Clausal Proofs of Mutilated Chessboards

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Certificates

What makes a problem hard?

Certificate angle: can one efficiently check an alleged solution?

Consider chess: does white begin and win?

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Consider the sudoku on the right:
Is searching for the solution harder than verifying a given solution?

	4		3					
						7	9	
			6					
			1	4		5		
9							1	
2								6
				7	2			
	5					8		
				9				

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Is searching for the solution harder than verifying a given solution?

Intuition: yes!

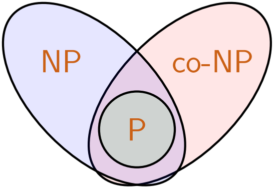
However, many problems for which we can efficiently check a solution turn out to be easy in practice.

1	4	*7*	3	*8*	*9*	*2*	*6*	*5*
5	*8*	*6*	*2*	*1*	*4*	7	9	*3*
3	*9*	*2*	6	*5*	*7*	*1*	*8*	*4*
8	*7*	*3*	1	4	*6*	5	*2*	*9*
9	*6*	*4*	*7*	*2*	*5*	*3*	1	*8*
2	*1*	*5*	*9*	*3*	*8*	*4*	*7*	6
6	*3*	*8*	*5*	7	2	*9*	*4*	*1*
7	5	*9*	*4*	*6*	*1*	8	*3*	*2*
4	*2*	*1*	*8*	9	*3*	*6*	*5*	*7*

Certificates and Complexity

Complexity classes of decision problems:

P : efficiently computable answers.
NP : efficiently checkable yes-answers.
co-NP : efficiently checkable no-answers.



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Solving the $P \stackrel{?}{=} NP$ question is worth $1,000,000 [Clay MI '00].

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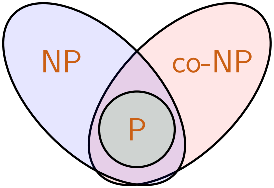

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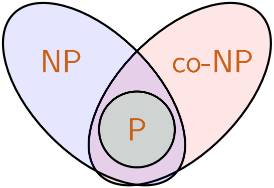

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What about co-NP?

How to find short proofs for interesting problems efficiently?

Original Motivation for Producing and Validating Proofs

Automated reasoning tools may give incorrect answers.

- Documented bugs in SAT, SMT, and QSAT solvers;
- Implementation errors often imply conceptual errors;
- Proofs now mandatory in some competitive events;
  [Balyo, Heule, and Järvisalo '17]
- Mathematical results require a stronger justification than a simple yes/no by a tool. Answers must be verifiable.

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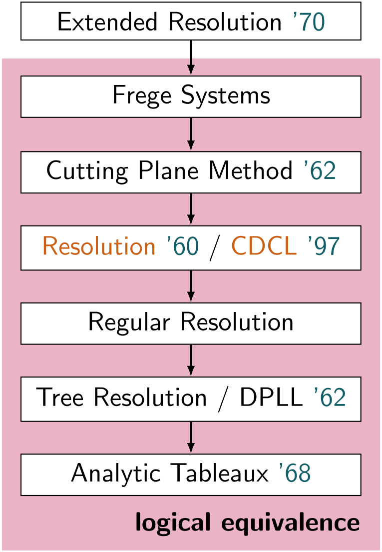
Requires a proof system to compactly express all techniques.

Proof Search in Strong Proof Systems

Existence of Short Proofs



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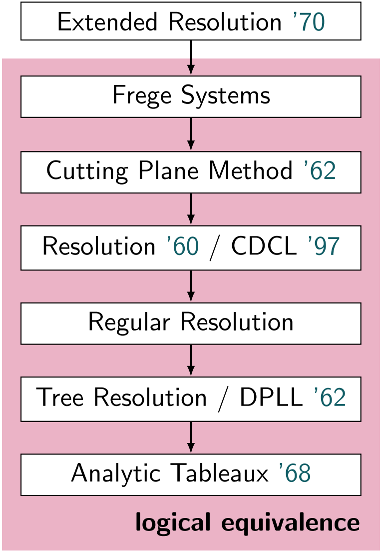
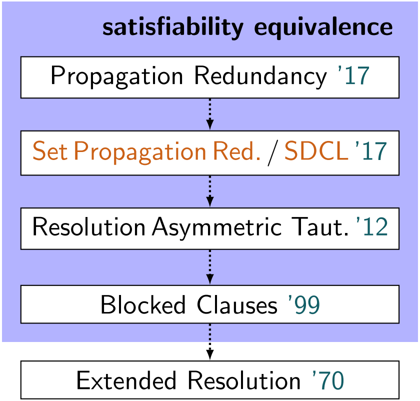


