Encoding Redundancy for Satisfaction-Driven Clause Learning

Marijn J.H. Heule

Benjamin Kiesl

Armin Biere

THE UNIVERSITY OF TEXAS AT AUSTIN

CISPA HELMHOLTZ CENTER FOR INFORMATION SECURITY

JOHANNES KEPLER UNIVERSITÄT LINZ

The Problem

Although SAT solvers can often handle gigantic formulas, they sometimes fail miserably on seemingly easy problems.

Outline

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SAT Solving in Practice: Gigantic Search Trees

- SAT: Given a propositional formula, is it satisfiable?
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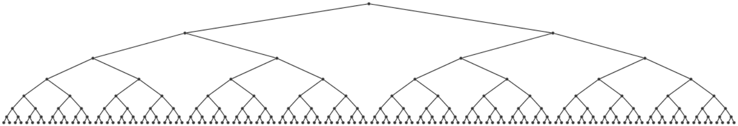

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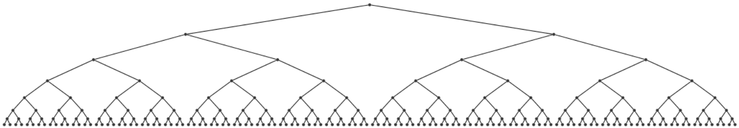

- Modern solvers often deal with millions of variables and clauses.

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- My little nephew could figure this out.
- What if we encode it into SAT and pass it to a solver?

SAT Solver: 21 Pigeons Into 20 Holes?

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"Arguably the single most studied combinatorial principle in all of proof complexity." [Nordström, SIGLOG News '15]

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➥ They need exponential time to solve these formulas.

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- What is needed to jump over this gap?
  1. a proof system that is stronger than resolution yet still mechanizable: PR proof system [Heule, K, Biere; CADE '17].
  2. a SAT solving paradigm harnessing the strength of PR: satisfaction-driven clause learning [Heule, K, Seidl, Biere; HVC '17].

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➥ Addition of redundant clauses can prune the search tree.

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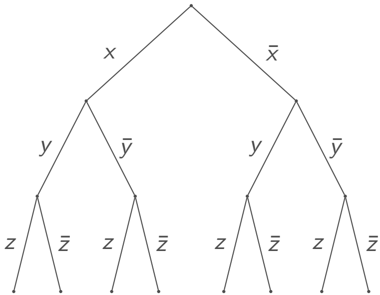
- Clause addition prunes the search tree of satisfying assignments.



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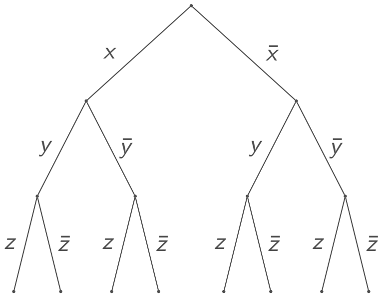

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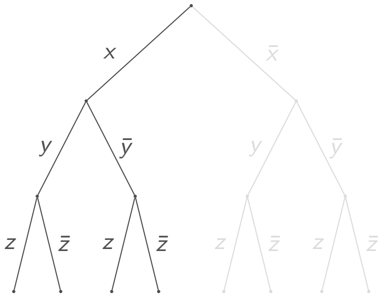

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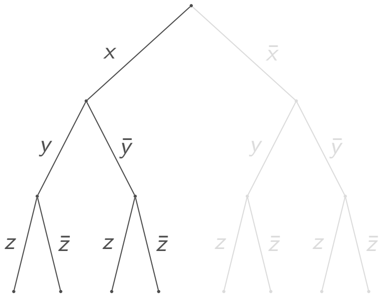

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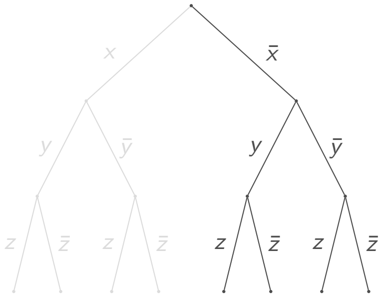

