The Resolution of Keller's Conjecture

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Overview

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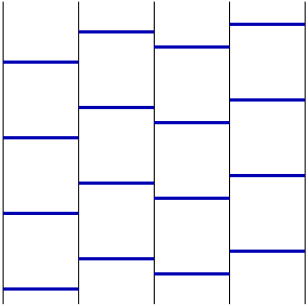
Introduction

Consider tiling a floor with square tiles, all of the same size. Is it the case that any gap-free tiling results in at least two fully connected tiles, i.e., tiles that have an entire edge in common?



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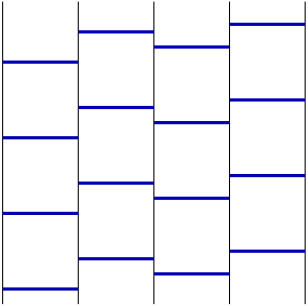
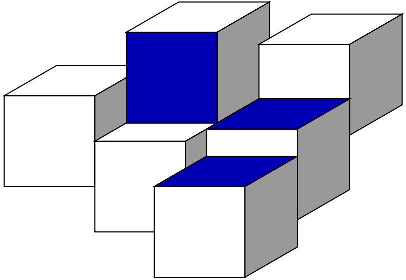
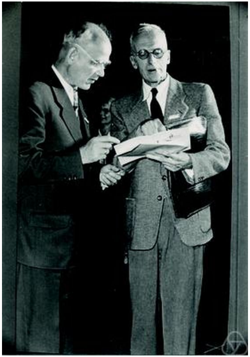




Keller's Conjecture

In 1930, Ott-Heinrich Keller conjectured that this phenomenon holds in every dimension.

Keller's Conjecture.
For all $n \geq 1$, every tiling of the n-dimensional space with unit cubes has two which fully share a face.



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Dimensions Resolved

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What about dimension 7?

Main Result

Theorem (Brakensiek, Heule, Mackey, and Narváez, 2020).
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- Proof involves resolving a maximum clique question about Keller graphs using SAT solving.
- The SAT formula is very difficult to solve, required extensive symmetry breaking.
- Total proof size is over 200 gigabytes! Verified by a proof checker.

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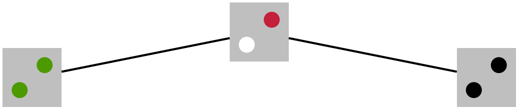
- Two vertices are adjacent if and only if 1) at least one corresponding dimension/dot has a complementary pair of colors; and 2) they differ in at least two dimensions/dots.



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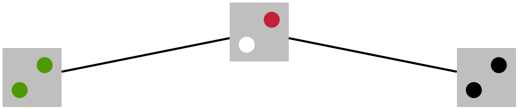
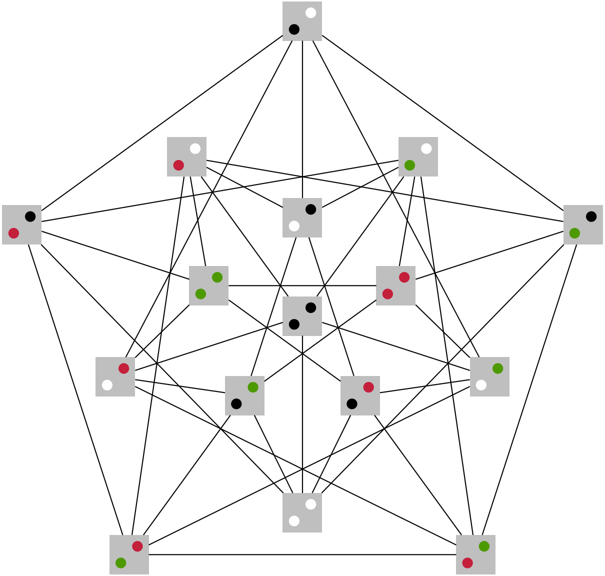


- Corrádi and Szabó's work (1990) showed that there is a counterexample to Keller's conjecture in some dimension n if one can show $G_{n,s}$ has a clique of size 2^n.

