Harmonic Functions and Collision Probabilities

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Abstract

There is a close relationship between harmonic functions – which have recently been proposed for path planning – and hitting probabilities for random processes. The hitting probabilities for random walks can be cast as a Dirichlet problem for harmonic functions, in much the same way as in path planning. This equivalence has implications both for uncertainty in motion planning and in the application of machine learning techniques to some robot problems. In particular, Erdmann's method can directly incorporate such hitting probabilities. In addition, the value functions obtained by reinforcement learning algorithms can be rapidly reconstructed by relaxation or resistive networks, once the extrema for such functions are known.

1 Introduction

Some recent research in robotics has focussed on the use of harmonic potential functions for path planning (Connolly et al. 1990, Akishita et al. 1990, Keymeulen and Decuyper 1991, Tarassenko and Blake 1991, Kim and Khosla 1992, Connolly and Burns 1993, Connolly and Grupen 1993, Singh et al. 1994, Connolly et al. 1995). This approach resembles other potential field approaches, except that the field is usually computed in a "global" manner, that is, over the entire region of interest. In particular, the method as described in (Connolly and Grupen 1993) uses a grid-based relaxation technique. This results in a potential such that, at each point, the potential function is the *average* of the potentials at neighboring points.

[*]Most of this work was performed while the author was at the Department of Computer Science at the University of Massachusetts, under NSF grants IRI-9116297 and IRI-9208920.

[†]International Journal of Robotics Research, 16(4):497-507.

Obstacle and goal points provide the boundary conditions for this potential, and are held fixed at high and low potential values, respectively. Because they are harmonic, such functions have extrema only at designated points (the boundary conditions), and do not suffer from the oft-cited "local-minima problem". These functions can be computed easily either by relaxation (Connolly and Grupen 1993) or resistive grids (Tarassenko and Blake 1991). Implementation details and experimental results for harmonic function path planning are described in (Connolly and Grupen 1993) and (Grupen et al. 1994). Such functions can also be used in conjunction with dynamic effects to plan and control repetitive manipulator movements (Connolly et al. 1995).

This paper discusses the relationship between harmonic function planning, its probabilistic interpretation, and recent work on uncertainty and learning in robotics. The probabilistic interpretation of harmonic functions is described in more detail in (Doyle and Snell 1984, Kemeny et al. 1976). Doyle and Snell's usage of the term "harmonic" will be adopted here: A function will be considered harmonic if it is the equilibrium voltage found in a resistive network. In the following discussion, the equivalence between probabilities and potentials is described in terms of lattices[1], since this is a representation compatible with grid-based approaches. Harmonic potential functions can be directly interpreted as collision probabilities. They can be used to introduce an appropriate drift in randomization, and their properties can be used to guarantee some useful properties for reinforcement learning systems. It may be of some use to consider these potentials as the stable equilibria for diffusion processes. That is, the potentials represent the asymptotic, steady-state values (e.g., concentrations or voltages) for some kind of diffusion (e.g., chemical or electronic).

2 Laplace's Equation

A harmonic function ϕ on a domain $\Omega \subset \mathbf{R}^n$ is a function which satisfies Laplace's equation:

$$\nabla^2 \phi = \sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_i^2} = 0 \tag{1}$$

where $\mathbf{x} \in \mathbf{R}^n$, and each x_i is a configuration space variable.

In the case of robot path planning, the boundary of Ω (denoted by $\partial\Omega$) consists of the boundaries of all obstacles and goals in a cspace[2] representation. Note that in this discussion, goals may be sets and not merely points. Moreover, there may be more than one goal set in the freespace domain. Harmonic functions satisfy the min-max principle: spontaneous creation of local mimima within the solution region is impossible (Weinstock 1974, Zachmanoglou and Thoe 1986). In the continuous case, this can be shown using the Gauss Integral Theorem (Courant and Hilbert 1989). This theorem states that for a harmonic function ϕ, the following integral holds:[3]

$$\int_{\partial\Gamma} \frac{\partial \phi}{\partial \mathbf{n}} dS = 0 \tag{2}$$

[1] It should be emphasized that the equivalence extends to the continuous case as well.

[2] Configuration space.

[3] This equation can be derived by applying Green's Formula to a function ϕ which is harmonic.

which is an integral of normal derivatives over the boundary ($\partial\Gamma$) of an arbitrary region $\Gamma \subset \Omega$. This theorem states that there is a balance between inward and outward flow on the boundary of any volume within the solution region (not including obstacles or goals). It expresses the conservation of mass in an incompressible fluid flow. Figure 1 illustrates an example of the application of the theorem. The shaded region is Γ, and along its boundary, the derivatives of the harmonic function (shown by the streamlines) taken along the normals to $\partial\Gamma$ (bold arrows) cancel each other out in the integral in Equation 2. Thus, there is always a "way out" from any point or region. This principle holds for any number of dimensions (Axler et al. 1991). Thus, gradient descent on a harmonic function cannot encounter the "local minimum problem" often cited in the robot path planning literature (Khatib 1985, Koditschek 1987, Arkin 1987, Khosla and Volpe 1988, Barraquand and Latombe 1991, Barraquand et al. 1992).

The discrete analogue to the Gauss Integral Theorem is Kirchhoff's Current Law, i.e., the sum of current flowing into and out of a node in a resistive network must be zero. This is equivalent to the mean value property, which is derived in the next section.

In the following discussion, all problems are assumed to be defined on a lattice (grid) of points in $\mathbf{Z}^n$ – this lattice corresponds to the configuration space of the robot (see Connolly and Grupen 1993 for a more detailed treatment). Functions will be defined on this lattice which have specific values at each lattice point. For both Laplace's equation and the theory of Markov processes, the following discussion also extends to the continuous case (see Kemeny et al. 1976).

