Clausal Proofs for Pseudo-Boolean Reasoning[⋆]

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Abstract. When augmented with a Pseudo-Boolean (PB) solver, a Boolean satisfiability (SAT) solver can apply apply powerful reasoning methods to determine when a set of parity or cardinality constraints, extracted from the clauses of the input formula, has no solution. By converting the intermediate constraints generated by the PB solver into ordered binary decision diagrams (BDDs), a proof-generating, BDD-based SAT solver can then produce a clausal proof that the input formula is unsatisfiable. Working together, the two solvers can generate proofs of unsatisfiability for problems that are intractable for other proof-generating SAT solvers. The PB solver can, at times, detect that the proof can exploit modular arithmetic to give smaller BDD representations and therefore shorter proofs.

1 Introduction

Like all complex software, modern satisfiability (SAT) solvers are prone to bugs. In seeking to maximize their performance, developers may attempt optimizations that are either unsound or incorrectly implemented. Requiring a solver to be formally verified is not feasible for current solvers. On the other hand, ensuring that each execution of the solver yields the correct result has become a standard requirement. For a satisfiable formula, the solver can generate a purported solution, and this can be checked directly. For an unsatisfiable formula, the solver can produce a proof of unsatisfiability in a logical framework that enables checking by an efficient and trusted proof checker. Proof generation is a vital capability when SAT solvers are used for formal correctness and security verification, and for mathematical theorem proving.

Most high-performance, proof-generating SAT solvers are based on conflict-driven, clause-learning (CDCL) algorithms [42]. Although the methods used by earlier solvers were limited to steps that could be justified within a resolution framework [43, 52], modern solvers employ a variety of optimizations that require a more expressive proof framework, with the most common being Deletion Resolution Asymmetric Tautology (DRAT) [31,50]. Like resolution proofs, a DRAT proof is a *clausal proof* consisting of a sequence of clauses, each of which preserves the satisfiability of the preceding clauses. An unsatisfiability proof starts with the clauses of the input formula and ends with an empty clause, indicating logical falsehood. The fact that this clause can be derived from the original formula proves that the original formula cannot be satisfied.

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Even with the capabilities of the DRAT framework, some solvers employ reasoning techniques for which they cannot generate unsatisfiability proofs. A number of SAT solvers can extract parity constraints from the input clauses and solve these as linear equations over the integers modulo 2 [6, 30, 37, 47]. Some can also detect and reason about cardinality constraints [6]. However, all these programs revert to standard CDCL when proof generation is required. To overcome the proof-generating limitations of current solvers, some have suggested using more powerful proof frameworks, for example, based on pseudo-Boolean constraints [27] or Binary Decision Diagrams [5]. Staying with DRAT avoids the need to develop, certify, and deploy new proof systems, file formats, and checkers.

Current CDCL solvers do not use the full power of the DRAT framework. In particular, DRAT supports adding *extension variables* to a clausal proof, in the style of extended resolution [48]. These variables serve as abbreviations for formulas over existing input and extension variables. Compared to standard resolution, allowing extension variables can yield proofs that are exponentially more compact [19], and the same holds for the extension rule in DRAT. In general, however, CDCL solvers have been unable to exploit this capability, with the exception that some of their preprocessing and inprocessing techniques [8, 34] require extension variables [39]. One solver attempted to introduce extension variables as it operated [3], but it achieved only modest success.

In 2006, Biere, Jussila, and Sinz demonstrated that the underlying logic behind algorithms for constructing Reduced, Ordered Binary Decision Diagrams (BDDs) [10] can be encoded as steps in an extended resolution framework [35, 46]. By introducing an extension variable for each BDD node generated, the logic for each recursive step of standard BDD operations can be expressed with a short sequence of proof steps. BDDs provide a systematic way to exploit the power of extension variables. The recently developed solver PGBDD [11, 12] (for "proof-generating BDD") builds on this work with a more general capability for existentially quantifying variables. It can generate unsatisfiability proofs for several classic challenge problems for which the shortest possible standard resolution proofs are of exponential size.

We show that BDDs can provide a bridge between *pseudo-Boolean reasoning* and clausal proofs. Pseudo-Boolean (PB) constraints have the form $\sum_{j=1,n} a_j\, x_j \rhd b$, where each variable x_j can be assigned value 0 or 1, the coefficients a_j and constant b are integers, and the relation symbol $\rhd$ is either $=$, $\geq$, or $\equiv \bmod r$ for some modulus r. Both parity and cardinality constraints can be expressed as PB constraints. A PB solver can employ Gaussian elimination or Fourier-Motzkin elimination [21, 51] to determine when a set of constraints is unsatisfiable. Our newly developed program PGPBS (for "proof-generating pseudo-Boolean solver") augments PGBDD with a pseudo-Boolean solver, combining the power of PB reasoning with DRAT proof generation.

To enable proof generation, the PB solver generates BDD representations of its intermediate constraints and has proof-generating BDD operations construct proofs that each of these constraints is logically implied by previous constraints. When the PB solver reaches a constraint that cannot be satisfied, e.g., the equation $0 = 2$, the constraint will be represented by the *false* BDD leaf $\bot$, which yields a proof step consisting of the empty clause. The resulting proof is checkable within the DRAT framework without any reference to pseudo-Boolean constraints or BDDs. Barnett and Biere [5] also proposed using BDDs when proving that the constraints generated by a PB solver were logically implied by their predecessors, but they proposed doing so in a separate proof framework rather than as the solver operates.

As an optimization, the PB solver can automatically detect cases where the unsatisfiability proof for an integer-constraint problem can use modular arithmetic. This leads to more compact BDD representations, and therefore shorter proofs.

We demonstrate the power of PGPBS's combination of BDDs and pseudo-Boolean reasoning by showing that that it can achieve polynomial scaling on two classes of problems for which CDCL solvers have exponential performance. These include parity constraints involving exclusive-or operations [17, 49] and cardinality constraints, including the mutilated chessboard [2] and pigeonhole problems [29]. Although PGBDD on its own can also achieve polynomial scaling for both classes of problems, incorporating pseudo-Boolean reasoning makes the solver much more robust. It can handle wider variations in the problem definition, how the problem is encoded as clauses, and the BDD variable ordering. It also operates with greater automation, requiring no guidance or hints from the user. These capabilities eliminate major shortcomings of PGBDD.