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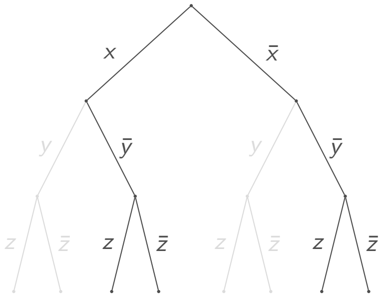

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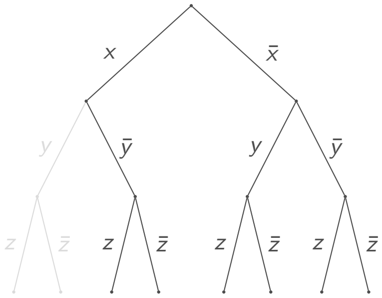

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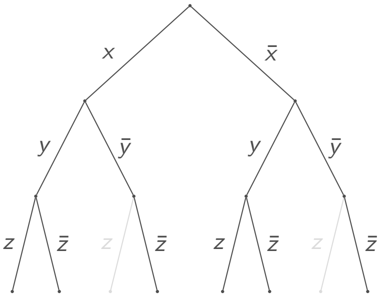

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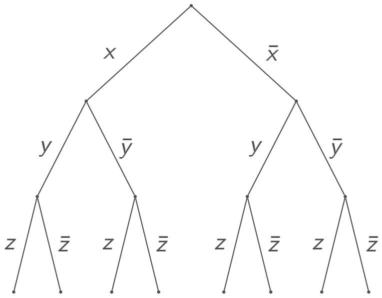

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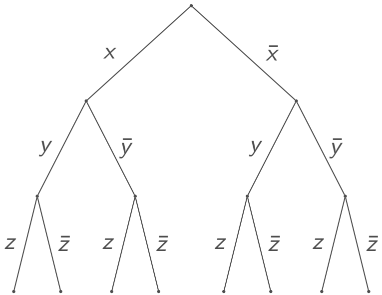

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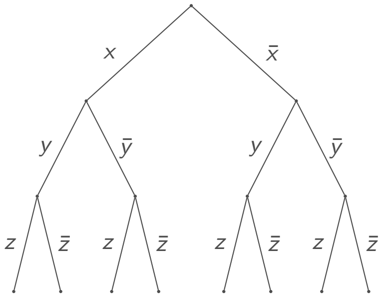

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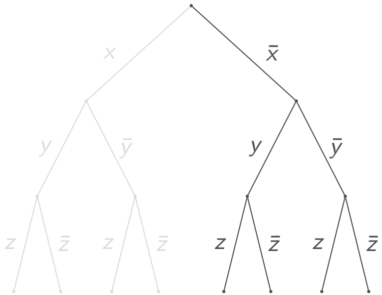

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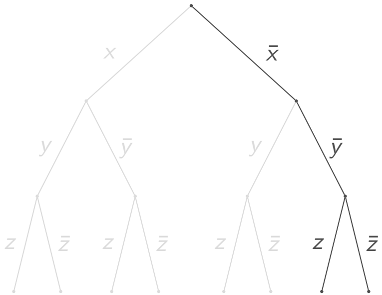

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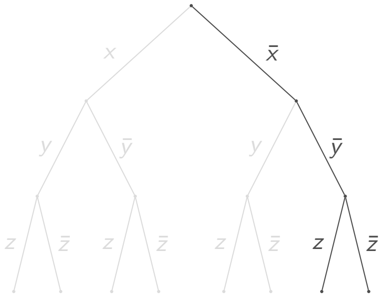

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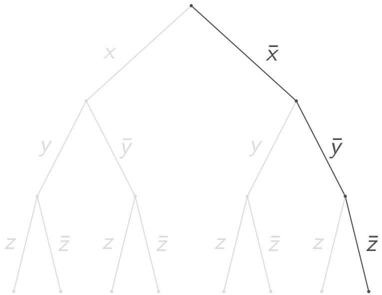

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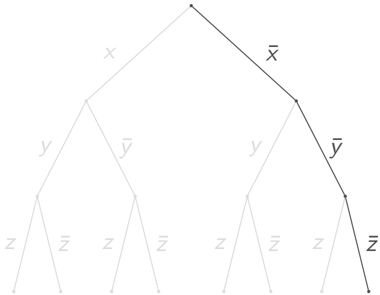

