variational Approach

$$\min_{f:M\to\mathbb{R}^2} \int \|\nabla f\|^2$$

Does this work?

$\boxed{\Delta f = 0}$

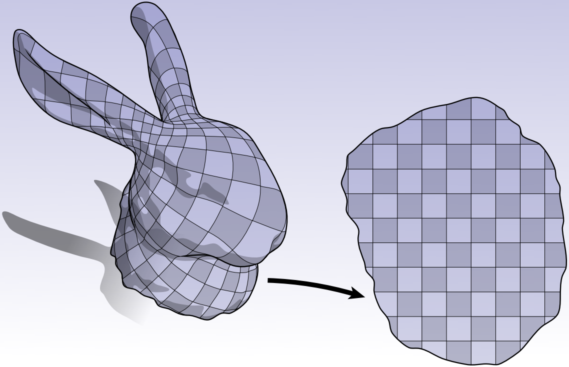
$$\min_{f:M\to\mathbb{R}^2} \int \|\nabla f\|^2$$

Does this work?

$$f(x) \equiv \text{const.}$$

$\boxed{\Delta f = 0}$

Harmonic Parameterization

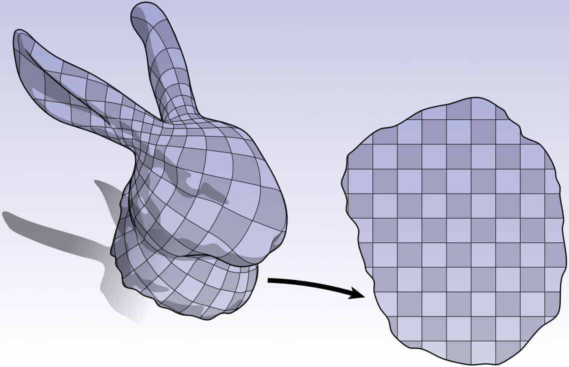


$$\min_{\substack{f:M\to\mathbb{R}^2 \\ f|_{\partial M}\ \text{fixed}}} \int \|\nabla f\|^2$$

[Eck et al., 1995]

$\boxed{\Delta f = 0}$

Harmonic Parameterization



$$\min_{\substack{f:M\to\mathbb{R}^2 \\ f|_{\partial M}\text{ fixed}}} \int \|\nabla f\|^2$$

[Eck et al., 1995]

$$\Delta f = 0 \text{ in } M\backslash\partial M, \text{ with } f|_{\partial M} \text{ fixed}$$

