Computing Roadmap

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In this paper we study the pr
the same connected component of
concerned with sets $S \subseteq \mathbb{R}^n$, our al
set $S \subseteq R^n$ can be joined by a semi
algorithm computes a one-dimensio
an embedding of S in a space $\hat{R}^n$ fo
R). $\mathcal{R}(S)$ is called the roadmap of
algorithm described in [Can88a], [C
stratified sets.

We measure the complexity of th
polynomials k, their maximum degr
bits c, and the number of variables n
plexity of our new algorithm is (k^n)
deterministic. Note that the combin
both cases is within a polylog factor
connected components $\Omega(k^n)$.

1 Introduction

Recently, attention has turned to comp
such as counting the number of connecte
was described by Schwartz and Sharir [S
decomposition provides a convenient su
mation. This idea was persued by Koze
deciding adjacency between cells, and r
the cell decomposition [BOKR86] (this
see also [Can91]). Their algorithm was
lel running time. From Kozen and Yap
of semi-algebraic sets are at hand, beca
regular cell complex.

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the form $B(A_1, \ldots, A_k)$ where $B : \{0,1\}^k \to \{0,1\}$ is a boolean function and each A_i is an atomic formula of one of the following types:

$$(F_i = 0) \quad (F_i \neq 0) \quad (F_i > 0) \quad (F_i < 0) \quad (F_i \geq 0) \quad (F_i \leq 0) \tag{1}$$

with $F_i \in \mathbf{Q}[x]$. In the analysis that follows it will be helpful to assume a certain form for the defining predicate.

Definition A formula $B(A_i, \ldots, A_k)$ is said to be in *monotone standard form* if the boolean function B is monotone, and all atomic formulae A_i are either $(F_i = 0)$ or $(F_i > 0)$.

An arbitrary formula can be converted to monotone standard form with a constant factor increase in size. Assume we are given a boolean circuit C to represent the function B. This circuit can be converted to a negation-free, and therefore monotone circuit C_M as follows. For each node $v \in C$ there are two in C_M, one of which represents v and the other $\neg v$. Now all primitive logical operations between nodes in C_M can be implemented with $\wedge$ and $\vee$. e.g. if $C_7 = C_3 \vee \neg C_5$ in the circuit C, in C_M, the node representing C_7 is the $\vee$ of the node representing C_3 and the node representing $\neg C_5$. We also need to compute $\neg C_7$ for later use, and this node is the $\wedge$ of the node representing $\neg C_3$ and the node representing C_5.

The circuit C_M defines a monotone boolean function of the original atomic formulae and their negations. The negations can be pushed into the atomic formulae by replacing $\neg(F_i > 0)$ with $(F_i \leq 0)$ etc. This formula can be converted to standard form by substituting for the inequalities $\leq, \neq, \geq$ with a union of a pair of inequalities using $>$ and $=$. Overall the number of atomic formulae increases by a factor of at most four compared to the original predicate.

Predicate complexity

We measure the complexity of a predicate with four quantities, the number of polynomials k, the number of variables n, the maximum degree of the polynomials d, and the maximum coefficient length c of the coefficients of the polynomials.

There remains one sticky point with regard to the boolean formula B. There is a certain cost associated with evaluating B, given the signs of the F_i's. This time is clearly linear for the first evaluation. The algorithm of [Can88a] requires frequent re-evaluation of B when a single F_i changes sign. Our complexity bounds will be valid if the time to re-evaluate B when a single F_i changes sign is $O(\log k)$. In [Can90] we show that this will be true if the function B is defined by a formula. Basically, we show there that an arbitrary boolean formula can be converted to an equivalent log-depth *trinary* formula (in polynomial time). The predicate B has been assumed to be a formula in most previous work on semi-algebraic sets. Our algorithm will still work if the function B is represented by a general circuit, but if the time to re-evaluate B when a single input changes is greater than $\log k$, then we must substitute this larger time for $\log k$ in the complexity bounds above.

2.1 Stratifications

Definition A *stratification* $\underline{S}$ of a set $S \subset \mathbf{R}^n$ is a partition of S into a finite number of disjoint subsets S_i called *strata* such that each S_i is a manifold.

