Marijn J.H. Heule
School of Computer Science
Carnegie Mellon University, Pittsburgh
marijn@cmu.edu

The Solution

Easier variants of notorious math problems

Computers have been used to solve several long-standing open math problems, including the Robbins conjecture (1996), the Boolean Pythagorean triples problem (2016), and Keller's conjecture (2020). Thanks to the continuing advances that are being made in the field of automated reasoning, we can be optimistic that many success stories will follow. However, the power of automated reasoning has its limitations, and thus computers are still far away from producing substantial mathematical achievements such as the proof of Fermat's last theorem. So what are the actual boundaries of automated reasoning? In this article, Marijn Heule discusses some easier variants of notorious math problems and pose them as challenges for both humans and computers.

Computer-assisted mathematics has a rich history with many exciting results. Major achievements range from Appel and Haken's proof of the Four Color Theorem [1] in 1976 to the success of Buzzard, Commelin, Massot and collaborators [3] in formalizing a complicated construction by Scholze just a year ago. Computers are better than humans in dealing with enormous case splits and in checking the details of mathematical proofs once properly formalized. My work in this direction focuses mostly on computer-assisted (frequently purely computer-reliant) mathematical discovery, trying to find proofs that would be practically impossible for humans to find. Although the proofs found in this manner are often too large to be interpreted by humans, it is very well possible that for some of these problems a small and humanly understandable proof does not even exist.

Automated reasoning has been effective in mathematical discovery and has facilitated the resolution of various mathematical problems. Interesting new methods have been applied successfully in recent years, including the use of neural networks [14]. One may therefore wonder whether automated reasoning is ready to solve problems that have intrigued mathematicians for many decades. Although the answer is probably *no*, that does not mean we should not try. The question is: What is the most promising way to pursue this endeavor?

I propose to study variants of three notorious mathematical problems: the Collatz conjecture [8], the Chromatic Number of the Plane (CNP) [13], and the most wanted Folkman graph [6]. The proposed variants are likely much easier than the original problems, but they are still unresolved. If we cannot even solve these easier variants, then the original problems are completely out of reach. In fact, disproving the proposed variants of Collatz or CNP would immediately solve the respective original problems. It is not evident whether these variants are more suitable for human or automated reasoning, although all variants can be formulated as search problems with existing techniques. I expect that they are a challenge for both humans and computers. Personally, I would gladly see them solved because they represent major roadblocks that currently stand in the way of progress on the original problems. To motivate research, I offer prizes for the solutions.

Collatz conjecture

The Collatz conjecture states that every natural number will eventually reach 1 when the following map, known as the Collatz map, is applied repeatedly: if the number is even, divide it by 2; if the number is odd, multiply it by 3 and add 1. For example, if we start with the number 3, then we follow the path $3 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$. Once 1 is reached, the repeated application of the map results in the cycle $1 \to 4 \to 2 \to 1$. Most mathematicians believe that the conjecture holds, which is also supported by experimental evidence: computers were used to validate that all numbers up to 2^{68} will eventually reach 1. However, a proof of the conjecture still appears far away. Paul Erdős famously stated that "mathematics may not be ready for such problems".

A couple of years ago, Scott Aaronson and I started a moonshot project to tackle the conjecture via automated reasoning. Scott designed a rewriting system that terminates on any input string if and only if the Collatz conjecture is true. The rewrite system consists of eleven rules. Unsurprisingly, state-of-the-art rewrite termination tools, such as AProVE, were not able to solve the termination problem. However, we were able to show termination of any subset of ten rules (proving termination of a strict subsystem is generally easier). Scott solved them by hand while I solved them via the help of satisfiability (SAT) solving — a crucial technique to prove hard termination problems. The subsystems were clearly easier than the full system, even though most of them were non-trivial. My initial approach required many hours on a computer cluster for the hardest subsystems. However, an implementation by my PhD student Emre Yolcu can now solve these subsystems in mere seconds [15].

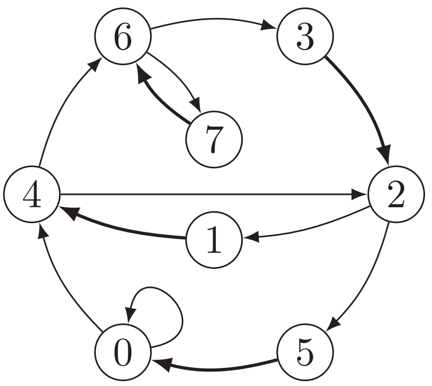

Figure 1 Transition graph of the iterates in the Collatz trajectories across residue classes modulo 8. Let $C(n)$ denote the Collatz map. The edge $u \to v$ is part of its transition graph if and only if there exists some $n \equiv u \pmod 8$ such that $C(n) = v \pmod 8$. Bold edges indicate transitions where $C(n) > n$.