2 Pseudo-Boolean Constraints

Let x_j, for $1 \leq j \leq n$, be a set of variables, each of which may be assigned value 0 or 1, and a_j, for $1 \leq j \leq n$, be a set of integer coefficients. Constant b is also an integer. A *pseudo-Boolean* constraint is of the form $\sum_{j=1,n} a_j \, x_j \rhd b$, with $\rhd$ defining the relation between the left-hand weighted sum and the right-hand constant. For an *integer equation*, $\rhd$ is $=$, i.e., the two sides must be equal. For an *ordering constraint*, $\rhd$ is $\geq$. For a *modular equation*, $\rhd$ is $\equiv \bmod r$, where r is the chosen modulus.

Three constraint types are of special importance for solving cardinality problems. An *at-least-one* (ALO) constraint is an ordering constraint with $a_j \in \{0, +1\}$ for all j, and $b = +1$. An *at-most-one* (AMO) constraint is an ordering constraint with $a_j \in \{-1, 0\}$ for all j, and $b = -1$. An *exactly-one* constraint is an integer equation with $a_j \in \{0, +1\}$ for all j and $b = +1$.

2.1 BDD Representations

Many researchers have investigated the use of BDDs to represent pseudo-Boolean constraints [1,24,33]. As examples, Figure 1 shows BDD representations of the three forms of constraints for $n = 10$ and $b = 0$, with $a_j = +1$ for odd values of j and -1 for even values. The modular equation has $r = 3$. The BDDs for both the integer equation (A) and ordering constraint (B) have an increasing number of nodes at each level for the first $n/2$ levels, with a node at level k for each possible value of the *prefix sum* $\sum_{j=1,k-1} a_j \, x_j$. As the level k approaches n, however, the number of nodes at each level decreases. If a prefix sum becomes too extreme on the negative side, it becomes impossible for the remaining values to cause the sum to reach $b = 0$. For the integer equation, a similar phenomenon happens if a prefix sum becomes too extreme on the positive side. For an ordering constraint, a sufficiently positive prefix sum will guarantee that the total sum will be at least 0. For the modular sum (C), the number of nodes at any level cannot exceed r—one for each possible value of the prefix sum modulo r.

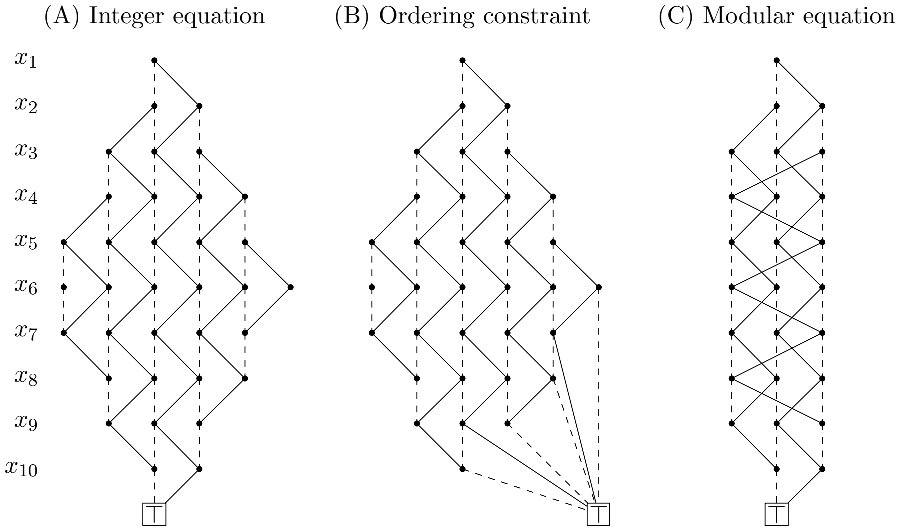


Fig. 1. Example BDD representations of pseudo-Boolean equations and ordering constraints. Solid (respectively, dashed) lines indicate the branch when the variable is assigned 1 (resp., 0). The leaf representing the *false* Boolean constant $\bot$ and its incoming edges are omitted.

Letting $a_{\max} = \max_{1 \leq j \leq n} |a_j|$, the BDD representation of an integer equation or ordering constraint will have at most $2\, a_{\max} \cdot n$ nodes at any level, while the representation of a modular equation will have at most r nodes at any level. Although large values of $a_{\max}$ ($a_{\max} \gg n$), can cause the BDDs to be of exponential size [1, 33], our use of them will assume that both $a_{\max}$ and r are small constants. The BDD representations will then be $O(n^2)$ for integer equations and ordering constraints, and $O(n)$ for modular equations. These bounds are independent of the BDD variable ordering.

Most BDD operations are implemented via the *Apply* algorithm [10], recursively traversing a set of argument BDDs to either construct a new BDD or to test some property of existing ones. The BDDs representing pseudo-Boolean constraints are *levelized*: every branch from a node at level j goes to a leaf node or to a node at level $j + 1$. We can therefore derive a bound on the maximum number of recursive steps to perform an operation on k argument BDDs, assuming both $a_{\max}$ and r are small constants. Due to the caching of intermediate results, the maximum number of steps at each level will be bounded by the product of the number of argument nodes at this level. The operation will therefore have worst-case complexity $O(n^{k+1})$ for integer equations and ordering constraints, while it will have complexity $O(k \cdot n)$ for modular equations.

2.2 Solving Systems of Equations with Gaussian Elimination

We use a formulation of Gaussian elimination that scales each derived equation, rather than dividing by the pivot value [4, 44]. Performing the steps therefore requires only addition and multiplication. This allows maintaining integer coefficients and automatically detecting a minimum, possibly non-prime, modulus for equation solving.

Consider a system of integer or modular equations E, where each equation $\mathbf{e}_i \in E$, is of the form $\sum_{j=1,n} a_{i,j}\, x_j = b_i$. Applying one step of Gaussian elimination involves selecting a *pivot*, consisting of an equation $\mathbf{e}_s \in E$ and a variable x_t such that $a_{s,t} \neq 0$. Then an equation $\mathbf{e}'_i$ is generated for each value of i:

$$\mathbf{e}'_i = \begin{cases} \mathbf{e}_i & a_{i,t} = 0 \\ -a_{i,t} \cdot \mathbf{e}_s + a_{s,t} \cdot \mathbf{e}_i, & a_{i,t} \neq 0 \end{cases} \tag{1}$$

where operations $+$ and $\cdot$ denote addition and scalar multiplication of equations. Observe that $a'_{i,t} = 0$ for all equations $\mathbf{e}'_i$. Letting $E \leftarrow \{\mathbf{e}'_i | i \neq s\}$, this step has reduced both the number of equations in E and the number of variables in the equations by one.