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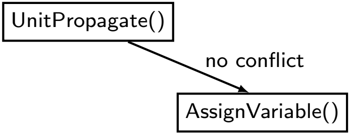

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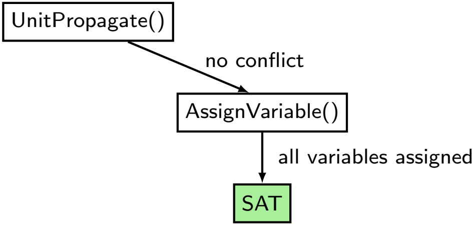

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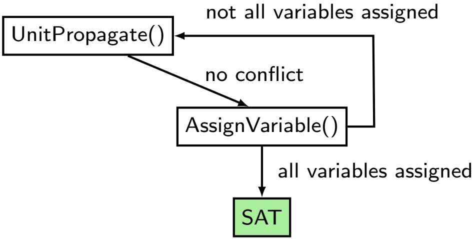

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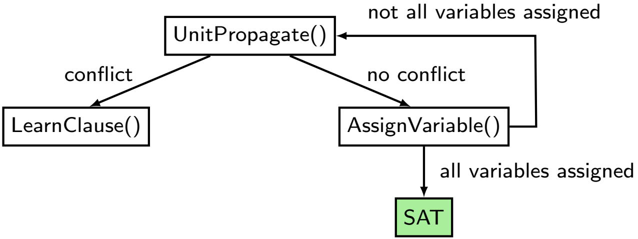

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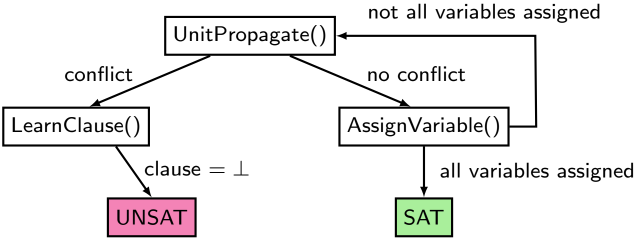

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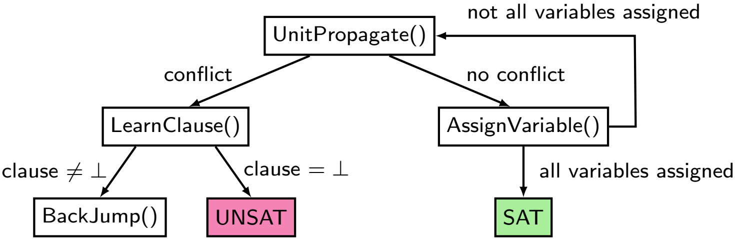

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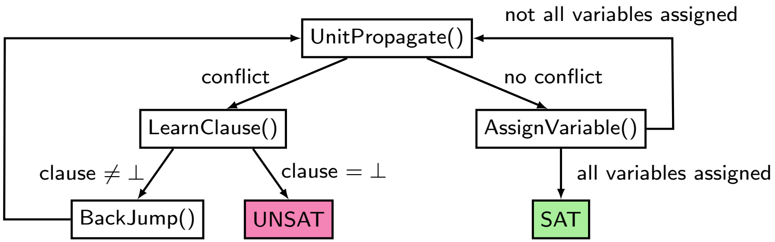

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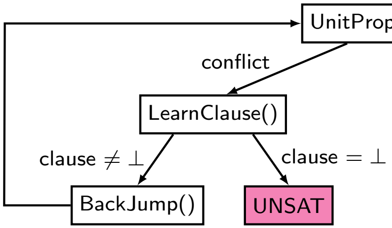

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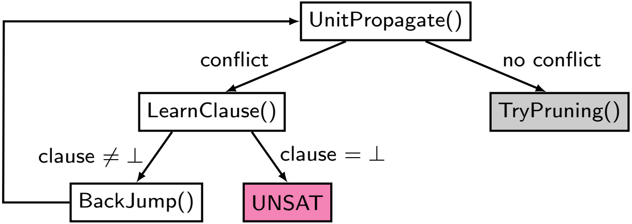