2.1 Potentials on a lattice

Denoting the harmonic function by ϕ, and using h as a small increment, the central difference approximations to each second directional derivative can be computed using Taylor series expansions in the appropriate variable:

$$\begin{aligned}
\phi(x+h) &= \phi(x) + h\phi_x(x) + \frac{h^2}{2}\phi_{xx}(x) + \\
&\quad \frac{h^3}{6}\phi_{xxx}(x) + \frac{h^4}{24}\phi_{xxxx}(\xi_+) \\
\phi(x-h) &= \phi(x) - h\phi_x(x) + \frac{h^2}{2}\phi_{xx}(x) - \\
&\quad \frac{h^3}{6}\phi_{xxx}(x) + \frac{h^4}{24}\phi_{xxxx}(\xi_-)
\end{aligned}$$

$$\begin{aligned}
\phi(x+h) - 2\phi(x) + \phi(x-h) = \frac{h^2}{2}\phi_{xx}(x) + \\
\frac{h^4}{24}\phi_{xxxx}(\xi)
\end{aligned}$$

Since $h = 1$ on the lattice, we have in two dimensions:

$$\phi(x+1,y) + \phi(x-1,y) + \phi(x,y+1) + \phi(x,y-1) - 4\phi(x) \approx \nabla^2\phi = 0 \tag{3}$$

Therefore, up to truncation error, the value of ϕ at a point in the lattice is the average of the values at the points in the (manhattan) neighborhood. This is known as the mean-value property, and applies in both the discrete and continuous cases (Axler et al. 1991). Although the above derivation pertains to two dimensions, the mean-value property holds for any number of dimensions (Axler et al. 1991). To compute harmonic functions on a lattice, one solves the system of equations defined by Equation 3 using iterative methods such as successive overrelaxation (SOR) (Burden et al. 1978). In a 4-dof path planner for a P-50 welding robot, using roughly 16,000 nodes, SOR converges in approximately 300ms on a Sparc 10 processor (see Grupen et al. 1994). One can also employ resistive, inductive, or capacitive circuits (Tarassenko and Blake 1991, McCann and Wilts 1949).

For the purposes of this paper, the definition of the term *harmonic* will be loosened somewhat (after Doyle and Snell 1984) to the case where the mean shown in Equation 3 is actually a *weighted mean*. This corresponds to a spatially distorted Laplacian, or to the case of a resistive network whose resistances are not necessarily uniform. Most of the interesting properties of harmonic functions, in particular the min-max and smoothness properties, still apply in this case.[4]

2.2 Potentials in Markov Processes

Consider a Markov chain on the lattice, and let $\mathcal{A}$ denote the absorbing set. Divide $\mathcal{A}$ into two disjoint subsets $\mathcal{O}$ and $\mathcal{G}$. Let $p(x)$ at any state x be the probability that the process, starting from x, will be absorbed in $\mathcal{O}$ before reaching any state in $\mathcal{G}$. Then $p(x)$ is known as a *hitting probability*. Assume for the moment that the probabilities of transition from the current state to any neighbor state are all equal ($\frac{1}{2n}$, where n is the number of dimensions in the lattice). The probability $p(x)$ can then be defined in two dimensions as:

$$\begin{aligned} p(x) = \frac{1}{4}(\quad & p(N|x \to N) + p(E|x \to E) \\ & + p(W|x \to W) + p(S|x \to S)) \end{aligned}$$

where N, E, W, S are the 4 nearest neighboring states and $x \to N$ denotes a transition from x to N. This sum follows from the fact that transitions are exclusive, but the probability of a transition is 1. Since the process is Markov, probabilities do not depend on prior time steps, so that $p(N|x \to N) = p(N)$, and likewise for the other neighbors. Thus, the hitting probability at any lattice point is the average of the hitting probabilities at the (manhattan) neighbors. This is the mean-value property, which can be expressed as:

$$p(x) = \frac{1}{n}\sum_{i=1}^{n} p(x_i) \tag{4}$$

[4]Smoothness holds in the continuous case. In the discrete case, one may enforce a desired degree of smoothness by interpolation, using the harmonic polynomials in $\mathbf{R}^n$ as a basis set.

where the x_i are the n neighbors of state x. This can also be seen by considering the transition matrix P for a Markov chain. One can think of P as a smoothing or averaging operator (Kemeny et al. 1976). For a general Markov chain, where the transition probabilities are not necessarily equal, Equation 4 becomes

$$p(i) = \sum_{j=1}^{m} P_{ij} p(j) \tag{5}$$

where i and j are states in the m-state Markov chain whose transition matrix is P. The sum of transition probabilities out of a state i must be no greater than 1, so that:

$$\sum_{j=1}^{m} P_{ij} \leq 1 \tag{6}$$

but note that when

$$\sum_{j=1}^{m} P_{ij} < 1 \tag{7}$$

an equivalent Markov chain P' can be constructed by adding a new "disappearing" state $m+1$ to the chain (see Kemeny et al. 1976), such that:

$$\sum_{j=1}^{m+1} P'_{ij} = 1 \tag{8}$$

Equation 5, appropriately modified, thus describes a true *weighted* mean. Clearly, the probability function p still obeys the min-max principle, and cannot exhibit any "local" extrema.