Lemma 3.1 *There exists a positive r_0 such that for all $r > r_0$, $S \cap D_r$ is a deformation retract of S.*

Proof Let $F = (F_1, \ldots, F_k)$ be a set of polynomials which define S. Let $\underline{\mathbf{R}^n}$ be a Whitney regular stratification of $\mathbf{R}^n$ which is compatible with the sign-invariant sets of F. All sign-invariant sets are unions of strata. Such a stratification is described in [BCR87]. By the semi-algebraic Sard theorem [BCR87], the map ρ has only finitely many critical values when restricted to any of these strata. Let r_0 be the largest critical value, then for all $r > r_0$, $S \cap D_r$ has the same homotopy type (hence number of connected components) as S. To see this, use ρ to lift a vector field on $[r, \infty)$ to one on $\mathbf{R}^n - \mathrm{Int}(D_r)$ which is is compatible with the stratification of $\mathbf{R}^n$. The flow along this vector field defines a retraction of S onto $S \cap D_r$. □

So to find connected components of S it suffices to find components of $S \cap D_r$ for sufficiently large r. In practice this is done by treating r as an indeterminate element of the base field. When it comes time to determine the sign of a base field element, which will be a polynomial in r, we use the sign of the highest degree term in r. This correctly gives the sign for sufficiently large r.

4 Reduction to a regular stratification

As was shown in [Can88a] one can obtain a regular stratification by taking the sign-invariant sets of a system of polynomials in sufficiently general position. In the present case, the given F_i will not be in general position. In [Can88a] a fixed perturbation was applied to their constant coefficients to achieve this. Now consider the following symbolic perturbation of the $F = (F_1, \ldots, F_k) : \mathbf{R}^n \to \mathbf{R}^k$. Define

$$F_{\epsilon a} = F + \epsilon a \tag{2}$$

where $a \in \mathbf{R}^k$ is supposed constant for the time being, and ϵ is a single infinitesimal. As shown in [Can88a] for almost all choices of the constant coefficients ϵa, the map $F_{\epsilon a}$ is transversal to the natural stratification by sign $(\underline{\mathbf{R}})^k$ of $\mathbf{R}^k$, where $\underline{\mathbf{R}} = \{\mathbf{R}^-, \{0\}, \mathbf{R}^+\}$. In particular, if we fix $a = (a_1, \ldots, a_k)$, then for almost all such choices, the map $F_{\epsilon a}$ is transversal to $(\underline{\mathbf{R}})^k$ for almost all ϵ. This implies that the sign-invariant sets of $F_{\epsilon a}$ form a Whitney regular stratification, a very important property for us later. We assume for now that such a_i's have been found. Later we will show how they can be determined either deterministically or probabilistically.

Assume the a_i's were chosen to be positive, the sign-invariant sets of (2) are the same as for

$$\begin{array}{l} F_1/a_1 + \epsilon \\ F_2/a_2 + \epsilon \\ \quad \vdots \quad \vdots \\ F_k/a_k + \epsilon \end{array} \tag{3}$$

Define $G_i = F_i/a_i$. Then another way to describe the sign-invariant stratification of the last paragraph is as the preimage under G of the stratification $(\underline{\mathbf{R}})^k$.

the $L(Z)$ for all strata, we obtain a codimension one set of a values for which G is not ansversal to $\sigma_{\delta\epsilon}$ for almost all δ and ϵ. □

Once we have this regular stratification, we can use a certain subset of the strata to proximate an arbitrary semi-algebraic set. Define $\mathbf{R}_{sep} = (-\infty, -\delta] \cup [-\epsilon, \epsilon] \cup [\delta, \infty)$, en $\mathbf{R}_{sep}$ is a closed set. Intuitively, this is a partition of the real line separating values at are nearly zero from those that are definitely non-zero. We will show later that the nnected components of the preimage $G^{-1}((\mathbf{R}_{sep})^k)$ are in one-to-one correspondence with e connected components of sign-invariant sets of F (if $\delta \gg \epsilon > 0$ are both sufficiently all).