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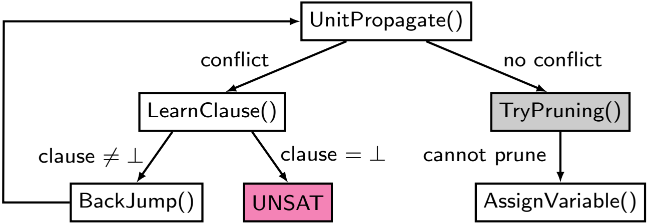

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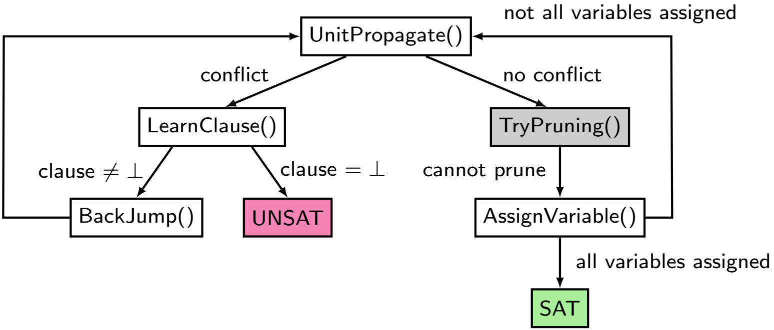

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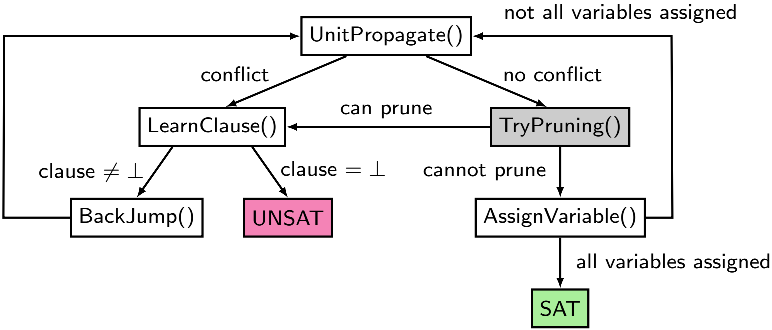

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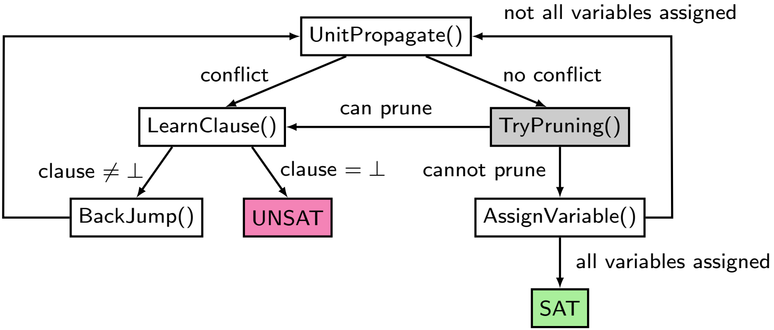

- Learned clauses are not necessarily implied (PR clauses).

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  - Originally ("positive reduct") [Heule, K, Seidl, Biere; HVC '17]: Take all satisfied clauses and remove unassigned literals, then add the clause that is blocked by the current assignment.

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➥ Wanted: Better encodings for pruning!

Background

Contribution

Encodings for Stronger Pruning: Some Preliminaries

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- $\text{touched}_\alpha(C)$ denotes the subclause of C that is assigned by α.
- Notion of implication via unit propagation:
  - Clauses: $F \vdash_1 C$ iff unit propagation derives a conflict on $F \wedge \bar{C}$.
  - Formulas: $F \vdash_1 G$ iff $F \vdash_1 C$ for all $C \in G$.

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Definition

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If the filtered positive reduct $\mathsf{f}_\alpha(F)$ *is satisfiable, then* F *and* $F \wedge \bar{\alpha}$ *are equisatisfiable.*

➥ Works very well in practice (see later)!