The (weighted) mean-value property is unique up to boundary conditions. To see this, consider a harmonic function $\phi(x)$ with respect to a particular Markov chain (hence it is known to satisfy the mean-value property), and suppose that there is another function $p(x)$ which also satisfies the mean-value property with respect to the same chain. Let

$$\epsilon(x) = p(x) - \phi(x) \tag{9}$$

at any state x, and by hypothesis, $\epsilon(x) = 0$ at the boundaries (absorbing states, in the Markov chain). Using the mean-value property for $p(x)$ and $\phi(x)$,

$$\begin{aligned} \epsilon(x) &= \frac{1}{n} \sum_{i=1}^{n} (p(x_i) - \phi(x_i)) \\ &= \frac{1}{n} \sum_{i=1}^{n} \epsilon(x_i) \end{aligned}$$

so that $\epsilon(x)$ also satisfies the mean-value property. Since $\epsilon(x) = 0$ at the boundaries, ϵ must be constant over all x (i.e., $\epsilon(x) = 0$ everywhere). Hence, $p(x) \equiv \phi(x)$. Hitting probabilities are therefore uniquely harmonic up to their boundary conditions.

There is a simple interpretation of this equivalence in the context of harmonic function path planning. Let $\mathcal{O}$ be the set of states which are occupied by obstacles, and let $\mathcal{G}$ be the set of goal states. The Dirichlet form for obstacles is assumed, where obstacle boundaries are fixed at $\phi(x) = 1$, and goal regions are fixed at $\phi(x) = 0$. In this case, $\phi(x)$ is simply the collision probability for a random walk starting at x. Therefore, gradient descent of a harmonic function so constructed will always minimize the probability of collisions with obstacles during a random motion (or perturbation). So if the actual trajectory of the effector has a random component (e.g., control error or external forces), its probability of accidental collision with obstacles is minimized by using a gradient descent on the harmonic function ϕ.

3 Uncertainty in Motion

Several authors have examined robot task execution in the presence of uncertainty in environmental modeling, robot motion, and obstacle motion (e.g., Donald 1990, Griswold and Eem 1990, Oommen et al. 1991, Erdmann 1992, Simmons and Koenig 1995). This discussion centers around Erdmann's work. Erdmann argues for the utility of randomization in robotic tasks as a way of overcoming uncertainty. The approach used by Erdmann relies on a randomization step in cases where uncertainty prevents a reliable measurement of effector position. Bounds can be computed on the expected time-to-completion for the randomization process. The approach can take estimated obstacle positions into account through the use of an arbitrary label function ℓ. An appropriate choice for ℓ is the hitting probability for the process.

Let $p_c(x)$ be the probability that a random walk starting at x collides with an obstacle before reaching the goal. As noted in (Erdmann 1992), a prerequisite to any randomizing step is that $p_c(x) < 1$ (i.e., that there is a nonzero probability of reaching the goal before hitting an obstacle). If $\mathcal{O}$ denotes the obstacle set, and $\mathcal{G}$ the goal set, then these probabilities can be computed directly by setting the following boundary conditions:

$$\begin{aligned} p_c(x) &= 1 \qquad x \in \mathcal{O} \\ p_c(x) &= 0 \qquad x \in \mathcal{G} \end{aligned}$$

and then computing the harmonic function which satisfies these constraints (see section 2). If $p_c(x) = 1$, then the goal is blocked, and some other strategy must be employed.

In (Erdmann 1992), the randomization process is modeled as a stochastic diffusion (Karlin and Taylor 1981), with drift $\bar{\mu}$ and variance σ. The label function ℓ is used to analyze expected velocity of the process. If L is the Brownian motion operator with drift:

$$v(x) = (L\ell)(x) \tag{10}$$

$$= \frac{1}{2}\sigma^2\nabla^2\ell + \bar{\mu}\nabla\ell \tag{11}$$

Letting $\ell(x) = p_c(x)$ (a harmonic function), a drift velocity can be computed which biases the randomization toward the goal. Since $p_c(x)$ is harmonic, the first term of Equation 11

vanishes:

$$v(x) = \bar{\mu} \nabla p_c(x)$$

By allowing μ to be a function of configuration, and then adopting Erdmann's convention that the expected infinitesimal velocity $v(x) = -1$ (with respect to the label function p_c), $\bar{\mu}(x)$ must be a vector in the opposite half-plane with respect to $\nabla p_c(x)$. In other words, the process drift should be in the direction of decreasing collision probabilities. Thus, the same framework which is used for coarse motion planning in (Connolly and Grupen 1993) can also be used for determining the drift in randomization. This allows randomization to take into account the estimated locations of obstacles in the workspace.

3.1 Examples

To illustrate the use of label functions in motion planning under uncertainty, a simple simulation of a 2-dof robot arm has been programmed. The simulator uses the gradient of a label function to command the arm, subject to a random perturbation of the motion of the arm by up to 5 times the commanded motion step. In the examples shown here, a randomization of 1.66 times the step size was used. Figure 2 shows the initial configuration used in the examples. The goal (denoted by the open circle and open square) is a specific configuration. Obstacles are drawn as a mesh, with the outermost layer removed so the paths (a sequence of points) can be easily seen.

Figure 3 shows a path obtained using a euclidean distance function as the label function. This function was computed using a wavefront propagation starting from the goal point (see, e.g. Barraquand et al. 1992). This path took 252 steps to reach the goal. Figure 4 shows the same scenario, but with a harmonic label function. The resulting path took 156 steps to complete. The difference in the number of steps arises in part because gradient descent of a distance function does not minimize hitting probabilities. In fact, in certain cases, distance functions obtained from the l1 or l2 norm will guarantee that many paths pass through certain isolated points. This is illustrated in Figure 5. Notice that any path originating to the right of the wedge must pass through one point at the tip of the wedge. If the effector's motion has a random component here, or if there is error in the estimation of the obstacle's location, then there will be a significant probability that the effector will hit the obstacle. Such collisions occur in Figure 3, resulting in many more steps required to reach the goal configuration. In contrast, the motion produced by the harmonic label function (Figure 4) is guided to regions of minimum hitting probability. In this case, no collisions occur.