For each sign-sequence $\sigma \in (\underline{\mathbf{R}})^k$, there is also "separated" sign-sequence σ_{sep} defined as

$$(\sigma_{sep})_i = \begin{cases} [\delta, \infty) & \text{if } \sigma_i = \mathbf{R}^+ \\ [-\epsilon, \epsilon] & \text{if } \sigma_i = \{0\} \\ (-\infty, -\delta] & \text{if } \sigma_i = \mathbf{R}^- \end{cases} \quad (5)$$

d the σ_{sep} are exactly the connected components of $(\mathbf{R}_{sep})^k$.

The set $\mathbf{R}_{sep}$ has a regular stratification, denoted $\underline{\mathbf{R}_{sep}}$:

$$\underline{\mathbf{R}_{sep}} = \{(-\infty, -\delta), \{-\delta\}, (-\delta, -\epsilon), \{-\epsilon\}, (-\epsilon, \epsilon), \{\epsilon\}, (\epsilon, \delta), \{\delta\}, (\delta, \infty)\} \quad (6)$$

t because we chose a carefully, the preimage $G^{-1}((\underline{\mathbf{R}_{sep}})^k)$ is a Whitney regular strati- ation of $G^{-1}((\mathbf{R}_{sep})^k)$. Since the preimage is also a closed, bounded set, and therefore npact, the algorithm of [Can88a] can be applied directly to compute its connected com- nents. This takes us a long way toward computing the connected components of a given ni-algebraic set, and leaves us only with the task of determining adjacencies between nected components of sign-invariant sets. For now we must show

eorem 4.2 *Let $G = (F_1/a_1, \ldots, F_k/a_k)$ and $\mathbf{R}_{sep}$ be defined as above. Then the con- ted components of sign-invariant sets of F are in one-to-one correspondence with the nected components of $G^{-1}((\mathbf{R}_{sep})^k)$ for almost all $a \in \mathbf{R}^k$ and for all sufficiently small $> \epsilon > 0$*

The result follows from the next two lemmas. There is a natural correspondence between h non-empty sign-invariant set $F_\sigma = F^{-1}(\sigma)$ and the set $G^{-1}(\sigma_{sep})$. The lemmas show t these two sets have the same homotopy type, by showing that they can be retracted o a common subset.

Consider a particular sign-invariant set F_σ of F, and number the F_i's such that $F_1, \ldots, F_m$ all zero on F_σ, and the remaining polynomials are non-zero. Assume also for simplicity all non-zero F_i's are positive on F_σ. We replace each inequality $F_i > 0$ for $i > m$ a new inequality $F_i \geq \delta$. Together with the inequality $\rho \leq r$ (ρ is the radius function ned earlier), this defines a closed set $F_\sigma^-(\delta)$ which is a subset of F_σ for $\delta > 0$. Since it so a subset of the compact ball of radius r, it is compact.

We first show that the homotopy type of $F_\sigma^-(\delta)$ is the same as that of F_σ for small gh δ. In fact we have

Lemma 4.3 *There exists a positive δ_0 such that for all positive $\delta < \delta$ deformation retract of F_σ.*

Proof Consider the set D in $\mathbf{R}^{n+1}$ defined as $D = \{(x_1, \ldots, x_n, \delta) |$ of D as the "graph" of $F_\sigma^-(\delta)$. D is certainly semi-algebraic, and so ha stratification compatible with the signs of the polynomials F_i and $(F_i$ $\pi_\delta : (x, \delta) \mapsto \delta$ has a finite number of critical values when restricted to $\delta_0 > 0$ to be the smallest positive critical value, and let δ be any posit δ_0. Then π_δ is regular on all strata for values in the range $(0, \delta]$. V a vector field on $(0, \delta]$ to a vector field on D which is compatible w D is compact when restricted to $[0, \delta]$, this gives us a deformation $D|_{(0,\delta]} = \pi_\delta^{-1}(0, \delta]$ onto the compact set $D_\delta = \pi_\delta^{-1}(\delta)$.