Finding Short Proofs



Express solving techniques compactly
[Järvisalo, Heule, and Biere '12]

Short proofs without new variables
[Heule, Kiesl, and Biere '17A]

Satisfaction-Driven Clause Learning [Heule, Kiesl, Biere '17B]

SDCL generalizes CDCL and finds proofs in the SPR proof system.

CDCL in a nutshell:

1. Main loop combines efficient problem simplification with cheap, but effective decision heuristics; ($>$ 90% of time)
2. Reasoning kicks in if the current state is conflicting;
3. The current state is analyzed and turned into a constraint;
4. The constraint is added to the problem, the heuristics are updated, and the algorithm (partially) restarts.

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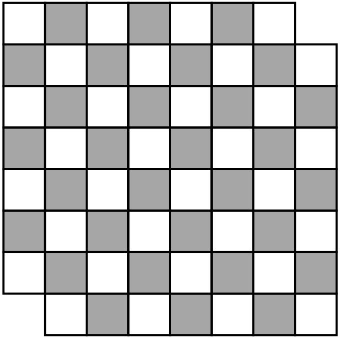
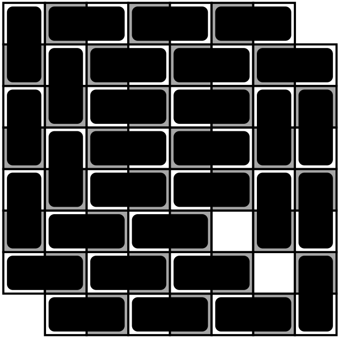
Short proofs for problems that are hard for resolution

including pigeonhole, Tseitin, and mutilated chessboard problems

Mutilated Chessboards: "A Tough Nut to Crack" [McCarthy]

Can a chessboard be fully covered with dominos after removing two diagonally opposite corner squares?





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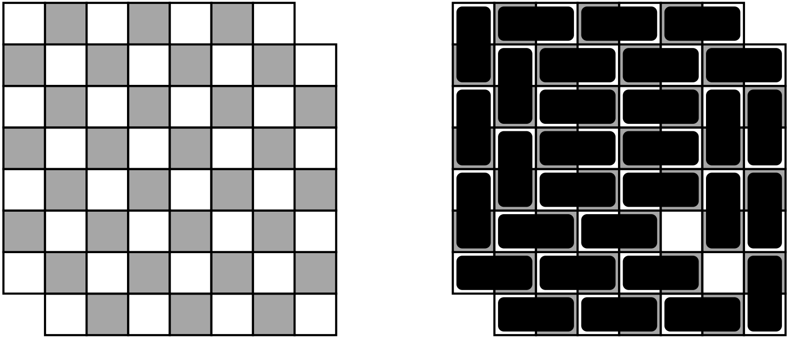
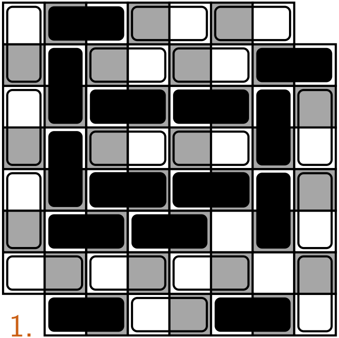


Easy to refute based on the following two observations:

- There are more white squares than black squares; and
- A domino covers exactly one white and one black square.

Without Loss of Satisfaction

One of the crucial techniques in SAT solvers is to generalize a conflicting state and use it to constrain the problem.