4 Learning and Robotics

Robot navigation and planning serves as a convenient application domain for reinforcement learning techniques (Dayan 1991, Bachrach 1992, Thrun and Möller 1992, Mahadevan and Connell 1992, Singh et al. 1994, Moore 1994, Fagg et al. 1994). In this context, robot control is often treated as a Markovian decision problem, solved using a variant of either temporal differencing (TD) (Sutton 1988) or Q-learning (Watkins 1989). In these cases, the problem

is one of finding the correct expected value function for the underlying Markov chain which represents the decision problem. Convergence proofs for these methods (Dayan 1994, Sutton 1988) rely in part on the fact that the expected value function $E[z|x]$ for a Markov chain (or martingale)[5] is defined by

$$E[z|x] = P^{\infty} z \tag{12}$$

where P is the transition matrix for the Markov chain, and z are the expected values at absorbing states. Typically, z represents the reward or penalty for entering any particular member of the set of absorbing states in the chain.

A *regular function* for the Markov chain defined by P is a function f such that (Kemeny et al. 1976):

$$f = Pf \tag{13}$$

That is, f is invariant under the operator defined by the transition matrix. As seen in Section 2.2, such functions are expected value functions for the Markov chain, subject to appropriate boundary conditions. Moreover, every regular function for a Markov chain is *harmonic* in the sense that such functions satisfy the weighted mean-value property defined by the transition probabilities of the Markov chain. For the purposes of this discussion, the terms "hitting probability" and "expected value function" are equivalent, since they refer to the probability of a process reaching set $\mathcal{G}$ before being absorbed by set $\mathcal{O}$.

Thus, there is a common mathematical foundation for both harmonic function path planning and techniques in reinforcement learning. Because they are harmonic, expected value functions must obey certain properties that do not appear to have been explicitly recognized in the robot learning literature. For example, the min-max property implies that expected value functions cannot exhibit local minima. The presence of such minima (as observed in Bachrach 1992) can be used as an indication that a learning process has not converged sufficiently for the desired task.

To illustrate this equivalence, harmonic functions were computed using random walk statistics, temporal differencing (TD(0)), and SOR on a simple environment. Figure 6 shows a sample path for each estimate of the environment's expected value function. The small open squares in the lower and upper left are goals. The large square in the middle and the outermost margin are obstacles. The small open circles represent the paths obtained by steepest descent, hence minimization of hitting probability. Equivalently, this maximizes goal-reaching probability, since if $p(x)$ is the hitting probability, $1 - p(x)$ is the probability that the process reaches the goal before hitting an obstacle.

Figure 7 shows the estimates of the expected value function produced by each method. The random walk statistics (leftmost surface) were generated by computing the ratio of obstacle hits (when such hits occurred before the goal was reached) to the total number of random walks for a given starting state. The TD(0) value function (center surface) was computed using TD(0) as described in (Sutton 1988), applied to random walks from a given starting state. In the TD case, the expected value was set to 1 at obstacle points and 0 at goal points. The random-walk and TD-based results were obtained by using 50 random walks starting at each state. For the SOR computation (rightmost surface), 53 iterations (over

[5] That is, the expected absorbing state value for a process starting at state x.

the entire grid) were used, where the boundary conditions were defined as in the TD case. Uniform transition probabilities were assumed in each case. Note the similarity between the surfaces. An intuitive sense of the probabilistic interpretation of such functions can be had by observing the behavior of the surface near obstacles and away from the goals, where the hitting probability is near 1, as one would expect.

Systems which use expected value functions should be able to take advantage of smoothness and the min-max principle by preserving the harmonic nature of these functions in subsequent processing steps. The following operations preserve the harmonic properties of a function ϕ (where c is a constant):

- dilation $\phi'(x) = \phi(cx)$
- translation $\phi'(x) = \phi(x + c)$
- scaling $\phi'(x) = c\phi(x)$
- bias $\phi'(x) = c + \phi(x)$

These properties can all be verified by applying the Laplacian to ϕ'. Clearly, any linear combination of harmonic functions is also harmonic. Thus, when collision probabilities (expected values) can be isolated, their properties can be preserved by using the above operations (see also Axler et al. 1991).

When applied to navigation in its simplest form, each expected value function will only be valid for a particular obstacle/goal configuration. As soon as this configuration changes, a new function must be computed (by uniqueness for mean-value functions). Changing the absorbing set necessarily changes the expected value function, requiring recomputation. Thus, if any aspect of the environment changes, the expected value function must be recomputed, regardless of the method chosen. It follows that robot navigation or planning methods which require extensive training will not work well in nonstationary environments. This suggests that, in addition to the tradeoffs enforced by dimensionality and state space size, the *rate of change* of absorbing states must also be considered when computing expected value functions.

This discussion describes a basic equivalence between methods such as TD (Sutton 1988), Monte-Carlo solution (Barto and Duff 1994), and direct methods such as SOR or resistive networks, all of which compute harmonic functions in some form. Typically, methods such as temporal differencing can also involve a transformation $\mathbf{X}$ which maps observations to states. In cases where $\mathbf{X}$ and z, and the state space (including transition probabilities) are fully determined and/or small enough, the expected value for each state can be computed directly (e.g., with a resisitive network). In many cases though, the state observation matrix $\mathbf{X}$ is not fully determined. In other cases, the state space will be too large to allow direct computation. Finally, the TD method can rely on the process itself to update the expected value function, and for this reason does not require any knowledge of transition probabilities. In the case of a direct computation using, e.g., SOR or resistive networks, transition probabilities must be determined or assumed beforehand (if they are not available). Thus, the following are prerequisites for direct computation:

- knowledge of the observation matrix $\mathbf{X}$.

- process transition probabilities.
- boundary conditions.
- "small" state spaces.

If these conditions do not hold, direct computation may be infeasible, and a scheme such as TD is an appropriate choice for computing the value function.