From Keller's Conjecture to Graph Theory: $G_{2,2}$



Toward Resolving Dimension 7

- In 2011, Debroni, Eblen, Langston, Myrvold, Shor and Weerapurage showed that the largest clique in $G_{7,2}$ has size 124.
- To confirm Keller's conjecture in dimension 7, one needs to prove that $G_{7,64}$ does not have a clique of size $2^7 = 128$.
- Between 2013 and 2017, Łysakowska and Kisielewicz showed that if one of $G_{7,3}$, $G_{7,4}$ or $G_{7,6}$ has no clique of size 2^7, then Keller's conjecture is true in dimension 7.

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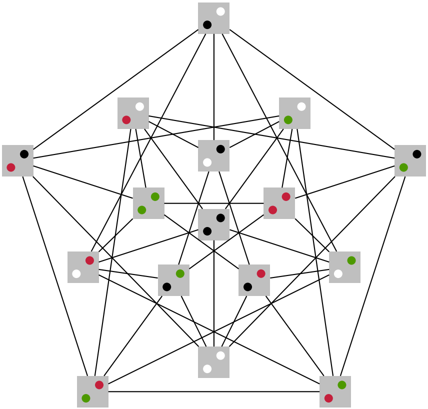
Succinct Encoding: Groups

$G_{n,s}$ can be partitioned into 2^n independent sets (groups)

Key Observation: If there is a clique of size 2^n, each group has exactly one vertex in the clique.



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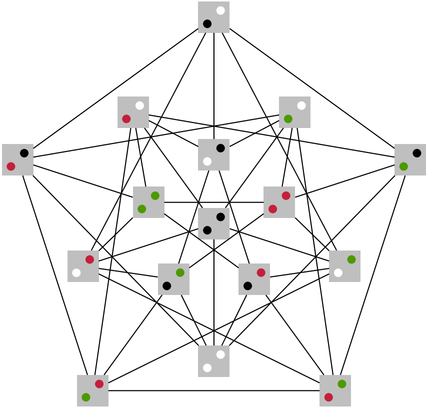

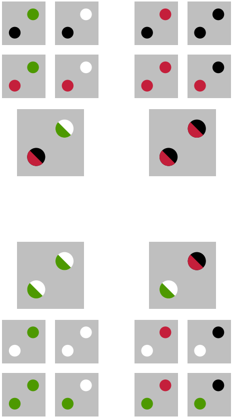

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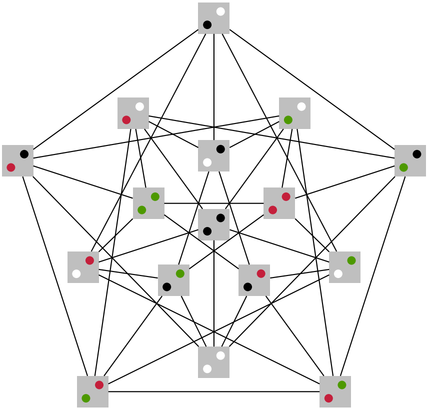

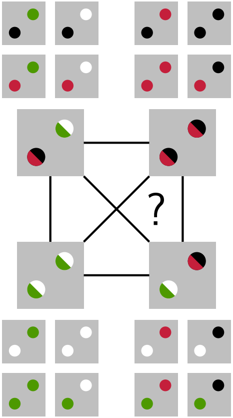

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Using auxiliary variables, these expressions can be encoded as succinct propositional formulas.