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  - positive reduct $\mathrm{p}_\alpha(F) = (x) \wedge (\bar{x}) \Rightarrow$ unsatisfiable $\Rightarrow$ can't prune α

Even Stronger Pruning: PR Reduct

- PR reduct (don't try to understand this):

Definition

Let F be a formula and α an assignment. Then, the PR reduct $\mathrm{pr}_\alpha(F)$ of F and α is the formula $G \wedge C$ where C is the clause that blocks α and G is the union of the following sets of clauses where all the s_i are new variables:

$$\{\bar{x}^p \vee \bar{x}^n \mid x \in var(F) \setminus var(\alpha)\},$$

$$\{\bar{s}_i \vee \mathrm{touched}_\alpha(D_i) \vee \mathrm{untouched}_\alpha(D_i)^p \mid D_i \in F\},$$

$$\{\overline{L^n} \vee s_i \mid D_i \in F \text{ and } L \subseteq \mathrm{untouched}_\alpha(D_i)$$
$$\text{such that } F\,|_\alpha \nvdash_1 \mathrm{untouched}_\alpha(D_i) \setminus L\}.$$

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Even Stronger Pruning: PR Reduct (continued)

- Allows for even stronger pruning than the filtered positive reduct.
- Precisely characterizes propagation redundancy.
  - ➥ Extremely general redundancy notion (NP-hard).
- Has other nice theoretical properties.
- Doesn't work well in practice
  - ➥ Constructing and solving take too long.

Evaluation: SDCL in Practice

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- Three of the most popular formula families hard for resolution.
- Proofs validated by formally verified proof checkers.
- Robust w.r.t. scrambling for Tseitin formulas and mutilated chessboards.

Experimental Data: Pigeonhole Principle

Formula	MLBT	Plain	Pos. Red.	F. Red
hole20	> 3600	> 3600	0.26	0.49
hole30	> 3600	> 3600	1.96	4.03
hole40	> 3600	> 3600	9.02	19.54
hole50	> 3600	> 3600	28.63	65.90

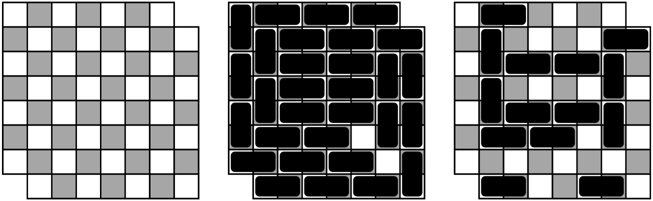
- MLBT – MapleLCMDistChronoBT (winner SAT Competition 2018)
- Plain – SaDiCaL in CDCL mode

Experimental Data: Tseitin Formulas

Formula	MLBT	Plain	Pos. Red.	F. Red
Urquhart-s3-b1	2.95	16.31	> 3600	0.02
Urquhart-s3-b2	1.36	2.82	> 3600	0.03
Urquhart-s3-b3	2.28	2.08	> 3600	0.03
Urquhart-s3-b4	10.74	7.65	> 3600	0.03
Urquhart-s4-b1	86.11	> 3600	> 3600	0.32
Urquhart-s4-b2	154.35	183.77	> 3600	0.11
Urquhart-s4-b3	258.46	129.27	> 3600	0.16
Urquhart-s4-b4	> 3600	> 3600	> 3600	0.14
Urquhart-s5-b1	> 3600	> 3600	> 3600	1.27
Urquhart-s5-b2	> 3600	> 3600	> 3600	0.58
Urquhart-s5-b3	> 3600	> 3600	> 3600	1.67
Urquhart-s5-b4	> 3600	> 3600	> 3600	2.91

Experimental Data: Mutilated Chessboards

Formula	`MLBT`	Plain	Pos. Red.	F. Red
mchess_15	51.53	2480.67	> 3600	13.14
mchess_16	380.45	2115.75	> 3600	15.52
mchess_17	2418.35	> 3600	> 3600	25.54
mchess_18	> 3600	> 3600	> 3600	43.88



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- Next step: SDCL for hard problems from cryptanalysis?