Repeated applications of the elimination step will terminate when either 1) all equations have been eliminated, or 2) an unsolvable equation is encountered. For case 1, the system has solutions, but these may, in general, assign values other than 0 and 1 to the variables. (Importantly, parity constraints are represented by modular equations with $r = 2$. Their solutions *will* be 0-1 valued, and so a SAT solver can make use of them [30, 37].) For case 2, if some elimination step generates an equation of the form $0 = b$ with $b \neq 0$, then this equation has no solution in any case, and therefore neither did the original system. Our proofs of unsatisfiability rely on reaching this condition.

For the modular case, all coefficients and the constants are kept within the range 0 to $r - 1$. For integer equations, the coefficients can grow exponentially in m. Fortunately, the cardinality problems we consider only require coefficient values -1, 0, and $+1$.

As we have seen, the BDD representations of modular equations have bounded width, making them both more compact and making the algorithms that operate on them more efficient than for integer equations. As we will see, the unsatisfiability proof generated by applying Gaussian elimination to a system of modular equations can be significantly more compact than for the same equations over integers. This gives rise to an optimization we call *modulus auto-detection.* The idea is to apply Gaussian elimination to a set of integer equations, recording the dependencies between the equations generated, but without performing any proof generation. Once the solver reaches an equation of the form $0 = b$ where $b \neq 0$, it chooses the smallest $r \geq 2$ such that $b \bmod r \neq 0$. It then generates a proof, reinterpreting the Gaussian elimination steps using modulo-r arithmetic. Since the only operations of (1) are multiplication and addition, the final equation will be $0 \equiv b \pmod{r}$, which has no solution. Here we can see that allowing r to be composite is both valid and may be optimal. For example, the smallest choice for $b = 30$ would be $r = 4$, rather than the prime $r = 7$. Auto-detection can be applied whenever Gaussian elimination encounters an unsolvable equation.

2.3 Solving Systems of Ordering Constraints with Fourier-Motzkin Elimination

Consider a set C, consisting of constraints $\mathbf{c}_i$ of the form $\sum_{j=1,n} a_{i,j}\, x_j \geq b_i$. Applying one step of Fourier-Motzin elimination [21, 51] to this system involves identifying a *pivot*, consisting of a variable x_t such that $a_{k,t} \neq 0$ for at least one value of k. The set is partitioned into three sets by assigning each constraint $\mathbf{c}_i$ to C^+, C^-, or C^0, depending on whether coefficient $a_{i,t}$ is positive, negative, or zero, respectively. For each pair i and i' such that $\mathbf{c}_i \in C^+$ and $\mathbf{c}_{i'} \in C^-$, a new constraint $\mathbf{c}_{i,i'}$ is generated as:

$$\mathbf{c}_{i,i'} = -a_{i',t} \cdot \mathbf{c}_i + a_{i,t} \cdot \mathbf{c}_{i'} \tag{2}$$

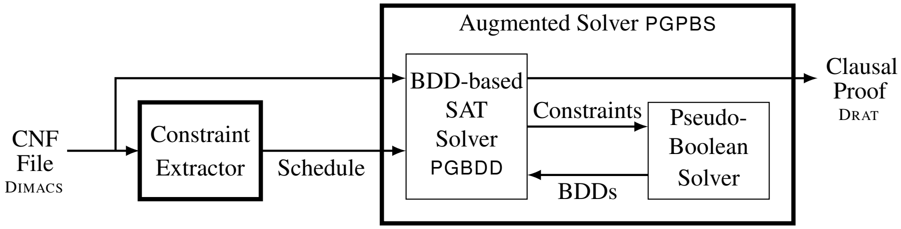


Fig. 2. Overall Structure of PGPBS. It augments the BDD-based SAT solver PGBDD with inferences from a pseudo-Boolean constraint solver. The constraint extractor is a separate program.

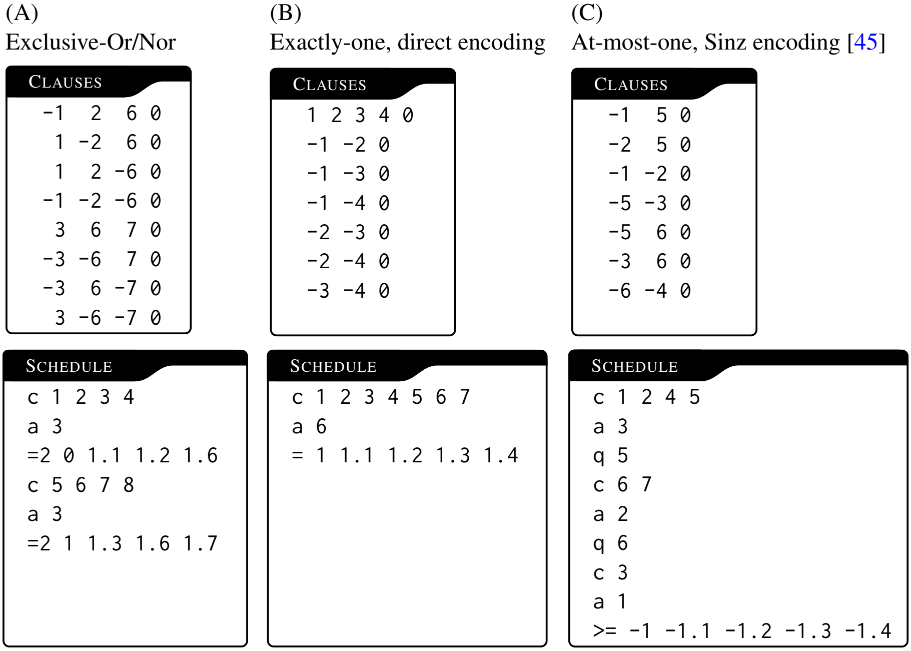
(Note that the multiplication is always by positive values, since $a_{i',t} < 0$.) Letting $C \leftarrow C^0 \cup \{\mathbf{c}_{i,i'} \mid \mathbf{c}_i \in C^+, \mathbf{c}_{i'} \in C^-\}$, all of these constraints have coefficient 0 for variable x_t. Therefore this step has reduced the number of variables in the constraints by one, but it may have increased the number of constraints.

As with Gaussian elimination, repeated application of the elimination step will terminate when either 1) all variables have been eliminated or 2) an unsolvable constraint is encountered. With case 1, the constraints can be satisfied, although possibly by assigning values other than 0 or 1 to some of the variables. An unsolvable constraint (case 2) is one where the sum of the positive coefficients is less than the constant term. If such a constraint is encountered, then the original system of constraints has no solution.