We also explored easier Collatz-like problems. Some turned out to be real challenges. For example, we studied whether the repeated application of the Collatz map will eventually reach a number $1 \pmod 8$, i.e., a number that leaves the remainder 1 when divided by 8. However, for some numbers, the map avoids the numbers congruent to $1 \pmod 8$ for many steps. Figure 1 shows the transitions across residue classes modulo 8 of the Collatz map. The figure shows that there are multiple cycles. Some cycles such as $6 \to 7 \to 6 \pmod 8$ and $0 \to 0 \pmod 8$ cannot be repeated indefinitely. As a consequence, the map cannot avoid the numbers congruent to $4 \pmod 8$, which can also be shown using our automated approach. Unfortunately, similar reasoning cannot deduce that one will eventually reach a number congruent to $1 \pmod 8$.

Studying the transition graph, arguably the easiest non-trivial variant, asks whether the Collatz map will eventually reach a number congruent to $5 \pmod 8$ or $7 \pmod 8$. Without $5 \pmod 8$ and $7 \pmod 8$, only two cycles remain: $1 \to 4 \to 2 \to 1$ and $1 \to 4 \to 6 \to 3 \to 2 \to 1$. The former cycle is decreasing, while the latter is increasing (any non-trivial variant needs at least one increasing cycle). Experiments also suggest that problem is easier: for the original Collatz, the observed worse case of the number of steps to termination is quadratic in the number of bits of the initial number, while for the variant without $5 \pmod 8$ and $7 \pmod 8$ it is linear [5]. The variant can also be expressed as a Collatz-like problem with the following map ($\perp$ denotes an undefined case at which the iterated application stops):

$$H(n) = \begin{cases} \dfrac{3n}{4} & \text{if } n \equiv 0 \pmod 4, \\ \dfrac{9n+1}{8} & \text{if } n \equiv 7 \pmod 8, \\ \perp & \text{otherwise.} \end{cases}$$

This map slightly increases the numbers congruent to $7 \pmod 8$, decreases numbers congruent to $0 \pmod 4$, and terminates on all other numbers. Reasoning about this map therefore seems significantly easier than the original Collatz conjecture as it terminates instantly on most numbers. However, there exist arbitrarily long sequences of map $H(n)$. Consider, for example, the starting number 466032, which results in a sequence that first goes down for a few steps, then up, down, and up again, before terminating:

$$466032 \to 349524 \to 262143 \to 294911 \to 331775 \to 373247 \to 419903 \to 472391 \to 531440 \to 398580 \to 298935 \to 336302 \to \perp .$$

Showing termination of $H(n)$ is my first challenge:

Challenge 1 ($\$500$). *Prove or disprove that iteratively applying map $H(n)$ will eventually reach $\perp$ for all $n \geq 1$.*

A counterexample in the form of a number n such that the repeated application of $H(n)$ will never reach $\perp$ can be converted into a counterexample for the Collatz conjecture and is therefore unlikely to exist.

Chromatic Number of the Plane

The Chromatic Number of the Plane (CNP) asks how many colors are required in a coloring of the plane to ensure that there exists no monochromatic pair of points with distance 1. Early work on CNP showed that the chromatic number is between 4 and 7. The lower bound is due to the Moser spindle: a 7-vertex *unit-distance graph* with 11 edges. A unit-distance graph is formed by a set of points in the plane where two points are connected by an edge whenever the distance between them is exactly 1. The upper bound can be shown by a tessellation of the plane using hexagon tiles with an outer radius slightly smaller than 1. Tiles consist of a single color and tiles of the same color can be placed more than distance 1 from each other. Figure 2 shows the Moser spindle and the tessellation.

In a breakthrough in 2018, Aubrey de Grey improved the lower bound by constructing — with the help of automated-reasoning tools — a 1581-vertex unit-distance graph with chromatic number 5 [4]. This result triggered the Polymath-16 project in which mathematicians around the globe teamed up to make further progress on this problem. In the weeks and months afterwards, I was able to gradually reduce the size of the smallest unit-distance graph with chromatic number 5 using SAT techniques. My best result so far is a graph with 510 vertices. Jaan Parts has the current record with 509 vertices [11].