More generally, the intent of this discussion is to show that the fundamental properties of Markov chains provide a common framework for connectionist learning techniques and harmonic function path planning. This in turn suggests that the choice of method is dependent on the characteristics of the problem to be solved. The relationship of TD to harmonic functions also suggests ways in which the special properties of harmonic functions (e.g., min-max property, smoothness) can be used to advantage in learning-based systems. Note also that when sufficient information is available, direct methods (e.g., resistive networks) can be used to reconstruct expected value functions from their boundary conditions. By the min-max property, the boundary conditions can easily be distinguished as the extrema of the value function. If transition probabilities are known or can be assumed, this allows the compression of learned (stationary) expected value functions after convergence, since only the boundary conditions are needed.

The relationship between Laplace's equation and Markov chains can be generalized when a reward or penalty is associated with each transition in the Markov chain. In this case, Laplace's equation can be replaced with Poisson's equation:

$$\nabla^2 \phi(\mathbf{x}) = f(\mathbf{x}) \tag{14}$$

where f is the reward/penalty as a function of state. This equivalence is explored in more detail in (Hordijk 1974).

5 Conclusion

In this paper, some properties of harmonic functions have been discussed in the context of related work in path planning, control under uncertainty, and machine learning. In particular, the equivalence of harmonic functions and collision probabilities has been illustrated. This connection is applied to motion uncertainty, and in particular to Erdmann's randomization method. The result is an explicit incorporation of obstacle information into the randomization process, and an interpretation of harmonic functions in terms of goal-reaching probabilities and in terms of drift in the random motion. This also illustrates the interpretation of harmonic functions as an extrapolation of obstacle geometry (Connolly and Grupen 1993).

Collision probabilities are often utilized in robot-learning systems, in one form or another. Because these probabilities are harmonic, local minima can be eliminated to enforce smooth behavior of the system. In particular, temporal differencing (and in fact, any scheme based on absorbing Markov chains) is in some sense equivalent to the computation of a specific harmonic function. The TD weights are related to the corresponding harmonic function by

the matrix $\mathbf{X}$ of state observation vectors. It is important to note that this function can only be fully determined if the transition probabilities for the Markov chain are known or assumed *a priori*. In temporal differencing, the observed process itself obeys these probabilities, whereas in a resistive grid (or other direct computation), they must be specified in advance. In any event, harmonic functions exhibit formal properties which are of use in applications based on the theory of Markov decision problems.

References

[1] Akishita S , Kawamura S , and Hayashi K 1990. Laplace potential for moving obstacle avoidance and approach of a mobile robot. In *1990 Japan-USA Symposium on Flexible Automation, A Pacific Rim Conference*, pages 139–142.

[2] Arkin R. C 1987. *Towards Cosmopolitan Robots: Intelligent Navigation in Extended Man-Made Environments*. PhD thesis, University of Massachusetts.

[3] Axler S , Bourdon P , and Ramey W 1991. *Harmonic Function Theory*, volume 137 of *Graduate Texts in Mathematics*. Springer-Verlag, New York.

[4] Bachrach J. R 1992. *Connectionist Modeling and Control of Finite State Environments*. PhD thesis, University of Massachusetts.

[5] Barraquand J , Langlois B , and Latombe J.-C 1992. Numerical potential field techniques for robot path planning. *IEEE Transactions on Systems, Man, and Cybernetics*, 22(2):224–241.

[6] Barraquand J and Latombe J.-C 1991. Robot motion planning: A distributed representation approach. *International Journal of Robotics Research*, 10(6):628–649.

[7] Barto A and Duff M 1994. Monte Carlo matrix inversion and reinforcement learning. In Cowan J. D , Tesauro G , and Alspector J , editors, *Neural Information Processing Systems 6*, pages 687–694. Morgan Kaufmann Publishers.

[8] Burden R. L , Faires J. D , and Reynolds A. C 1978. *Numerical Analysis*. Prindle, Weber and Schmidt, Boston.

[9] Connolly C. I and Burns J. B 1993. A model for the functioning of the striatum. *Biological Cybernetics*, 68(6):535–544.

[10] Connolly C. I , Burns J. B , and Weiss R 1990. Path planning using Laplace's Equation. In *Proceedings of the 1990 IEEE International Conference on Robotics and Automation*, pages 2102–2106. IEEE.

[11] Connolly C. I and Grupen R. A 1993. The applications of harmonic functions to robotics. *Journal of Robotic Systems*, 10(7):931–946.

[12] Connolly C. I , Grupen R. A , and Souccar K. X 1995. A hamiltonian approach to kinodynamic planning. In *Proceedings of the 1995 IEEE International Conference on Robotics and Automation*, page to appear. IEEE.

[13] Courant R and Hilbert D 1989. *Methods of Mathematical Physics*, volume 2. John Wiley and Sons, New York.

[14] Dayan P 1991. Navigating through temporal difference. In Lippmann R. P , Moody J. E , and Touretzky D. S , editors, *Advances in Neural Information Processing Systems 3*, pages 464–470. Morgan Kaufmann Publishers.

[15] Dayan P 1994. The convergence of TD(λ) for general λ. *Machine Learning*, 8:341–362.

[16] Decuyper J and Keymeulen D 1991. A reactive robot navigation system based on a fluid dynamics metaphor. In *Parallel Problem Solving from Nature*, Lecture Notes in Computer Science, pages 348–355. Springer Verlag.

[17] Donald B. R 1990. Planning multi-step error detection and recovery strategies. *International Journal of Robotics Research*, 9(1):3–60.

[18] Doyle P and Snell J. L 1984. *Random Walks and Electric Networks*. Carus Monographs in Mathematics. American Mathematical Society.