Reminder: Model Equations

$\boxed{\Delta f = 0}$ *Laplace equation*

Linear solve

$\boxed{\Delta f = g}$ *Poisson equation*

Linear solve

$\boxed{f_t = \Delta f}$ *Heat equation*

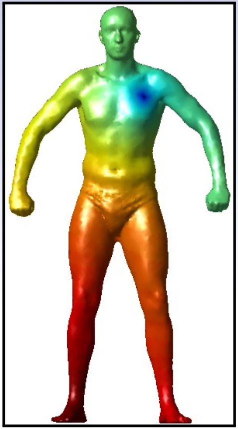
ODE time-step

$\boxed{\Delta \phi_i = \lambda_i \phi_i}$ *Vibration modes*

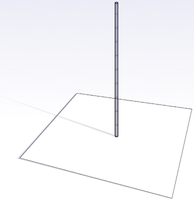
Eigenproblem

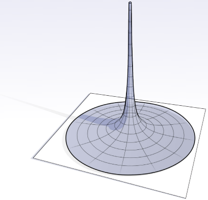
$\boxed{\Delta f = g}$

Recall: Green's Function



$$\Delta g_p = \delta_p \text{ for } p \in M$$

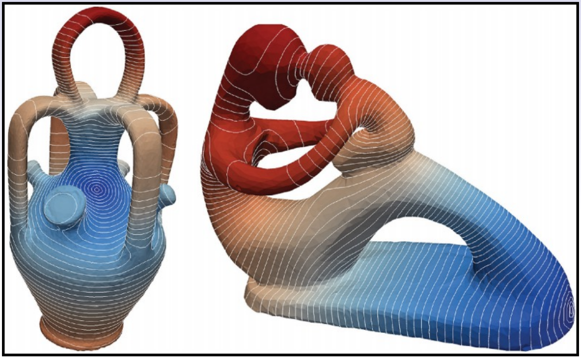




$\boxed{\Delta f = g}$

Application: Biharmonic Distances

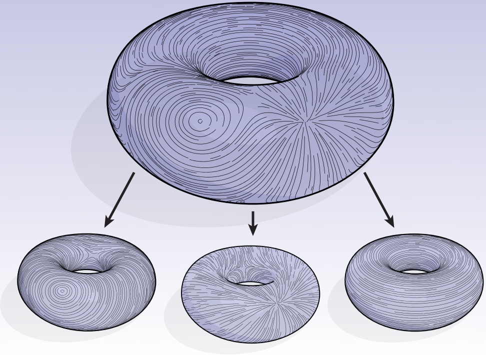
$$d_b(p,q) \equiv \|g_p - g_q\|_2$$



[Lipman et al., 2010], formula in [Solomon et al., 2014]

$\boxed{\Delta f = g}$

Hodge Decomposition



$$\vec{v}(x) = R^{90^\circ}\nabla g + \nabla f + \vec{h}(x)$$

- Divergence-free part: $R^{90^\circ}\nabla g$
- Curl-free part: ∇f
- Harmonic part: $\vec{h}(x)$ ($= \vec{0}$ if surface has no holes)

$\boxed{\Delta f = g}$

Computing the Curl-Free Part

$$\min_{f(x)} \int_\Sigma \|\nabla f(x) - \vec{v}(x)\|^2 \, dA$$

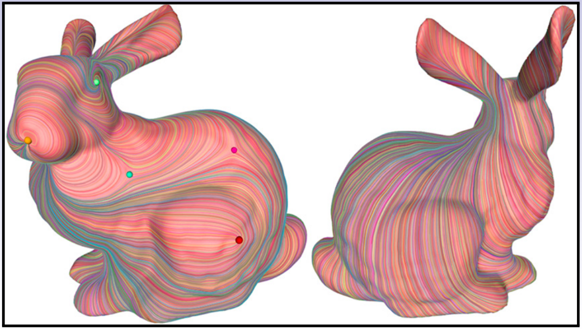
$\updownarrow$ `<calculus>`

$$\boxed{\Delta f(x) = \nabla \cdot \vec{v}(x)}$$

Get divergence-free part as $\vec{v}(x) - \nabla f(x)$ (when $\vec{h} \equiv \vec{0}$)

$\boxed{\Delta f = g}$

Application: Vector Field Design

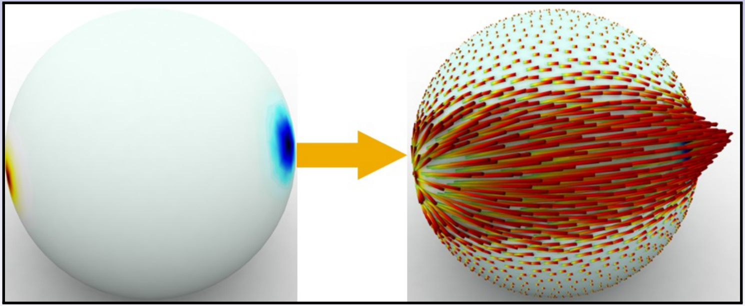


$$\Delta f = -\bar{K} \longrightarrow \vec{v}(x) = \nabla f(x)$$

[Crane et al., 2010, de Goes and Crane, 2010]

$\boxed{\Delta f = g}$

Application: Earth Mover's Distance



$$\min_{\vec{J}(x)} \int_M \|\vec{J}(x)\|$$

$$\text{such that } \vec{J} = R^{90^\circ} \nabla g + \nabla f + \vec{h}(x)$$

$$\Delta f = \rho_1 - \rho_0$$

[Solomon et al., 2014]

Reminder: Model Equations

$$\boxed{\Delta f = 0}$$

Laplace equation

Linear solve

$$\boxed{\Delta f = g}$$

Poisson equation

Linear solve

$$\boxed{f_t = \Delta f}$$

Heat equation

ODE time-step

$$\boxed{\Delta \phi_i = \lambda_i \phi_i}$$

Vibration modes

Eigenproblem

$\boxed{f_t = \Delta f}$

Generalizing Gaussian Blurs

Gradient descent on $\int \|\nabla f(x)\|^2\, dx$:

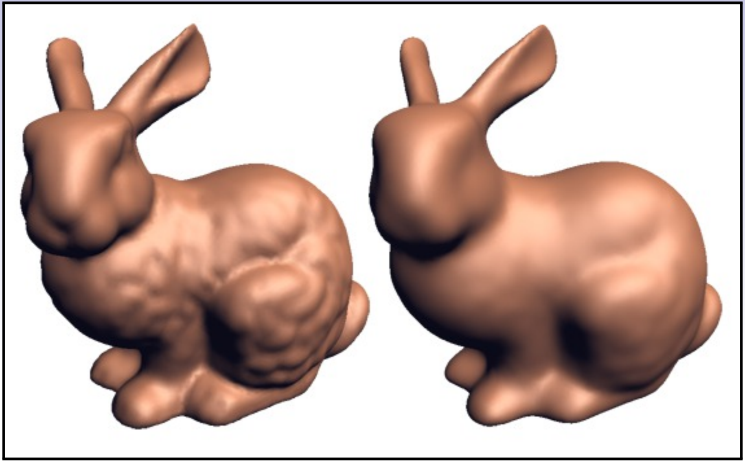
$$\frac{\partial f(x,t)}{\partial t} = \Delta_x f(x,t)$$

with $f(\cdot, 0) \equiv f_0(\cdot)$.

Image by M. Bottazzi

$\boxed{f_t = \Delta f}$

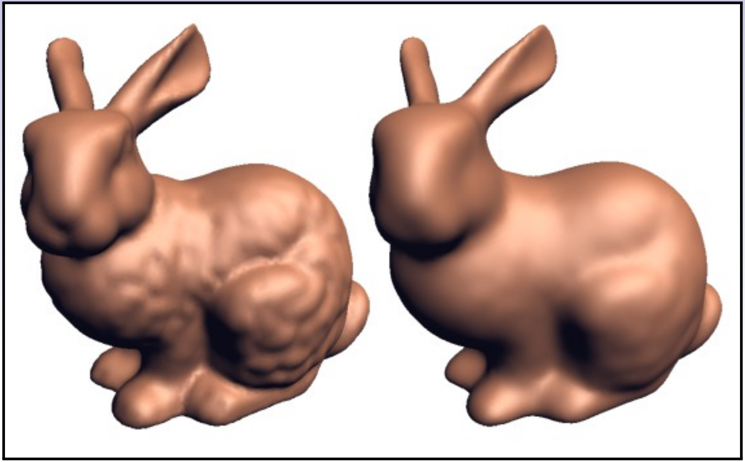
Application: Implicit Fairing



Idea: Take $f_0(x)$ to be the coordinate function.

$\boxed{f_t = \Delta f}$

Application: Implicit Fairing



Idea: Take $f_0(x)$ to be the coordinate function.

Detail: Δ changes over time.

[Desbrun et al., 1999]

$\boxed{\Delta f = g}$

Alternative: Screened Poisson Smoothing

Simplest incarnation of [Chuang and Kazhdan, 2011]:

$$\min_{f(x)} \alpha^2 \|f - f_0\|^2 + \|\nabla f\|^2$$

$$\updownarrow$$

$$(\alpha^2 I - \Delta) f = \alpha^2 f_0$$

$\boxed{f_t = \Delta f \to \Delta f = g}$

Interesting Connection

(Semi-)Implicit Euler:

$$(I - hL)u_{k+1} = u_k$$

Screened Poisson:

$$(\alpha^2 I - \Delta)f = \alpha^2 f_0$$

$\boxed{f_t = \Delta f \to \Delta f = g}$

Interesting Connection

(Semi-)Implicit Euler:

$$(I - hL)u_{k+1} = u_k$$

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$$(\alpha^2 I - \Delta)f = \alpha^2 f_0$$

One time step of *implicit Euler* is *screened Poisson*.

$\boxed{f_t = \Delta f \to \Delta f = g}$

Interesting Connection

(Semi-)Implicit Euler:

$$(I - hL)u_{k+1} = u_k$$

Screened Poisson:

$$(\alpha^2 I - \Delta)f = \alpha^2 f_0$$

One time step of *implicit Euler* is *screened Poisson*.

Accidentally replaced one PDE with another!