Fourier-Motzkin elimination would appear to be hopelessly inefficient. The number of constraints can grow exponentially as the elimination proceeds, and the coefficients can grow doubly exponentially. Fortunately, the cardinality problems we consider have the property that for any variable x_t, there is at most one constraint $\mathbf{c}_i$ having $a_{i,t} = +1$, at most constraint $\mathbf{c}_{i'}$ having $a_{i',t} = -1$, and no other constraint with a non-zero coefficient at position t. This property is maintained by each elimination step, and so the number of constraints will decrease with each step, and the coefficients will be restricted to the values -1, 0, and $+1$.

3 Overall Operation

Figure 2 illustrates the program structure. The pair of programs—extractor and solver—supports the standard flow for proof-generating SAT solvers, reading the input conjunctive normal form (CNF) formula expressed in the standard DIMACS format and generating proofs in the standard DRAT format. No other guidance or hint is provided. The constraint extractor identifies pseudo-Boolean constraints encoded as clauses in the input file and generates a *schedule* indicating how clauses should be combined and quantified to derive BDD representations of the constraints. PGPBS augments the SAT solver PGBDD with a PB solver. PGBDD supplies the constraints to the PB solver, which applies either Gaussian elimination or Fourier-Motzkin elimination. The PB solver generates BDD representations of the constraints it generates, and, since the BDD library generates proof steps while performing BDD operations, it can generate a proof that each new constraint is logically implied by previous constraints. When the PB solver encounters an unsolvable constraint, an empty clause is generated, completing the proof.



(A)
Exclusive-Or/Nor

```
CLAUSES
-1  2  6 0
 1 -2  6 0
 1  2 -6 0
-1 -2 -6 0
 3  6  7 0
-3 -6  7 0
-3  6 -7 0
 3 -6 -7 0
```

```
SCHEDULE
c 1 2 3 4
a 3
=2 0 1.1 1.2 1.6
c 5 6 7 8
a 3
=2 1 1.3 1.6 1.7
```

(B)
Exactly-one, direct encoding

```
CLAUSES
1 2 3 4 0
-1 -2 0
-1 -3 0
-1 -4 0
-2 -3 0
-2 -4 0
-3 -4 0
```

```
SCHEDULE
c 1 2 3 4 5 6 7
a 6
= 1 1.1 1.2 1.3 1.4
```

(C)
At-most-one, Sinz encoding [45]

```
CLAUSES
-1  5 0
-2  5 0
-1 -2 0
-5 -3 0
-5  6 0
-3  6 0
-6 -4 0
```

```
SCHEDULE
c 1 2 4 5
a 3
q 5
c 6 7
a 2
q 6
c 3
a 1
>= -1 -1.1 -1.2 -1.3 -1.4
```

Fig. 3. Examples of pseudo-Boolean constraints extracted from CNF representations. Schedules use a stack notation indicating clauses, conjunction and quantification operations, and constraints.

3.1 Constraint Extraction

The constraint extractor uses heuristic methods to identify how the input clauses match standard patterns for exclusive-or/nor, ALO, and AMO constraints. The heuristics are independent of any ordering of the clauses or variables, although they do depend on the polarities of the literals. The generated schedule indicates how to combine clauses and to quantify variables to give the different constraints. The schedule uses a stack notation, having the following commands:

`c` $c_1, \ldots, c_k$	Generate and push the BDDs for the specified clauses.
`a` m	Pop the top $m+1$ elements. Combine with m AND operations. Push the result.
`q` $v_1, \ldots, v_k$	Quantify the top element by the specified variables.
C $b\ a_1 . v_1, \ldots, a_k . v_k$	Confirm that the top stack element implies the constraint

The different constraint types C are '=' for integer equations, '=2' for mod-2 equations, and '>=' for integer orderings. Each constraint line lists the constant b and then indicates the non-zero terms as a combination of coefficient and variable, separated by '.'.

Figure 3 provides a series of examples illustrating the operation of the extractor. A k-way exclusive-or or exclusive-nor (A) is encoded with 2^{k-1} clauses (here $k = 3$), listing all combinations of the negated variables having even (XOR) or odd (XNOR) parity. The schedule lists the clause numbers, forms their conjunction, and indicates a mod-2 equation. The constant b is 1 for exclusive-or and 0 for exclusive-nor.

An exactly-one constraint (B) can be expressed as a combination of an ALO constraint and an AMO constraint. The extractor assumes that any clause with all literals having positive polarity encodes an ALO constraint. In this example, a k-way AMO constraint ($k = 4$) is encoded directly as a set of $k\,(k-1)/2$ binary clauses.

An AMO constraint can be also encoded with auxiliary variables (B) in variety of ways, including that devised by Sinz [45]. The extractor examines how variables occur in binary clauses. Those that occur only with negative polarity are assumed to be constraint variables, while those that have mixed polarity are assumed to be auxiliary variables. As is shown, the generated schedule for an AMO constraint encoded with auxiliary variables employs *early quantification* [13] to linearize the conjuncting of clauses and the quantification of auxiliary variables.

The heuristics used for identifying auxiliary variables and partitioning the clauses into distinct constraints apply to a wide range of AMO constraints, including those using hierarchical encodings [16, 36] and those considered in other constraint extraction programs [9]. Our method can be overly optimistic, labeling some subsets of clauses incorrectly. Fortunately, any such error will be quickly identified when the solver attempts to prove that the BDD generated by conjuncting the clauses and quantifying the auxiliary variables implies the BDD generated for the constraint.

3.2 Solver Operation

The SAT solver portion of PGPBS can generate BDD representations of input clauses and perform conjunction and existential quantification operations on BDDs [11, 12]. As the solver manipulates BDDs to track the solution state, it also generates clauses according to resolution and extension proof rules. The state of the solver at any time is captured by a set of *terms* $T_1, T_2, \ldots, T_n$, where each term T_i consists of:

- A root node u_i in the BDD.
- The extension variable associated with this node, also written as u_i.
- A unit clause, included in the proof clauses, consisting of extension variable u_i, asserting that the Boolean function represented by BDD node u_i evaluates to true for any variable assignment that satisfies the input clauses.
- Implicitly, the set $\theta(u_i)$ of all *defining clauses* that were added to the proof when introducing the extension variables for the nodes in the BDD subgraph having root u_i. These are included in the generated clauses and provide the semantic model for the BDD within the proof framework.

The BDD package supports proof-generating BDD operations ApplyAnd, used to perform conjunction, and ProveImplication, used to generate proofs of implication. The ApplyAnd operation takes as arguments BDD roots u and v, and it generates a BDD representation with root w of their conjunction. It also generates a proof of the clause $\overline{u} \vee \overline{v} \vee w$, proving the implication $u \wedge v \rightarrow w$. The ProveImplication operation performs implication testing without generating any new BDD nodes. It takes as arguments BDD roots u and v, and it generates a proof of the clause $\overline{u} \vee v$, proving that $u \rightarrow v$. An error is signaled if the implication does not hold.