Encoding Size

Keller Graph	Cube Count	Variable Count	Clause Count
$G_{7,3}$	279 936	39 424	200 320
$G_{7,4}$	2 097 152	43 008	265 728
$G_{7,6}$	35 831 808	50 176	399 232

the number of clauses is smaller than the number of cubes

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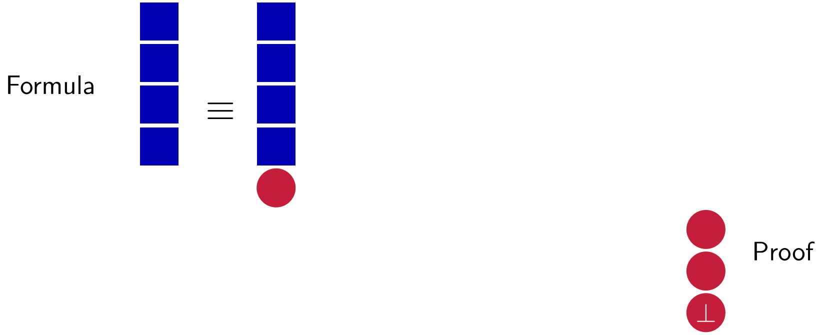

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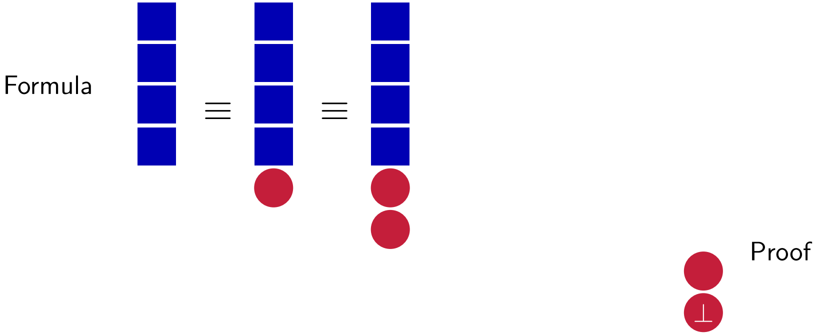

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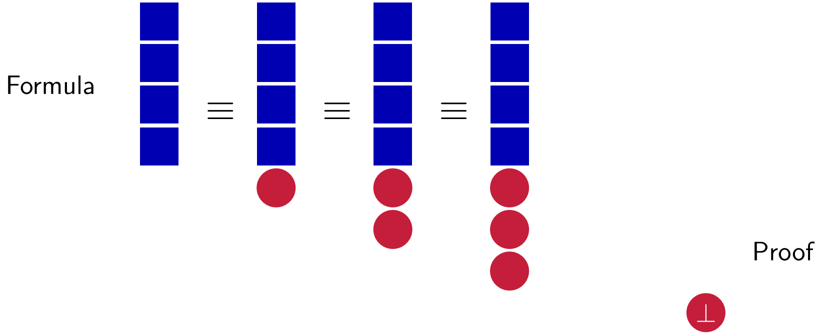

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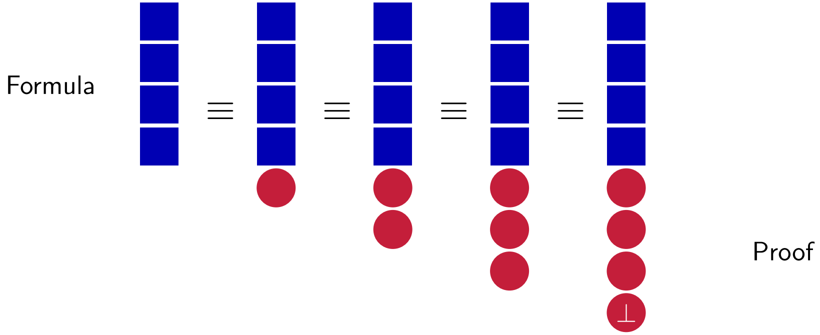