But if we define $\pi_x : (x, \delta) \mapsto x$, then the projection $\pi_x(G|_{(0,\delta]})$ is is just $F_\sigma^-(\delta)$. Furthermore, composing π_x with the deformation re paragraph gives us a deformation retraction of F_σ onto $F_\sigma^-(\delta)$. □

To guarantee that δ is small enough, we leave δ as an indeterminat field (like r), and when it comes time to evaluate the sign of a base fi a polynomial in r and δ, we first find the term of lowest degree in δ, t with this degree in δ we take the sign of the highest degree term in r. preceding all evaluations with the quantification $\exists r_0\, \forall r > r_0\, \exists \delta_0\, \forall \delta <$

Operationally, this is also equivalent to working in a real field exte is larger than any element of $\mathbf{R}$ and ϵ is smaller than any element of the However, in the algorithms that follow, all the numerical calculations polynomials from $\mathbf{R}[r, \delta]$. We do not need r and δ to be values that only "sufficiently large" or "sufficiently small" real values that the sig are correctly computed. If we work only over the reals, life is much e notions of compactness and connectivity apply.

In the last lemma we defined compact sets $F_\sigma^-(\delta)$ with the useful are compact and are deformation retracts of sign-invariant sets. T *connected component* of a sign-invariant set F_σ contains a single compc we define a set $F_\sigma^+(\delta, \epsilon)$ which is a "neighborhood" of $F_\sigma^-(\delta)$ and c it. To define $F_\sigma^+(\delta, \epsilon)$, we take each inequality in an F_i and replac inequalities:

$$(F_i + \epsilon a_i \geq 0) \wedge (F_i - \epsilon a_i \leq 0) \quad \text{if} \quad \sigma_i = \{0\}$$
$$(F_i - \delta a_i \geq 0) \quad \text{if} \quad \sigma_i = \mathbf{R}^+$$

and notice that $F_\sigma^+(\delta, \epsilon)$ is now compact. By the previous general posi regularly stratified by the signs of the polynomials that define it. N is a subset of $F_\sigma^+(\delta, \epsilon)$.

Lemma 4.4 *Assume δ is chosen to satisfy lemma 4.3. There exists a for all $0 < \epsilon < \epsilon_0$, the set $F_\sigma^-(\delta)$ is a deformation retract of $F_\sigma^+(\delta, \epsilon)$.*

Proof We only sketch this proof since it is almost identical to the Let D in $\mathbf{R}^{n+1}$ defined as $D = \{(x_1, \ldots, x_n, \epsilon) \mid x \in F_\sigma^+(\delta, \epsilon)\}$, treat

For all sufficiently small $\epsilon > 0$, the set D has a Whitney regular stratification $\underline{D}$ into sign-invariant sets of the polynomials that define it. There is some ϵ_0 which is the smallest positive critical value of the projection $\pi : (x, \epsilon) \mapsto \epsilon$ restricted to $\underline{D}$. For $\epsilon < \epsilon_0$, We can use π to lift a vector field on $(0, \epsilon]$ and thereby define the desired retraction. □

Notice that $F_\sigma^+(\delta, \epsilon)$ is exactly the set $G^{-1}(\sigma_{sep})$, the non-singular approximation of $F^{-1}(\sigma)$. Since F_σ and $F_\sigma^+(\delta, \epsilon)$ have a common retract $F_\sigma^-(\delta)$, they have the same homotopy type and hence the same number of connected components. This completes the proof of theorem 4.2.

Finally, we observe that the sets $F_\sigma^+(\delta, \epsilon)$ and $F_{\sigma'}^+(\delta, \epsilon)$ are disjoint for σ and σ' distinct. This follows because if the union of a component of $F_\sigma^+(\delta, \epsilon)$ and a component of $F_{\sigma'}^+(\delta, \epsilon)$ were connected, then its image under G would have to be connected also. But that image must lie in the union of the disjoint sets σ_{sep} and σ'_{sep}, and it must intersect both, which is impossible.

So the connected components of $G^{-1}((\mathbf{R}_{sep})^k)$, which are the union of connected components of $F_\sigma^+(\delta, \epsilon)$'s, correspond exactly to the connected components of sign-invariant sets of F. Since $G^{-1}((\mathbf{R}_{sep})^k)$ is a compact set which is regularly stratified as $G^{-1}((\mathbf{R}_{sep})^k)$, we can apply the algorithm of [Can88a], modified to work over arbitrary real coefficient fields as described in [Can88c].