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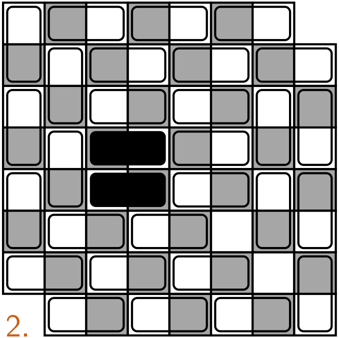


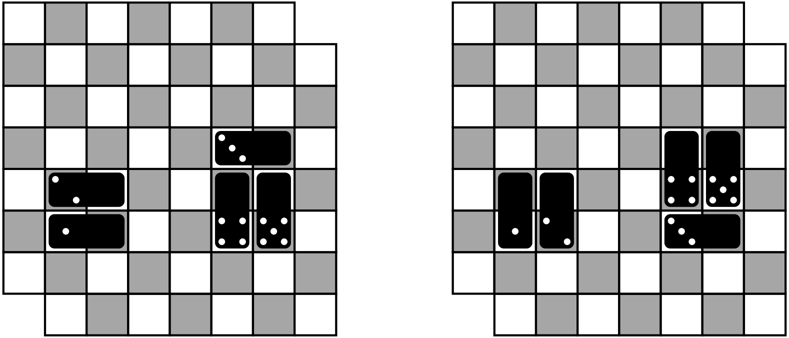
2.

The used proof system can have a big impact on the size:

1. Resolution can only reduce the 30 dominos to 14 (left); and
2. "Without loss of satisfaction" can reduce them to 2 (right).

Mutilated Chessboards: An alternative proof

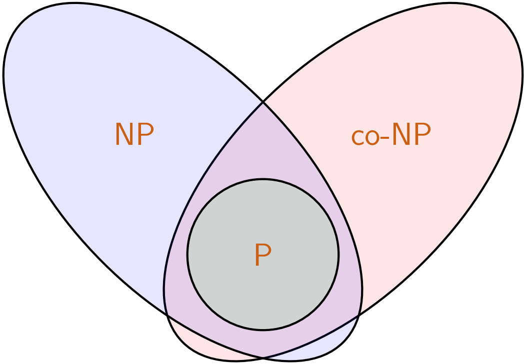
Satisfaction-Driven Clause Learning (SDCL) is a new solving paradigm that finds proofs in the PR proof system [HVC'17]



SDCL can detect that the above two patterns can be blocked

- This reduces the number of explored states exponentially
- We produced SPR proofs that are linear in the formula size

Complexity Classes Closer in Practice



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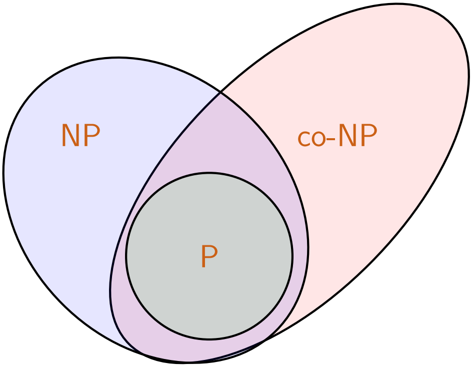

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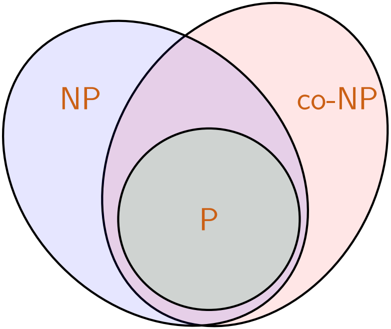

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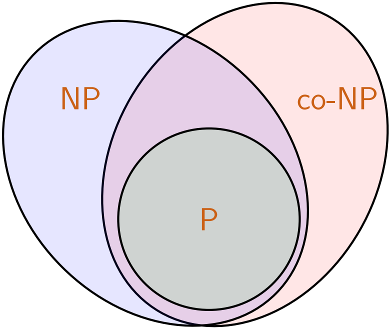

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short proofs exist for many co-NP-complete problem instances.

"co-NP is the new NP!"

Future Work: Arbitrarily Complex Solvers

Verifying efficient automated reasoning tools is a daunting task:

- Tools are constantly modified and improved; and
- Even top-pier and "experimentally correct" solvers turned out to be buggy. [Järvisalo, Heule, Biere '12]

Verified checkers of certificates in strong proof systems:

- Don't worry about correctness or completeness of tools;
- Facilitates making tools more complex and efficient; while
- Full confidence in results. [Heule, Hunt, Kaufmann, Wetzler '17]

ACL2

Isabelle

Formally verified checkers now also used in industry