I also looked into improving the lower bound to 6. At some point, I was hope-

Figure 2 Visualizations of early bounds of CNP: The Moser spindle graph and a 7-color tessellation of the plane.

ful that I was close: I found several large graphs (around 100 000 vertices) for which SAT techniques were unable to find a valid 5-coloring. I tried to prove the absence of 5-colorings with massive parallel computation, but failed. Meanwhile, I was studying 4-colorings of dense unit-distance graphs and observed that points that are certain distances apart from each other always had a different color. This notion is also known as a virtual edge. There was a clear pattern in the found virtual edges: their length was an odd number divided by an odd number. In a later experiment I searched for colorings of the large graphs using edges for any distance that is an odd number divided by an odd number. Surprisingly, the solver was able to find valid 5-colorings. Hence the large graphs have chromatic number at most 5, because adding edges can only increase the chromatic number.

At this point I became intrigued by odd-distance graphs: two points are connected if they are an odd distance apart. Such graphs have been studied by Moshe Rosenfeld and various co-authors. One interesting result is that there exists an odd-distance graph with chromatic number 5 that consists of only 21 vertices [2], which is shown in Figure 3. Alexander Soifer conjectures that there are odd-distance graphs in the plane with an infinitely large chromatic number [13]. However, no odd-distance graph with chromatic number 6 is known. Constructing one is therefore my second challenge.

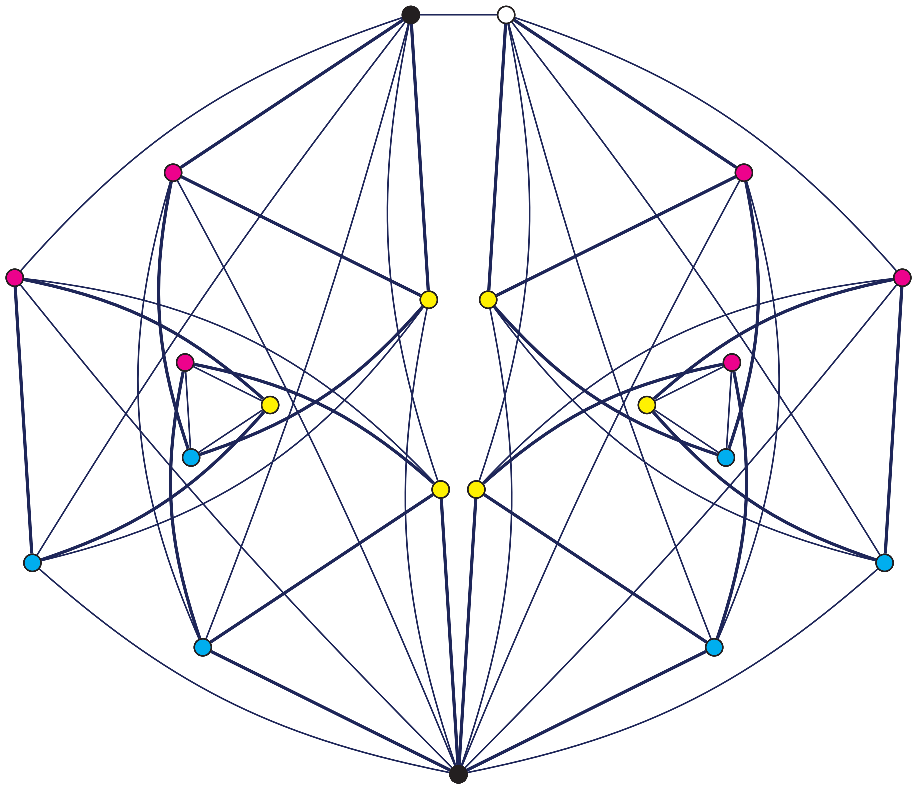

Figure 3 The smallest known odd-distance graph with chromatic number 5. The shortest edges have length 1 and the bold edges have length 3.

Challenge 2 (\$500). *Construct an odd-distance graph with chromatic number 6 or prove that none exists.*

What makes odd-distance graphs with chromatic number 6 interesting? The patterns observed in 4-colorings of dense unit-distance graphs can also be observed in 4-colorings of odd-distance graphs with significantly fewer vertices (points). Therefore, knowing which points form an odd-distance graph with chromatic number 6 can be a big help in constructing a unit-distance graph with chromatic number 6 (if it exists). Note that if no odd-distance graph with chromatic number 6 exists, then there is clearly no unit-distance graph with chromatic number 6. Hence the chromatic number of the plane would be 5.