When the solver encounters a clause command in the schedule file, it generates a term T_i for each of the specified input clauses C_i and pushes the term onto a stack. It also generates the proof $\theta(u_i), C_i \vdash u_i$, i.e., that function represented by BDD node u_i will evaluate to true for any variable assignment that satisfies the clause.

When the solver encounters a conjunction or quantification command, it creates a new term by performing the specified operation and proving that it is implied by earlier terms. Given newly generated BDD root u_{n+1}, it must prove that u_{n+1} is *implication redundant* with respect to the existing terms. That is, if u_{n+1} was generated by applying some operation to terms $T_{i_1}, T_{i_2}, \ldots, T_{i_k}$, then it must generate a proof of the clause $\overline{u}_{i_1} \vee \overline{u}_{i_2} \vee \cdots \vee \overline{u}_{i_k} \vee u_{n+1}$. This clause can then be resolved with the unit clauses associated with the existing terms to yield the unit clause u_{n+1}, allowing a new term T_{n+1} to be added. If some step generates a term T_{n+1} with BDD representation $u_{n+1} = \bot$, it will also generate the empty clause, completing a proof of unsatisfiability.

The PB solver portion of PGPBS can generate BDD representations of the intermediate constraints it creates. The SAT solver generates a new term for each of these BDDs. The proof generator need not have any understanding of the operation of the PB solver, and vice-versa. Suppose some set of input clauses encodes a pseudo-Boolean constraint, possibly using auxiliary variables, as was illustrated in Figure 3. The SAT solver performs the series of conjunction and quantification operations specified by the schedule to reduce the clauses to a single term T_n consisting of BDD root u_n and unit clause u_n. The auxiliary variables have been quantified away, and so u_n depends only on the constraint variables. It passes the constraint to the PB solver, which generates its BDD representation with root u_{n+1}. The SAT solver uses the PROVEIMPLICATION operation to generate the clause $\overline{u}_n \vee u_{n+1}$. This can be resolved with unit clause u_n to generate the unit clause u_{n+1}, and so the BDD representation of the constraint becomes term T_{n+1}. (Typically, the two BDDs are identical and so the implication holds trivially.) This process is repeated to convert the input formula into a set of pseudo-Boolean constraints, each represented as a term in the SAT solver.

Once the SAT solver has converted all of the input clauses into constraints, it passes control to the PB solver. From that point on, the SAT solver serves in a support role, generating proofs to justify the steps of the PB solver. As the PB solver operates, it generates a BDD representation of each new constraint: for each equation $\mathbf{e}'_i$ generated by Gaussian elimination (1) or each ordering constraint $\mathbf{c}_{i,i'}$ generated by Fourier-Motzkin elimination (2). For a new BDD with root u_{n+1} generated from constraints represented by terms T_i and T_j, it uses the APPLYAND operation to generate the conjunction w of the BDDs with roots u_i and u_j, as well as a proof of the clause $\overline{u}_i \vee \overline{u}_j \vee w$. It then uses the PROVEIMPLICATION operation with arguments w and u_{n+1} to generate a proof of the clause $\overline{w} \vee u_{n+1}$. It can then resolve the unit clauses for terms T_i and T_j with the generated clauses to generate a proof of the unit clause u_{n+1}, and so the BDD representation of the constraint becomes term T_{n+1}. When some step of the PB solver generates an unsolvable equation or ordering constraint, it encodes the constraint as the *false* BDD leaf $\bot$, and the SAT solver will generate the empty clause.

As an optimization, we implemented an operation APPLYANDPROVEIMPLICATION combining the functions of APPLYAND and PROVEIMPLICATION. It takes as arguments BDD roots u, v, and w and generates a proof that $u \wedge v \rightarrow w$ without constructing the BDD representation of $u \wedge v$. We found this reduced the total proof lengths by over $2\times$.

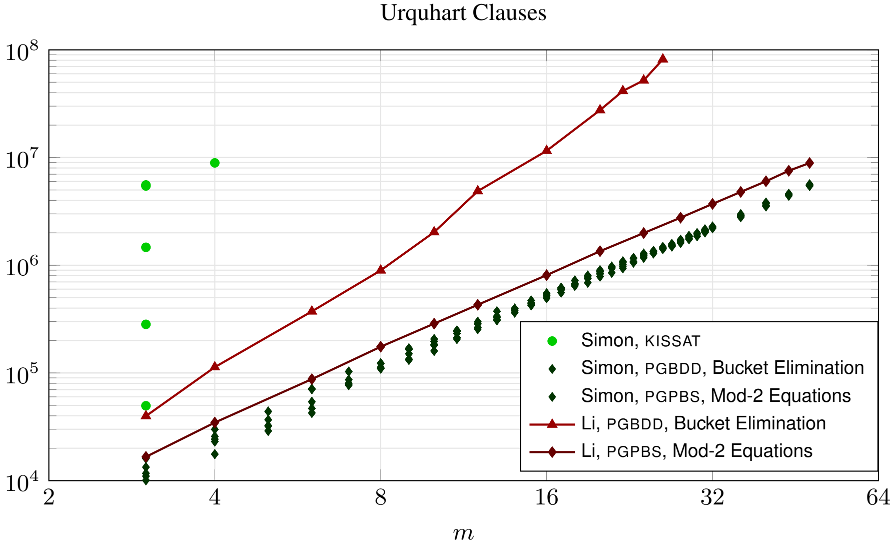


Fig. 4. Total number of clauses in proofs of two sets of Urquhart formulas.

4 Experimental Results

PGPBS is written in Python with its own BDD package and pseudo-Boolean constraint solver.[3] The Gaussian elimination solver employs a standard greedy pivot selection heuristic, attributed to Markowitz [23, 41], that seeks to minimize the number of nonzero coefficients created. The Fourier-Motzin solver uses a similar heuristic for selecting pivot variables.

The operation of PGPBS follows the flow illustrated in Figure 2, with constraints extracted directly from the input CNF file, and with the generated schedule driving the operation of the solver. Some measurements were taken using a BDD variable ordering according to their numbering in the input file, while others used a random BDD variable ordering to assess the sensitivity to the variable ordering. All generated proofs were checked with an LRAT proof checker [20]. We used KISSAT, winner of the 2020 SAT competition [7], as a representative CDCL solver. All measurements labeled "PGBDD" are for the earlier version of the solver, without pseudo-Boolean reasoning [11, 12].