[19] Erdmann M 1992. Randomization in robot tasks. *International Journal of Robotics Research*, 11(5):399–436.

[20] Fagg A. H , Lotspeich D , and Bekey G. A 1994. A reinforcement-learning approach to reactive control policy design for autonomous robots. In *Proceedings of the 1994 IEEE International Conference on Robotics and Automation*, pages 39–44. IEEE.

[21] Griswold N. C and Eem J 1990. Control for mobile robots in the presence of moving obstacles. *IEEE Transactions on Robotics and Automation*, 6(2):263–268.

[22] Grupen R. A , Coelho J. A , Connolly C. I , Gullapalli V , Huber M , and Souccar K 1994. Toward physical interaction and manipulation: Screwing in a light bulb. In *AAAI 1994 Spring Symposium on Physical Interaction and Manipulation*.

[23] Hordijk A 1974. *Dynamic programming and Markov potential theory*. Mathematisch Centrum, Amsterdam.

[24] Karlin S and Taylor H. M 1981. *A Second Course in Stochastic Processes*. Academic Press, New York.

[25] Kemeny J. G , Snell J. L , and Knapp A. W 1976. *Denumerable Markov Chains*, volume 40 of *Graduate Texts in Mathematics*. Springer-Verlag, New York.

[26] Khatib O 1985. Real-time obstacle avoidance for manipulators and mobile robots. In *Proceedings of the 1985 IEEE International Conference on Robotics and Automation*, pages 500–505. IEEE.

[27] Khosla P and Volpe R 1988. Superquadric artificial potentials for obstacle avoidance and approach. In *Proceedings of the 1988 IEEE International Conference on Robotics and Automation*, pages 1778–1784. IEEE.

[28] Kim J.-O and Khosla P 1991. Real-time obstacle avoidance using harmonic potential functions. In *Proceedings of the 1991 IEEE International Conference on Robotics and Automation*, pages 790–796. IEEE.

[29] Koditschek D. E 1987. Exact robot navigation by means of potential functions: Some topological considerations. In *Proceedings of the 1987 IEEE International Conference on Robotics and Automation*, pages 1–6. IEEE.

[30] Mahavedan S and Connell J 1992. Automatic programming of behavior-based robots using reinforcement learning. *Artificial Intelligence*, 55:311–365.

[31] McCann G. D and Wilts C. H 1949. Application of electric-analog computers to heat-transfer and fluid-flow problems. *Journal of Applied Mechanics*, 16(3):247–258.

[32] Moore A. W 1994. The parti-game algorithm for variable resolution reinforcement learning in multidimensional state-spaces. In Cowan J. D , Tesauro G , and Alspector J , editors, *Neural Information Processing Systems 6*, pages 711–718. Morgan Kaufmann Publishers.

[33] Oommen B. J , Andrade N , and Iyengar S. S 1991. Trajectory planning of robot manipulators in noisy work spaces using stochastic automata. *International Journal of Robotics Research*, 10(2):135–148.

[34] Simmons R and Koenig S 1995. Probabilistic navigation in partially observable environments. In *IJCAI '95*, pages 1080–1087.

[35] Singh S. P , Barto A. G , Grupen R , and Connolly C 1994. Robust reinforcement learning in motion planning. In Cowan J. D , Tesauro G , and Alspector J , editors, *Advances in Neural Information Processing Systems 6*, pages 655–662. Morgan Kaufmann Publishers.

[36] Sutton R. S 1988. Learning to predict by the methods of temporal differences. *Machine Learning*, 3(1):9–44.

[37] Tarassenko L and Blake A 1991. Analogue computation of collision-free paths. In *Proceedings of the 1991 IEEE International Conference on Robotics and Automation*, pages 540–545. IEEE.

[38] Thrun S. B and Möller K 1992. Active exploration in dynamic environments. In Moody J. E , Hanson S. J , and Lippmann R. P , editors, *Advances in Neural Information Processing Systems 4*. Morgan Kaufmann Publishers.

[39] Watkins C. J. C. H 1989. *Learning from Delayed Rewards*. PhD thesis, Cambridge University.

[40] Weinstock R 1974. *Calculus of Variations with Applications to Physics and Engineering.* Dover Publications, Inc.

[41] Zachmanoglou E. C and Thoe D. W 1986. *Introduction to Partial Differential Equations with Applications.* Dover Publications, Inc., New York.

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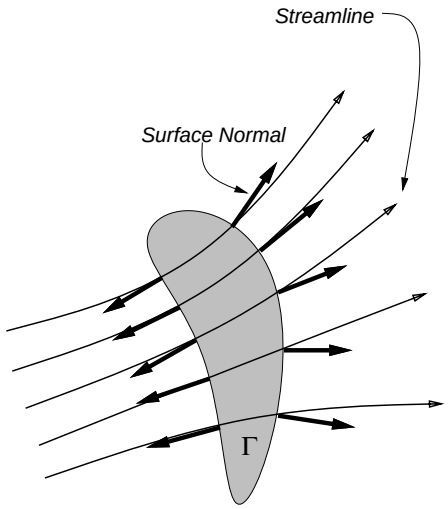

Figure 1: An illustration of the Gauss Integral Theorem: bold arrows are normals, light arrows are streamlines.