The Most Wanted Folkman Graph

Finally, let's consider another old but slightly more recent graph problem. It involves a graph property, called the monochromatic triangle property. A graph has this property if every coloring of its edges with two colors, say red and blue, includes a red or blue triangle. It is relatively easy to check that a fully connected graph with 6 vertices (i.e., a 6-clique) has the monochromatic tri-

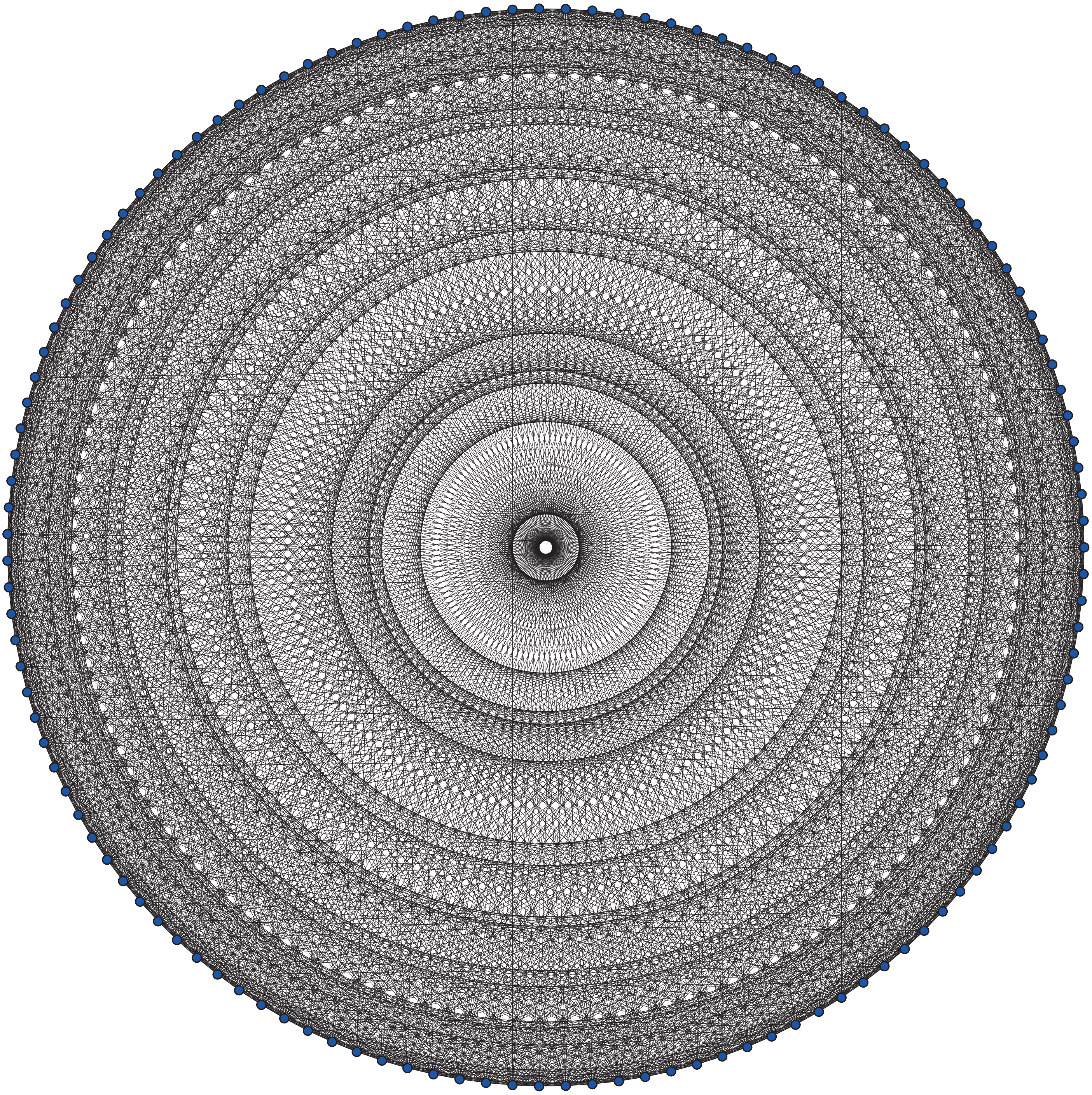

Figure 4 Graph G_{127} which is likely a Folkman graph.

angle property. This is also why Ramsey number $R(3) = 6$. In 1967, Erdős and Hajnal asked the following: does there exist a graph without a 4-clique as an induced subgraph that has the monochromatic triangle property? Folkman proved in 1970 that such graphs exist, but his argument involved graphs of enormous size. Graphs with the monochromatic triangle property that are 4-clique-free are known as Folkman graphs, and the Most Wanted Folkman Graph problem asks for the smallest such graph.