Eikonal equation for geodesics:

$$\|\nabla \phi\|_2 = 1$$

$$\implies \text{Need } \textit{direction} \text{ of } \nabla \phi.$$

Eikonal equation for geodesics:

$$\|\nabla \phi\|_2 = 1$$

$$\implies \text{Need } \textit{direction} \text{ of } \nabla \phi.$$

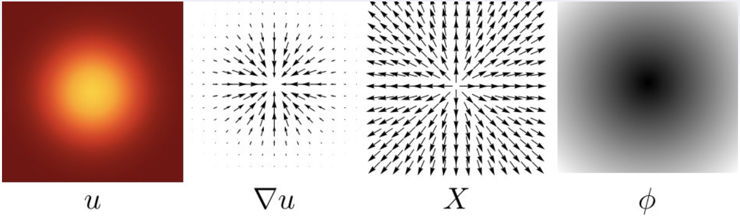
Idea:

Find u such that ∇u is *parallel* to geodesic.

$\boxed{f_t = \Delta f \text{ and } \Delta f = g}$

Application: The "Heat Method"

1. Integrate $u' = \nabla u$ (heat equation) to time $t \ll 1$.
2. Define vector field $X \equiv -\frac{\nabla u}{\|\nabla u\|_2}$.
3. Solve least-squares problem $\nabla \phi \approx X \iff \Delta \phi = \nabla \cdot X$.



Blazingly fast!
[Crane et al., 2013b]

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Eigenproblem

$$\boxed{\Delta\phi_i = \lambda_i \phi_i}$$

Laplace-Beltrami Eigenfunctions

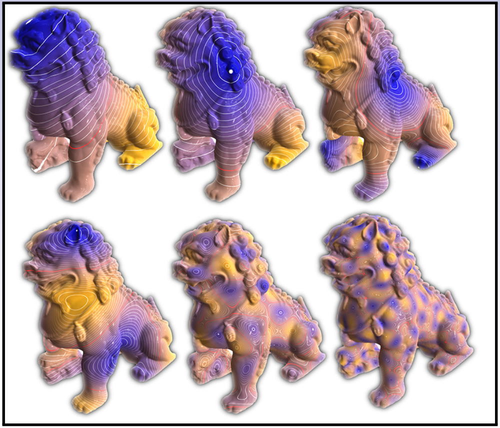


Image by B. Vallet and B. Lévy

Use eigenvalues and eigenfunctions to characterize shape.

$\boxed{\Delta\phi_i = \lambda_i \phi_i}$

Intrinsic Laplacian-Based Descriptors

All computable from eigenfunctions!

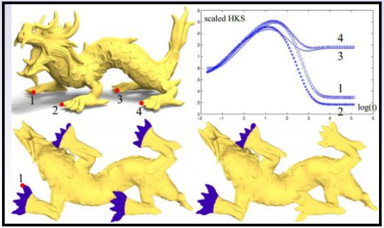
- $\text{HKS}(x;t) = \sum_i e^{\lambda_i t}\phi_i(x)^2$ [Sun et al., 2009]
- $\text{GPS}(x) = \left(\frac{\phi_1(x)}{\sqrt{-\lambda_1}}, \frac{\phi_2(x)}{\sqrt{-\lambda_2}}, \ldots\right)$ [Rustamov, 2007]
- $\text{WKS}(x;e) = C_e \sum_i \phi_i(x)^2 \exp\left(-\frac{1}{2\sigma^2}(e - \log(-\lambda_i))\right)$ [Aubry et al., 2011]

Many others—or ***learn*** *a function of eigenvalues!*
[Litman and Bronstein, 2014]

$\boxed{f_t = \Delta f}$

Example: Heat Kernel Signature

Heat diffusion encodes geometry for **all** times $t \geq 0$!



[Sun et al., 2009]

$$\mathrm{HKS}(x;t) \equiv k_t(x,x)$$

"Amount of heat diffused from x to itself over at time t."

- Signature of point x is a function of $t \geq 0$
- *Intrinsic* descriptor

$\boxed{\Delta\phi_i = \lambda_i\phi_i}$

HKS via Laplacian Eigenfunctions

$$\Delta\phi_i = \lambda_i\phi_i, f_0(x) = \sum_i a_i\phi_i(x)$$

$$\frac{\partial f(x,t)}{\partial t} = \Delta f \text{ with } f(x,0) \equiv f_0(x)$$