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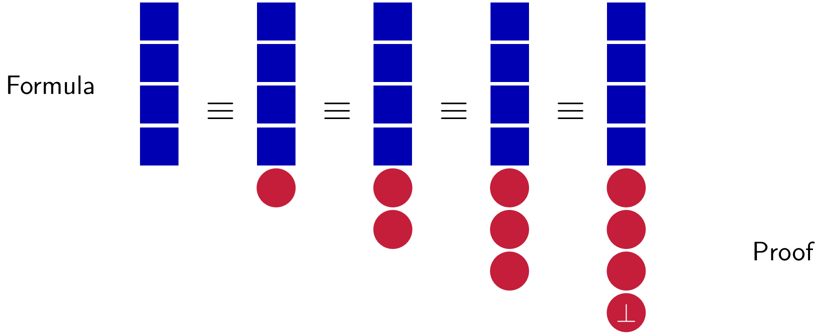


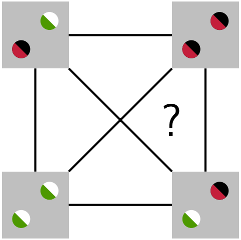
- Checking the redundancy of a clause in polynomial time
- Clausal proofs are easy to emit from modern SAT solvers
- Symmetry breaking can be expressed using clausal proofs

Symmetry Breaking Introduction

Without loss of generality we can assume that

- Both dots of the right top cube are black
- The bottom left dot of the bottom left cube is white

before symmetry breaking



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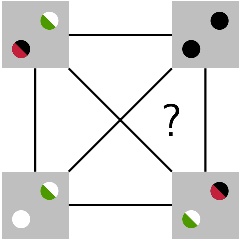

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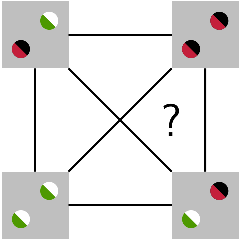

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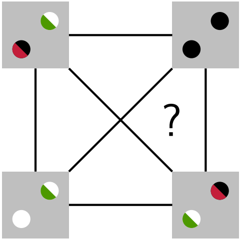
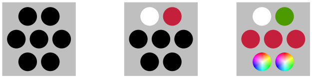


This problem becomes trivial after symmetry breaking

Symmetry Breaking Overview

The symmetry breaking consists of three parts:

1. Manual proof that we can assume the following three cubes:



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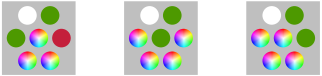

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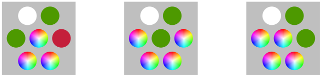
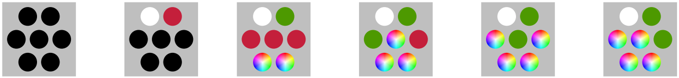


3. Enumerate and filter all options for the rainbow dimensions/dots

Case Split

Given the cubes, in how many ways can we color rainbow dots?



Worst case for r rainbow dots without symmetry breaking is s^r

With filtering using symmetry breaking these can be reduced to:

- $s = 3$: 21 525 (instead of $3^{13} = 1\,594\,323$)
- $s = 4$: 37 128 (instead of $4^{13} = 67\,108\,864$)
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We express this symmetry breaking in the clausal proof

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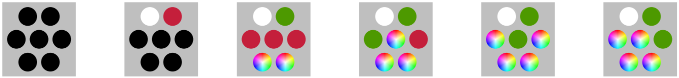

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One case was very hard and we split it into smaller cases