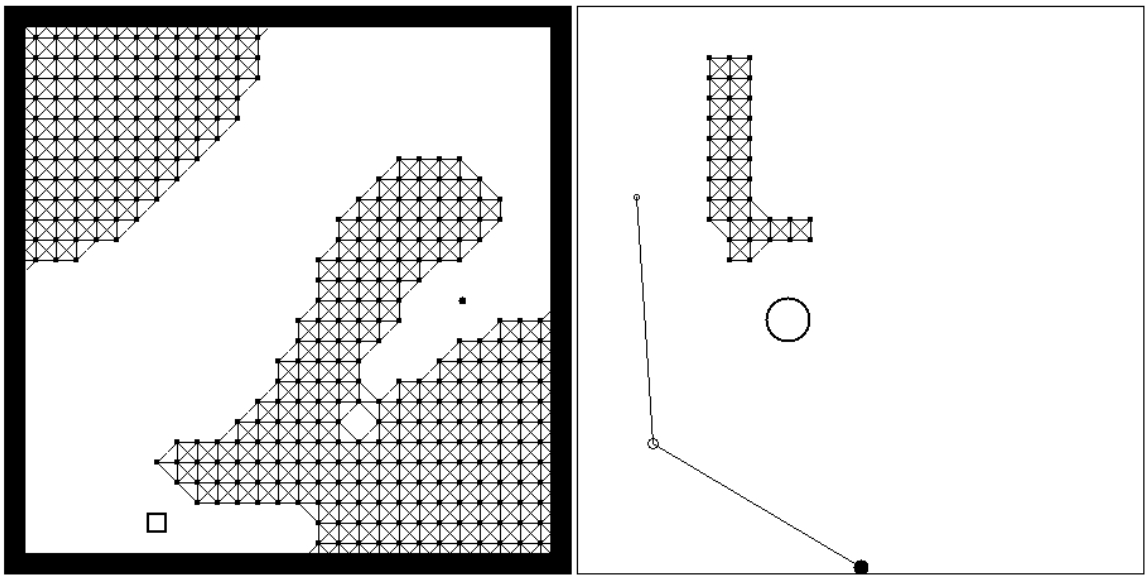

Figure 2: 2-DOF arm example, initial configuration: right pane is cartesian space, left pane is configuration space. Open circle (right) and open square (left) represent goal configurations. The mesh designates obstacles, including joint limits.

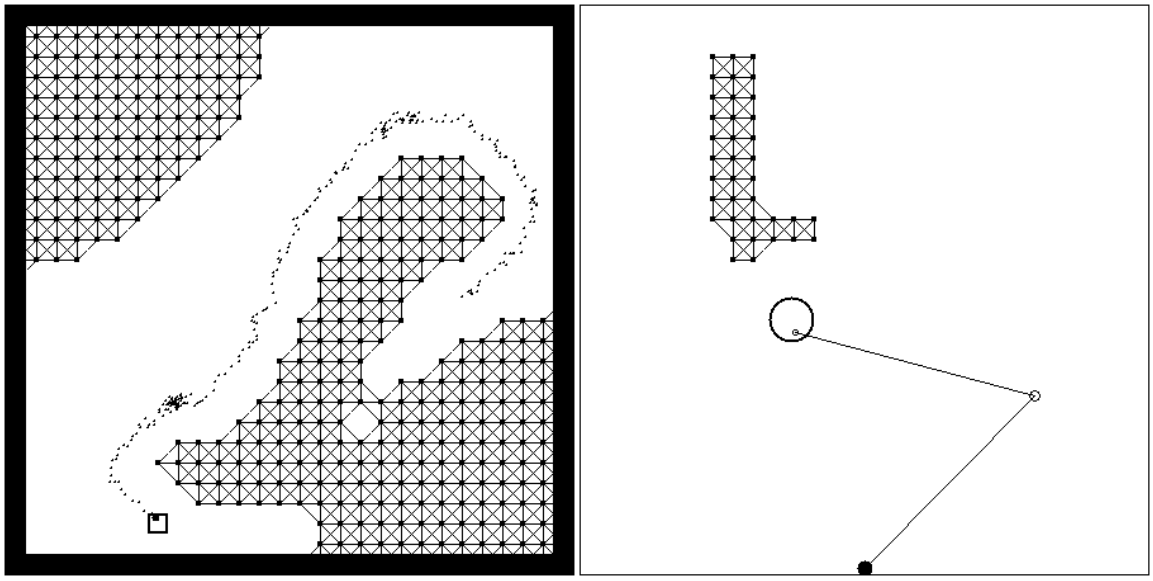

Figure 3: 2-DOF arm example, path and final configuration using an l2 distance map as the label function.

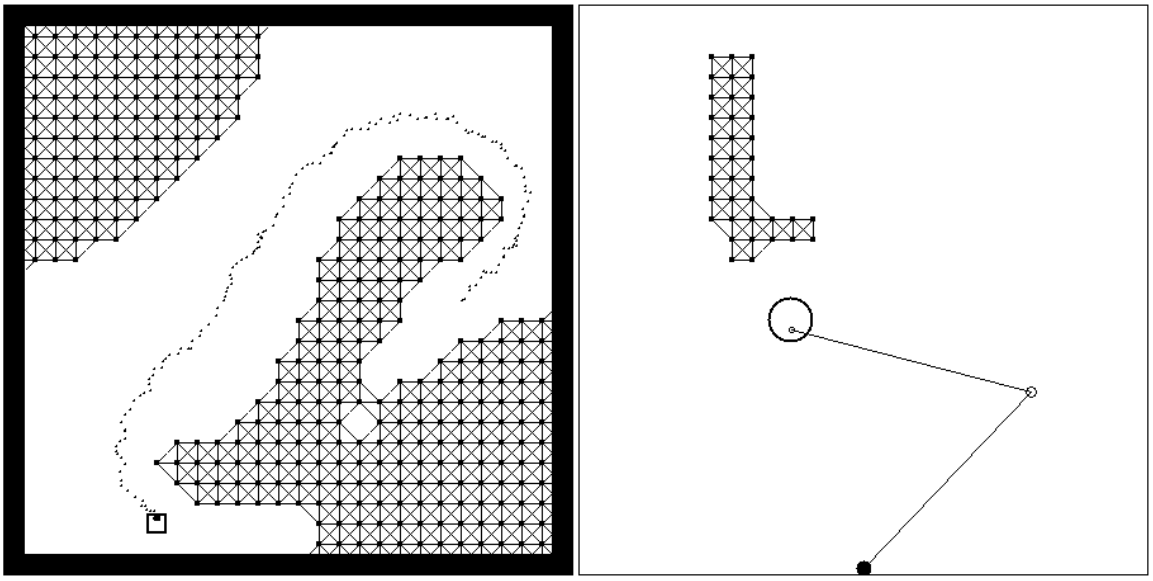

Figure 4: 2-DOF arm example, path and final configuration using a harmonic function as the label function.