$\boxed{\Delta\phi_i = \lambda_i\phi_i}$

HKS via Laplacian Eigenfunctions

$$\Delta\phi_i = \lambda_i\phi_i, f_0(x) = \sum_i a_i\phi_i(x)$$

$$\frac{\partial f(x,t)}{\partial t} = \Delta f \text{ with } f(x,0) \equiv f_0(x)$$

$$\Longrightarrow f(x,t) = \sum_i a_i e^{\lambda_i t}\phi_i(x)$$

$\boxed{\Delta\phi_i = \lambda_i\phi_i}$

HKS via Laplacian Eigenfunctions

$$\Delta\phi_i = \lambda_i\phi_i, f_0(x) = \sum_i a_i\phi_i(x)$$

$$\frac{\partial f(x,t)}{\partial t} = \Delta f \text{ with } f(x,0) \equiv f_0(x)$$

$$\implies f(x,t) = \sum_i a_i e^{\lambda_i t}\phi_i(x)$$

$$\begin{aligned}\implies \mathrm{HKS}(x;t) &\equiv k_t(x,x) \\ &= \sum_i e^{\lambda_i t}\phi_i(x)^2\end{aligned}$$

$$\boxed{\Delta\phi_i = \lambda_i\phi_i}$$

Application: Shape Retrieval

*Solve problems like **shape similarity search**.*

"Shape DNA" [Reuter et al., 2006]:
Identify a shape by its vector of Laplacian eigenvalues

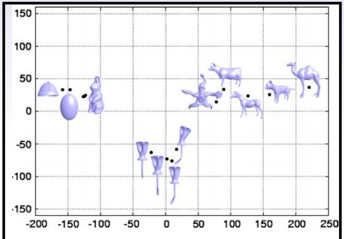
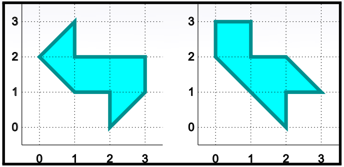
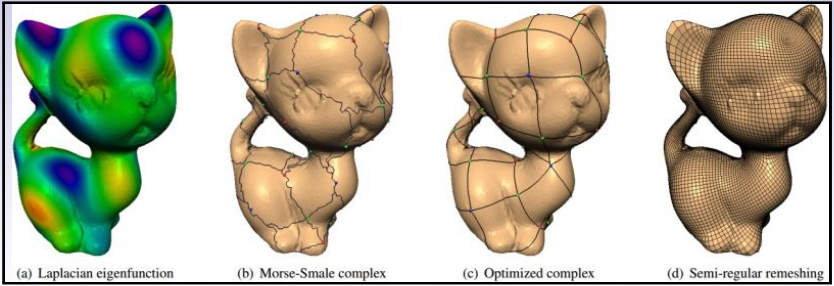


Fig. 19. 2d MDS plot of mesh Shape-DNAs.



$\boxed{\Delta\phi_i = \lambda_i\phi_i}$

Different Application: Quadrangulation

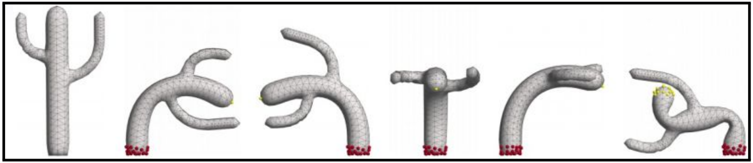


Connect critical points (well-spaced) of ϕ_i in *Morse-Smale complex*.

[Dong et al., 2006]

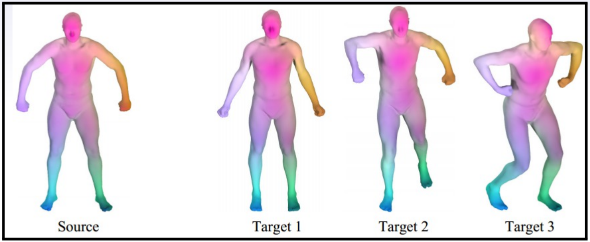
Other Ideas I

- **Mesh editing:** Displacement of vertices and parameters of a deformation should be *smooth* functions along a surface [Sorkine et al., 2004, Sorkine and Alexa, 2007] (and many others)



Other Ideas II

- **Surface reconstruction:** Poisson equation helps distinguish inside and outside [Kazhdan et al., 2006]
- **Regularization for mapping:** To compute $\phi : M_1 \to M_2$, ask that $\phi \circ \Delta_1 \approx \Delta_2 \circ \phi$ [Ovsjanikov et al., 2012]



For Slides

`http://ddg.cs.columbia.edu/SGP2014/LaplaceBeltrami.pdf`

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