We measure the performance of the solvers in terms of the total number of clauses in the generated proofs of unsatisfiability. This metric tracks closely with the solver runtime and has the advantage that it is machine independent. We set an upper limit of 100 million clauses for the proof sizes for the three measured solvers.

4.1 Urquhart Parity Formulas

Urquhart [49] defined a family of formulas that require resolution proofs of exponential size. Over the years, two sets of SAT benchmarks have been labeled as "Urquhart Prob-

[3] PGPBS, PGBDD, and the code for generating and testing a set of benchmarks, are available at https://github.com/rebryant/pgpbs-artifact and as https://doi.org/10.5281/zenodo.5907086.

lems" [15, 38]. The formulas are defined over a class of degree-5, undirected, bipartite graphs, parameterized by a size m, such that the number of nodes in the graph is $2m^2$. To transform a graph into a formula, each edge $\{i, j\}$ in the set of edges E has an associated variable $x_{\{i,j\}}$. (We use set notation to emphasize that the order of the indices does not matter.) Each vertex is assigned a polarity $p_i \in \{0, 1\}$, such that the sum of the polarities is odd. The clauses then encode that the sum for all values of i and j of $x_{\{i,j\}} + p_i$ equals 0 modulo 2. This is false of course, since each edge is counted twice in the sum, and the sum of the polarities is odd.

The two families of benchmarks differ in how the graphs are constructed. Li's benchmarks are based on the explicit construction of *expander* graphs [26, 40], upon which Urquhart's lower bound proof is based. Simon's benchmarks are based on randomly generated graphs and thus depend on the random seed. We generated five different formulas for each value of m. It is unlikely that Simon's graphs satisfy the expander property, but they are still very challenging benchmarks for most SAT solvers.

Figure 4 shows the performance of the solvers, measured as the number of clauses as a function of m, for both Simon's and Li's benchmarks. The smallest instances of the benchmark have $m = 3$. As can be seen KISSAT is able to generate proofs for the Simon version for four cases with $m = 3$ and one with $m = 4$, but it is unable to handle any other cases, including not even the minimum instance for Li's benchmark. Measurements are shown for PGBDD running bucket elimination, a simple algorithm that processes clauses and intermediate terms with conjunction and quantification operations according to the levels of the topmost variables [22, 35]. It achieves polynomial scaling on both benchmarks, with only mild sensitivity to the random seeds. Running PGPBS with modulo-2 equation solving improves the performance even further, such that we were able to handle both families of benchmarks up to $m = 48$. Considering that the problem grows quadratically in m, this represents a major improvement over KISSAT.

4.2 Other Parity Constraint Benchmarks

Chew and Heule [17] introduced a benchmark based on Boolean expressions computing the parity of a set of Boolean values $x_1, \ldots, x_n$ using two different orderings of the inputs, with a randomly chosen variable negated in the second computation. The SAT problem is to find a satisfying assignment that makes the two expressions yield the same result—an impossibility due to the negated variable. With KISSAT, we found the results were very sensitive to the choice of random permutation, and so we ran the solver for five different random seeds for each value of n. We were able to generate proofs for instances with n up to 47, but we also encountered cases where the proofs exceeded the 100-million clause limit starting with $n = 40$. The overall scaling is exponential.

Chew and Heule showed they could generate proofs for this problem that scale as $n \log n$. Using bucket elimination, PGBDD is able to obtain polynomial performance, handling up to $n = 3{,}000$ with a proof of 61 million clauses. PGPBS is able to apply Gaussian elimination with modulus $r = 2$, obtaining even better performance than did Chew and Heule. For $n = 10{,}000$, Chew and Heule's proof has 14 million clauses while the proof generated by PGPBS has less than 7 million.

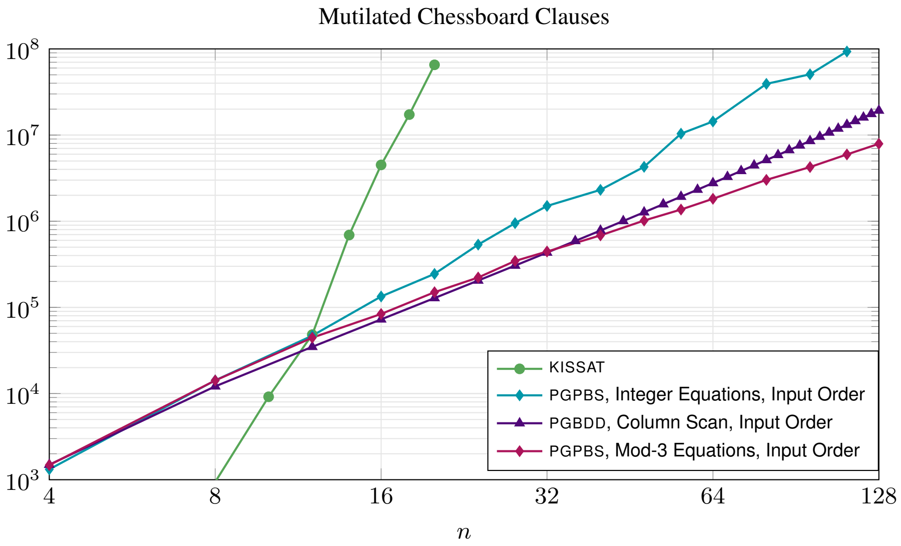


Fig. 5. Total number of clauses in proofs of $n \times n$ mutilated chess board problems.

Elffers and Nordström created the TSEITINGRID family of benchmarks for the 2016 SAT competition, based on grid graphs having fixed width but variable lengths [25]. These are designed to be challenging for SAT solvers while having polynomial scaling. The 2020 SAT competition included two instances of this benchmark, with 7×165 and 7×185 grids. None of the entrants could generate an unsatisfiability proof for either instance within the 5000 second time limit. On the other hand, PGPBS can readily handle both, generating proofs with less than 500,000 clauses and requiring at most 63 seconds. Indeed, PGPBS can solve the largest published instance, having a 7×200 grid, in 76 seconds. Clearly, parity constraint problems pose no major challenge for PGPBS.

4.3 Variants of the Mutilated Chessboard

The mutilated chessboard problem considers an $n \times n$ chessboard, with the corners on the upper left and the lower right removed. It attempts to tile the board with dominos, with each domino covering two squares. Since the two removed squares had the same color, and each domino covers one white and one black square, no tiling is possible. This problem has been well studied in the context of resolution proofs, for which it can be shown that any proof must be of exponential size [2].