Erdős was the first to push for finding the smallest Folkman graph. He offered a cash prize of $\$100$ to the first person who could show the existence of a Folkman graph with fewer than 10 billion vertices. Spencer in 1988 received the prize by proving that there exist Folkman graphs with fewer than 3 billion vertices. Afterward, Erdős offered another cash prize of $\$100$ for showing that there exists a Folkman graph with fewer than a million vertices. Lu received that prize from Ronald Graham in 2007 after constructing a Folkman graph with just 9697 vertices [10]. Graham continued Erdős's quest by offering another cash prize of $\$100$ for showing the existence of a Folkman graph with fewer than a hundred vertices. The current record is a Folkman graph consisting of 786 vertices found by Lange, Radziszowski and Xu [9].

The search for the smallest Folkman graph ran into an interesting twist in recent years. While it has been difficult for many years to find Folkman graphs of modest size, there are several small candidates for Folkman graphs (around 100 vertices) for which we cannot prove the monochromatic triangle property. The potentially most effective automated method for proving the monochromatic triangle property of small graphs is the use of SAT techniques, which is one of the reasons why I became interested in this problem. The SAT approach to the problem works as follows: given a graph, a propositional formula is constructed asking whether the monochromatic triangles can be avoided. If this formula is unsatisfiable, it means that the property cannot be avoided and that it therefore holds. Although the SAT formulas of the small candidates are not large, they turn out to be very hard.

Let's consider one graph in particular: G_{127} [12]. As the name suggests, G_{127} has 127 vertices. Let's call them $v_1, \ldots, v_{127}$. Two distinct vertices v_i and v_j are connected if and only if $|i-j| \equiv 5^k (\mathrm{mod}\, 127)$ for some $k \geq 0$, which leads to 2666 edges. Figure 4 shows G_{127}, which is 4-clique-free and believed to be a Folkman graph [12]. The SAT formula stating whether G_{127} has the monochromatic triangle property has several aspects in common with the SAT formula for the Pythagorean Triples problem [7], whose solution made me the proud recipient of a $\$100$ prize by Ronald Graham. For example, both formulas are similar in size, and all clauses in the two formulas are of length 3.

There is, however, a major difference: While local search techniques are able to quickly satisfy almost all clauses of the formula for the Pythagorean Triples problem, such techniques cannot get close to a satisfying assignment for the G_{127} formula. This would suggest that the G_{127} formula is very unsatisfiable (meaning that there are many potential ways to refute the formula) and that showing unsatisfiability (proving the monochromatic triangle property) should be relatively easy. However, in my experience, solving this formula with SAT techniques is much harder than the Pythagorean Triples problem, which required four CPU years and resulted in a proof of 200 terabytes [7]. Some other techniques might be more suitable for this problem, if they can exploit the observation that the formula is very unsatisfiable. Still, I consider it a serious challenge.

Challenge 3 ($\$500$). *Prove or disprove that any bi-coloring of G_{127} has a monochromatic triangle.*

Of the small candidates for Folkman graphs, G_{127} is interesting because of its many symmetries. It has also been observed by others [12] that many vertices can be dropped from G_{127}, likely preserving the monochromatic triangle property. Experimental evidence suggests that up to 33 vertices can be dropped, resulting in graphs with just 94 vertices. However, proving the monochromatic triangle property of these smaller graphs is expected to be more difficult than proving it for G_{127}.

Final remarks

The Collatz conjecture is famous not only because of its perceived hardness but also because of its elegance. Many variants have been proposed [8] and my first challenge is another one. What makes this challenge interesting is that it appears much easier while no obvious termination argument exists. For most other variants this is not the case: either they appear as hard as Collatz or an argument of (non-)termination is known. Successes on Collatz-like problems have been limited so far. Focusing on an easier variant could hopefully result in some interesting insights.

Out of the proposed variants, the second challenge is likely the easiest one. The main reason is that some progress is finally being made regarding the chromatic number of the plane. Before de Grey's breakthrough, practically no advances had been made in 68 years. I expect that smaller and smaller unit-distance graphs will be found with chromatic number 5 and that eventually also some interesting candidates for chromatic number 6 will be constructed. However, at that point, we might end up in a situation similar to the most wanted Folkman graph: proving that the chromatic number is 6 might become the hardest part.

The history of the most wanted Folkman graph is rich with lots of progress. If the third challenge can be met, then it might even be possible to prove the monochromatic triangle property of various of its subgraphs with less than 100 vertices (the challenge posed by Graham). The cash prizes by Erdős inspired many researchers in this direction. Let's continue this tradition.

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