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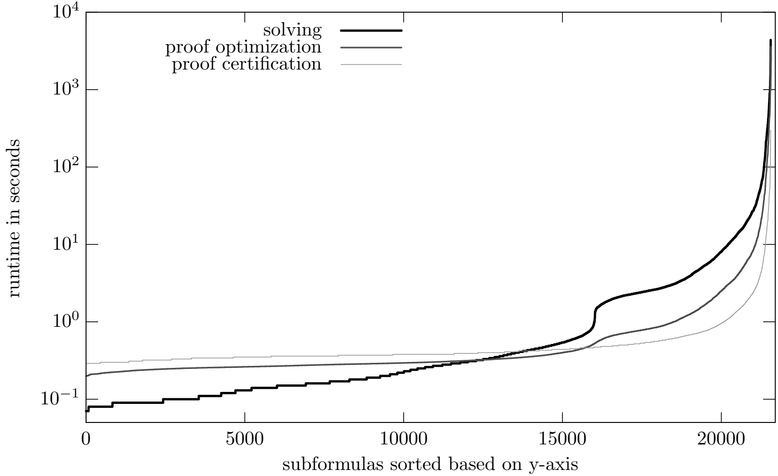
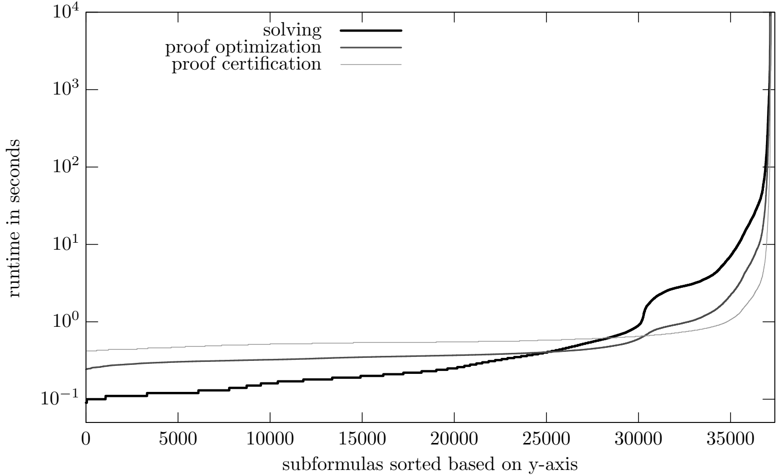
Experimental Setup

- Each case is solved using CaDiCaL
- Parallel execution on Xeon E5-2690 processors with 24 cores
- CaDiCaL emits proofs in the DRAT format
- DRAT proofs are optimized using DRAT-trim
- The formally-verified ACL2check certifies the optimized proofs

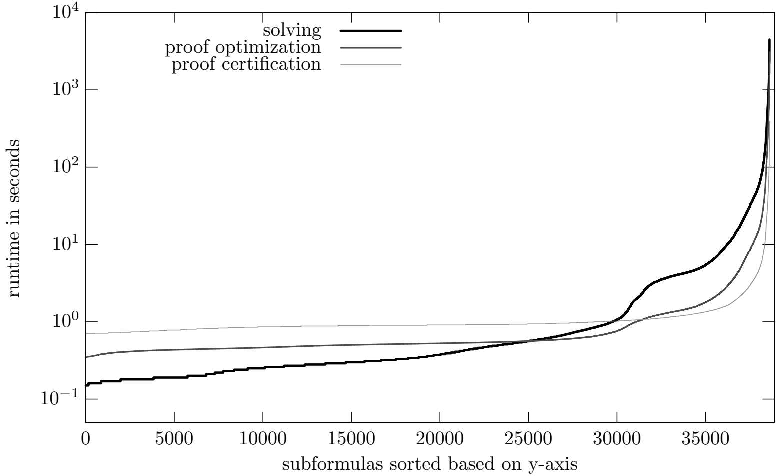
Results on $G_{7,3}$



Results on $G_{7,4}$



Results on $G_{7,6}$



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Conclusions

We resolved the remaining case of Keller's conjecture

- No clique of size 128 in $G_{7,3}$, $G_{7,4}$, and $G_{7,6}$
- Designed a SAT compact encoding
- Combined parallel SAT solver and symmetry breaking
- Constructed a clausal proof of unsatisfiability
- Certified the proof with a formally-verified checker

Future Work

Toward a full formal proof of Keller's conjecture:

- Formalize Keller's conjecture
- Prove the relation between Keller graphs and the conjecture
- Prove the correctness of the encoding

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Open questions:

- What is the largest clique in $G_{7,3}$, $G_{7,4}$, $G_{7,6}$?
- Is the clique of 256 in $G_{8,2}$ unique (modulo symmetries)?
- Why is there a transition between dimensions 7 and 8?

Fin: A Clique of Size 256 in $G_{8,2}$ (Mackey, 2002)