5 Determining adjacencies between sign components

In the last section, we modeled connected components of sign-invariant sets F_σ with regularly stratified compact "neighborhood" sets $F_\sigma^+(\delta, \epsilon)$. In this section, we show that these neighborhood sets can also be used to determine adjacencies between components. We will need to make use of a "big" neighborhood of one set and a "small" neighborhood of the other.

First we remark that if disjoint sets A and B have a connected union, then either $A \cap \overline{B} \neq \phi$, or $B \cap \overline{A} \neq \phi$. For if this were not true, both A and B would be open in the union $A \cup B$, and therefore both closed in $A \cup B$. So to check whether connected A and B have a connected union, it suffices to check whether either $A \cap \overline{B} \neq \phi$, or $B \cap \overline{A} \neq \phi$.

Let A be a connected component of F_σ, and let $A^+(\delta_1, \epsilon_1)$ be the corresponding connected component of $F_\sigma^+(\delta_1, \epsilon_1)$. Let B be a connected component of a second sign-invariant set $F_{\sigma'}$, and $B^+(\delta_0, \epsilon_0)$ be the corresponding connected component of $F_{\sigma'}^+(\delta_0, \epsilon_0)$. Note the use of two different sets of infinitesimals. The next lemma shows that for suitable choices of the infinitesimals, adjacency of A and B can be checked by testing overlap of $A^+(\delta_1, \epsilon_1)$ and $B^+(\delta_0, \epsilon_0)$:

Lemma 5.1 *With $A^+(\delta_1, \epsilon_1)$ and $B^+(\delta_0, \epsilon_0)$ as defined above, and for all sufficiently small $\delta_1 \gg \epsilon_1 \gg \delta_0 \gg \epsilon_0 > 0$,*

$$(A \cap \overline{B} \neq \phi) \iff A^+(\delta_1, \epsilon_1) \cap B^+(\delta_0, \epsilon_0) \neq \phi$$

Proof First we clarify what we mean by "sufficiently small". Let $\mathring{\forall}\alpha$ mean that there is some positive α' such that the formula that follows the quantifier is true for all positive α less than α'. The lemma states that

$$\mathring{\forall}\delta_1 \, \mathring{\forall}\epsilon_1 \, \mathring{\forall}\delta_0 \, \mathring{\forall}\epsilon_0 \, (\ldots$$

We can simplify this formula by doing $B^-(\delta_0)$ be the component of $F_{\sigma'}^-(\delta_0)$ tha

$$\mathring{\forall}\epsilon_0 \, (A^+(\delta_1, \epsilon_1) \cap B^+(\delta_0, \epsilon_0) \neq$$

To see this, notice that the set $(A^+(\delta_1$ restricted to $\epsilon_0 \in [0, \epsilon'_0]$, and so achieve is greater than zero, both formulae are need to show that

$$\mathring{\forall}\delta_0 (A^+(\delta_1, \epsilon_1) \cap B^-(\delta_0) =$$

Let p be a point in $(A^+(\delta_1, \epsilon_1) \cap B)$, an p, excluding those F_i's which are zero at The first formula will be true for all δ_0 le *any* δ_0, then there is a point p in $(A^+(\delta_1$ since $B^-(\delta_0)$ is a subset of B, so the se

$$\mathring{\forall}\delta_1 \mathring{\forall}\epsilon_1 (A^+(\delta_1, \epsilon_1) \cap$$

Suppose first that $A \cap \overline{B} \neq \phi$, and let p p is in the (relative) interior of $A^+(\delta_1, \epsilon_1$ arguments. We know that all neighbor checks this. Specifically, for every ϵ_1 U of p. This follows because all the p (some F_i's may be zero of course, but Since U intersects B, so does $A^+(\delta_1, \epsilon_1$

Conversely, suppose $A \cap \overline{B} = \phi$. W $\delta_1 > 0$. Pick a $\delta_1 > 0$. Now choose positive ϵ_1 values. Then for at least on

Why? Suppose the intersection $A^+(\delta_1, v_i) \cap B$. The sequence (p_i) lie convergent subsequence. But let p be continuity of the polynomials defining converges to $p \in A$. This shows $A \cap \overline{B}$ graph. So the above assumption was fa fact almost all) $v_i > 0$. □

To compute with this quantificatic ements of the ground field. This is e $\mathbf{R}(\delta_1, \epsilon_1, \delta_0, \epsilon_0)$ where δ_1, ϵ_1, δ_0 and ϵ_0 smaller than any positive element of

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Let S be the ori
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and each $(F_i > 0)$ wi
formula to be monot
$G^{-1}(\sigma_{\delta_0\epsilon_0})$.