The standard CNF encoding defines a Boolean variable for each possible horizontal or vertical domino placement. For each square, it encodes an exactly-one constraint for the set of dominos that could cover that square. Both the number of variables and the number of clauses scale as $\Theta(n^2)$. Figure 5 shows the performance of the different solvers as a function of n. KISSAT scales exponentially, hitting the 100-million clause limit with $n = 20$. The plot labeled "Column Scan" demonstrates that PGBDD performs very well on this problem when given a carefully crafted schedule and the proper variable ordering [11], requiring less than 20 million clauses for $n = 128$.

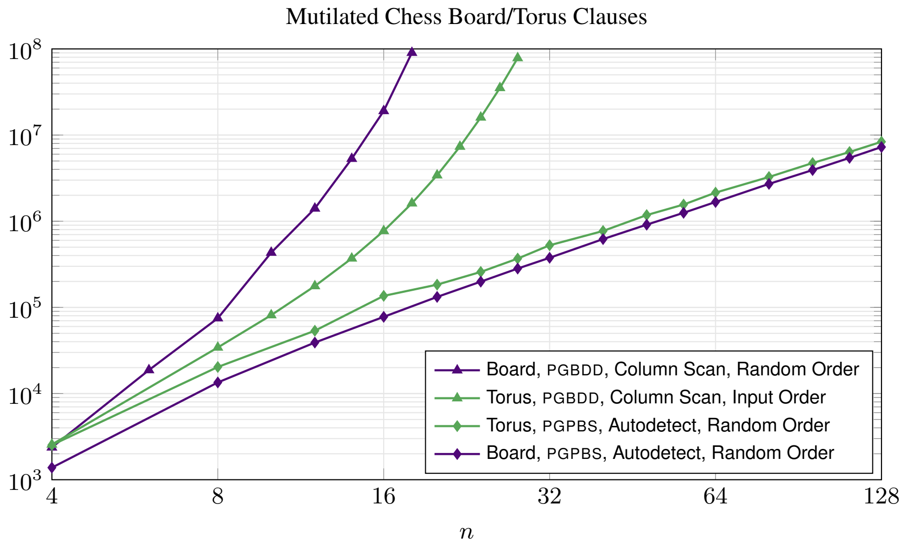


Fig. 6. Stress Testing: Changing the topology and variable ordering for mutilated chess. Autodetection enables the PB solver to use modulo-3 arithmetic.

The plot labeled "Integer Equations, Input Ordering" shows that PGPBS can achieve polynomial scaling on this problem when performing Gaussian elimination on integer equations. It does not scale as well as column scanning, reaching $n = 96$ before hitting the clause limit. (The unevenness of the plot appears to be an artifact of the randomization used to break ties during pivot selection.)

Looking deeper, we can see that solver avoids the worst-case performance for Gaussian elimination on this problem. Let us assume that the omitted corners are both white, and so the board has k black squares and $k - 2$ white squares, where $k = n^2/2$. Each variable occurs in one equation for a black square and in one for a white square. If we were to sum all of the equations for the black squares, we would get $\sum_{j=1,m} x_j = k$, where m is the number of variables. Similarly, summing the equations for the white squares gives $\sum_{j=1,m} x_j = k - 2$. Subtracting the second equation for the first gives the unsolvable equation $0 = 2$. These sums and differences can be performed using pseudo-Boolean equations with coefficients 0 and $+1$. Although Gaussian elimination combines equations in a different order, it maintains the property that the coefficients are limited to values -1, 0, and $+1$.

The plot labeled "Mod-3 Equations, Input Ordering" demonstrates the benefit of modular arithmetic when solving systems of equations. The equation $0 = 2$, obtained by integer Gaussian elimination for this problem, has no solution for any odd modulus; modulus auto-detection chooses $r = 3$. This optimization achieves better scaling, due to the bounded width of the BDD representations. Indeed, it outperforms the best results obtained with PGBDD, generating a proof with less than 8 million clauses for $n = 128$. For the remaining measurements, we assume that modulus auto-detection is enabled.

The plots of Figure 6 illustrate how pseudo-Boolean reasoning makes PGPBS more robust than PGBDD. First, we consider the extension of the mutilated chessboard prob-

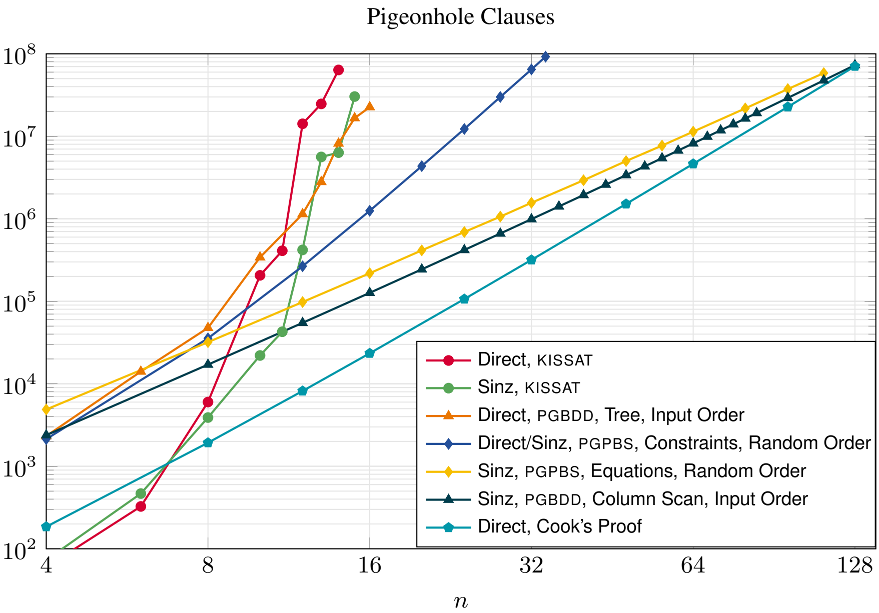


Fig. 7. Total number of clauses in proofs of pigeonhole problem for n holes

lem to a *torus*, with the sides of the board wrapping around both vertically and horizontally. As the plot labeled "Torus, PGBDD, Column Scan, Input Order" indicates, the performance of column scanning disintegrates for this seemingly minor change. The compact state encoding exploited by column scanning works only when there is a single frontier as the variables are processed from left to right. Second, the plot labeled "Board, PGBDD, Column Scan, Random Order" illustrates that column scanning is highly sensitive to the chosen BDD variable ordering. On the other hand, the four versions using auto-detected modular equations are only mildly sensitive to the topology (torus or board) or the variable ordering (input or random). For both topologies, the clause counts for the two different orderings (input and random) are so close to each other that they cannot be distinguished on the log-log scale. and so we show only the results for random orderings. These results show that pseudo-Boolean reasoning overcomes several major weaknesses of the pure Boolean methods of PGBDD. With its PB solver, PGPBS requires no guidance from the user regarding how to process the clauses, nor does it require any guidance or heuristics to choose a good BDD variable ordering. Furthermore, it is less sensitive to the problem definition.