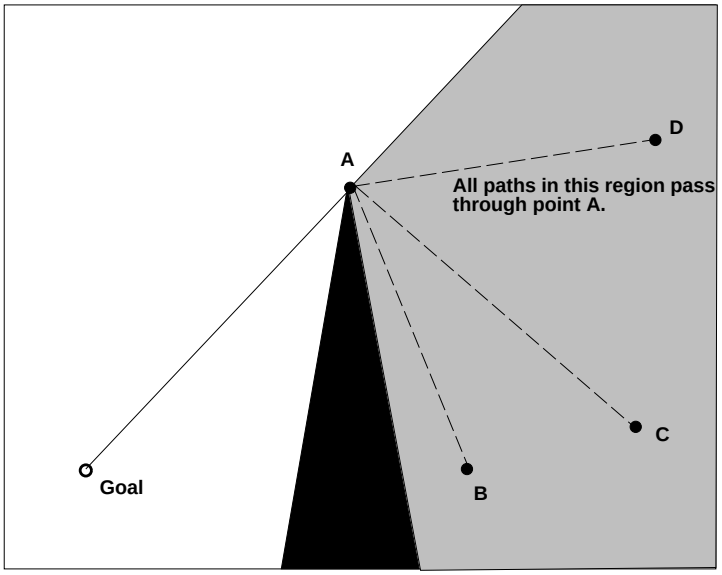

Figure 5: Shadow effect created by a euclidean distance function. The black wedge represents an obstacle.

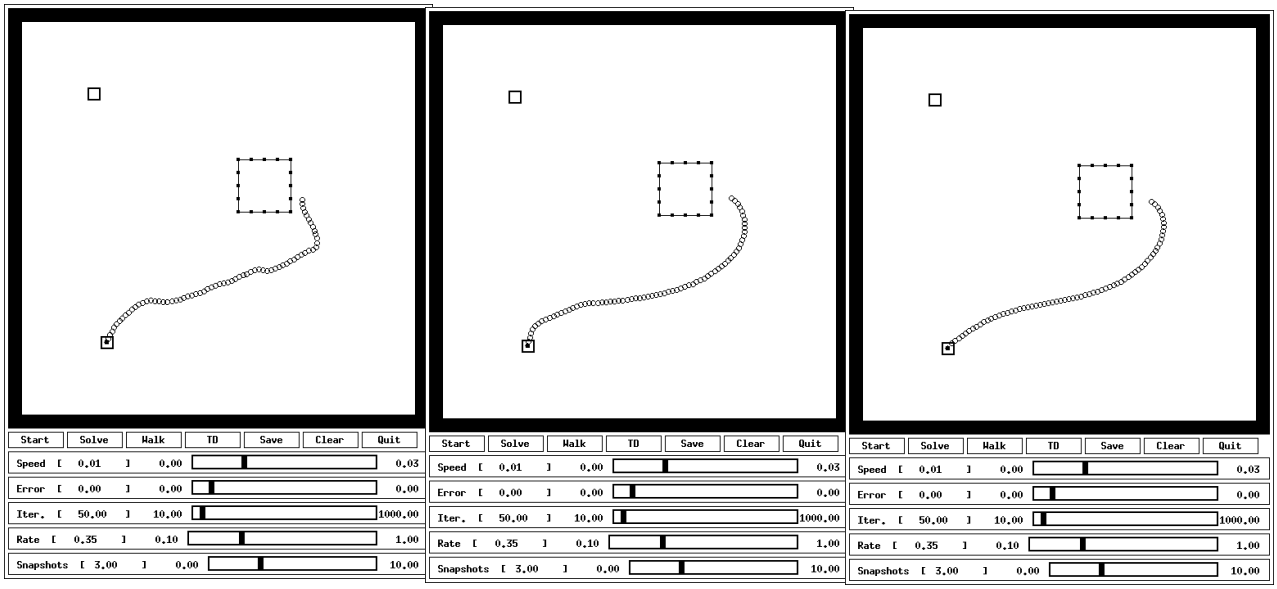

Figure 6: Environment with paths generated by random walk statistics (left), TD(0) (middle) and an SOR-generated harmonic function (right).

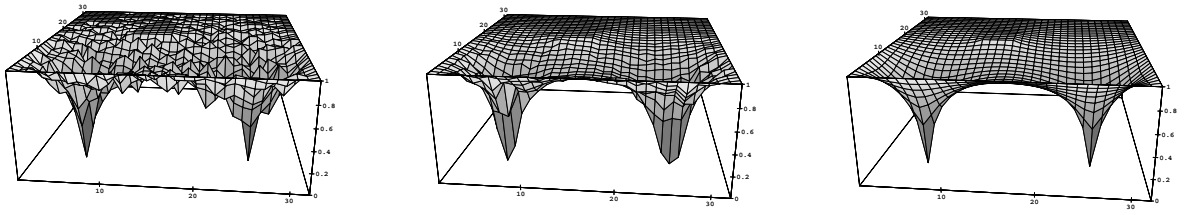

Figure 7: Harmonic functions computed using random walk statistics (left), TD(0) (middle) and SOR (right).