Similarly, we can
result relating the c
S_{sep1}:

Corollary 5.2 *Ther*
and connected compo

Proof Let $A_1, \ldots$
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We saw in the la
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Now consider the ur
intersects A_i. We co
and A_j if and only if

If $A_i \cup A_j$ is con
instance that $A_i \cap \overline{A}$
$A_i^+(\delta_1, \epsilon_1) \cap A_j^+(\delta_0, \epsilon_0$

Conversely, supp
$A_i^+(\delta_0, \epsilon_0)$ and $A_j^+(\delta_0$
two remaining possit
$A_i^+(\delta_0, \epsilon_0) \cap A_j^+(\delta_1, \epsilon_1$
□

5.1 Transform

To summarize, the fc
general semi-algebrai

3. Each query polynomial contains at most $O(n^2)$ infinitesimals, even if all of $a_1, \ldots, a_k$ are infinitesimal. This seems surprising at first, but one must remember that by using infinitesimal a_i's, we have guaranteed that all the algebraic surfaces defined by the input polynomials meet transversally. In particular, any collection of more than n surfaces will not have a common intersection point. If the query polynomials were only generated by intersection points, there could be at most $O(n)$ infinitesimals in each one. But the roadmap algorithm also generates hyperplanes which are defined by n input surfaces. These also meet the input surfaces transversally (except at one point each), and so an intersection point can actually depend on $O(n^2)$ input surfaces.

Making these observations allows us to determine the running time for the deterministic version of the algorithm, which uses infinitesimal a_i's. The bounds in [Can88a] and [Can88c] show that the roadmap algorithm for a particular input requires $O(k^n \log k) d^{O(n^2)}$ evaluations of query polynomials. The above arguments indicate that each polynomial has degree $d^{O(n^2)}$ in $O(n^2)$ variables, or $d^{O(n^4)}$ coefficients, and they require $d^{O(n^4)}$ time to evaluate (coefficient length bounds are much better than this, and dont affect the complexity bounds). The running time is the product of the number of query polynomial evaluations and the time for each which is $(k^n \log k) d^{O(n^4)}$.

For the randomized algorithm, we need to figure out the number of random bits required in the choice of a_i's. We could try to figure out explicitly the conditions for a particular a to be a good choice, but there is a simpler argument we can use, which takes advantage of the fact that our calculation can be expressed as an algebraic decision tree. A particular a must be a good choice if *all the query polynomials in the decision tree are non-zero* at that a (excepting query polynomials which are identically zero, which can be ignored). This follows because for such an a, there is an open, connected neighborhood $N(a)$ such that all the query polynomials have the same sign over all of $N(a)$ as they do at a. Thus the algorithms output is that same for all these choices. But almost all of the points in $N(a)$ must be good choices, since good points are dense. The algorithm must produce the correct output at these points, hence it produces the correct output at a.

So it suffices to choose a to avoid the zero sets of all the query polynomials. The query polynomials have degree $d^{O(n^2)}$ and there are potentially $(kd)^{O(n^2)}$. The union of the zero sets gives us a bad set which is an algebraic set in the space $\mathbf{R}^k$ of possible a-values which has degree $(kd)^{O(n^2)}$. By Schwartz's lemma, we will have probability p^n of hitting the bad set if we choose the a_i's randomly with $\log(p^{-1}(kd)^{O(n^2)})$ bits. Fixing p, we see that $O(n^2 \log(kd))$ bits suffice for the a_i's. This contributes the extra factor of $\log k$ to the running time of the original roadmap algorithm, and gives a randomized running time of $(k^n \log^2 k) d^{O(n^2)} c^2$.

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