4.4 Pigeonhole Problem

The pigeonhole problem is one of the most studied problems in propositional reasoning. Given a set of n holes and a set of $n+1$ pigeons, it asks whether there is an assignment of pigeons to holes such that (1) every pigeon is in some hole, and (2) every hole contains at most one pigeon. The answer is no, of course, but any resolution proof for this must be of exponential length [29].

The problem can be encoded into CNF with Boolean variables $p_{i,j}$, for $1 \leq i \leq n$ and $1 \leq j \leq n+1$, indicating that pigeon j is placed in hole i. A set of n AMO constraints indicates that each hole can contain at most one pigeon, and $n+1$ ALO constraints indicate that each pigeon must be placed in some hole. We experimented with two different encodings for the AMO constraints: the direct encoding requiring $n\,(n+1)/2$ clauses per hole, and the Sinz encoding [45], requiring $3n-1$ clauses.

Figure 7 shows the total number of clauses (input plus proof) as functions of n for this problem. KISSAT performs poorly, reaching the 100-million clause limit with $n = 14$ for the direct encoding and $n = 15$ for the Sinz encoding. Using PGBDD, we were unable to find any strategy that gets beyond $n = 16$ with a direct encoding. Our best results came from a "tree" strategy, simply forming the conjunction of the input clauses using a balanced tree of binary operations. For the Sinz encoding, on the other hand, we devised a column scanning technique similar to the method used to solve the mutilated chessboard problem. This approach scales very well, empirically measured as $\Theta(n^3)$. The proofs stay below 100 million clauses up to $n = 128$, although it can only reach $n = 17$ with a random variable ordering (plot not shown).

Using pseudo-Boolean reasoning with Fourier-Motzkin elimination, we were able to achieve polynomial scaling, reaching $n = 34$ with both encodings and for both input and random ordering. The four results are so similar that they are indistinguishable on a log-log plot, and so we show the average for the two encodings with random orderings. Observe that each variable $p_{i,j}$ occurs with coefficient -1 in the AMO constraint for hole i and with coefficient $+1$ in the ALO constraint for pigeon j. Thus, as described in Section 2.3, each step of Fourier-Motzkin elimination reduces the number of constraints by at least one, with the coefficients restricted to the values -1, 0, and $+1$. Indeed, it can be seen that the solver, in effect, sums the n AMO and $n+1$ ALO constraints to get the unsolvable constraint $0 \geq 1$. The scaling of proof sizes, empirically measured as $\Theta(n^5)$, is limited by the $O(n^2)$ growth of the BDD representations for the ordering constraints, as was illustrated in Figure 1C.

The plot labeled "Sinz, PGPBS, Equations, Random Order" demonstrates the effect of adding constraints to enforce exactly-one constraints on both the pigeons and the holes. The solver applies modulus auto-detection to give a modulus of $r = 2$. Modulo-2 reasoning enables the solver to match the performance of column scanning, with the further advantages of being fully automated and being insensitive to the variable ordering. However, it requires additional constraints in the input file.

Finally, the plot labeled "Direct, Cook's Proof" shows the complexity of Cook's extended-resolution proof of the pigeonhole problem [19], encoded in DRAT format. Although it is very concise for small values of n, its scaling as $\Theta(n^4)$ lies between the $\Theta(n^3)$ achieved by column scanning and equation solving, and the $\Theta(n^5)$ achieved by constraint solving. Of these, only Cook's proof and the solution by constraint solving are directly comparable, in that only these use a direct encoding and have only the minimum set of AMO and ALO constraints.

In summary, pseudo-Boolean reasoning makes this problem tractable with full automation, and it has minimal sensitivity to the variable ordering. Generating proofs by solving systems of ordering constraints is more challenging than by solving automatically detected modular equations, but both achieve polynomial scaling.

4.5 Other Cardinality Constraint Problems

Codel et al. [18] defined a general class of problems that includes the mutilated chessboard and the pigeonhole problems as special cases. Given a bipartite graph with vertices L and R such that $|L| < |R|$, the problem is to find a perfect matching, i.e., a subset of the edges such that each vertex has exactly one incident edge. For the mutilated chessboard, L and R correspond to the white and black squares, respectively, with edges based on chessboard adjacencies. For pigeonhole, L corresponds to the holes and R to the pigeons, and the graph is the complete bipartite graph $K_{n,n+1}$. No instance of this matching problem has a solution, since the sets of nodes are of unequal size.

Twelve instances of this problem were included in the 2021 SAT competition, based on randomly generated graphs with $n = |L|$ ranging from 15 to 20 and with $|R| = n + 1$. Different methods were used to encode the AMO constraints, and some included clauses to convert both sets of constraints into exactly-one constraints. In the competition, all of the solvers could easily handle the benchmarks with $n = 15$, most could handle $n = 16$, with typical runtimes of around 1000 seconds, but none could solve any of the larger problems. PGPBS can easily handle all of the benchmarks, requiring at most 13 seconds and generating proofs with less than 500,000 clauses.

5 Conclusions

Incorporating pseudo-Boolean reasoning into a SAT solver enables it to handle classes of problems encoded in CNF that are intractable for CDCL solvers. By having the PB solver generate BDD representations of its intermediate results, a BDD-based, proof-generating SAT solver can generate clausal proofs of unsatisfiability on behalf of the PB solver in the standard, DRAT proof framework. Compared to the SAT solver operating on its own, including a PB solver enables greater automation with less sensitivity to problem definition, encoding method, and variable ordering.

We have shown that applying pseudo-Boolean reasoning to unsatisfiable instances of parity and cardinality constraint problems can yield proofs that scale polynomially. Solving systems of equations over the integers modulo 2 yields 0-1 valued solutions, and so parity reasoning can also be used on satisfiable problems [6, 30, 37, 47]. On the other hand, Gaussian elimination over integers or with modulus $r > 2$, as well as Fourier-Motzkin elimination, are not guaranteed to find 0-1 valued solutions. When seeking solutions with cardinality reasoning, it seems more effective to use methods that adapt CDCL-based search to pseudo-Boolean constraints [14].

The method described here can be generalized to incorporate other reasoning methods into a proof-generating SAT solver. As long as intermediate results can be expressed as BDDs, a proof can be generated that the result of each step logically follows from the preceding steps. Thus, we could incorporate other pseudo-Boolean reasoning methods, such as cutting planes [28, 32], or we could add totally different reasoning methods.

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