8803 Connections between Learning, Game Theory, and Optimization

Homework # 3 **Due: December 7th 2010**

This homework is due by the start of class on December 7th 2010. You can either submit the homework via the course page on T-Square or hand it in at the beginning of the class on December 7th 2010. Start early!

Groundrules:

- Your work will be graded on correctness, clarity, and conciseness.
- You may collaborate with others on this problem set. However, you must *write your own solutions* and *list your collaborators/sources* for each problem.

Problems:

1. **Submodular functions:**

   (a) **[Coverage functions]** Given a finite universe U, let S_1, S_2, . . . , S_n be subsets of U. Define $f : 2^{[n]} \to R_+$ by

   $$f(A) = |\cup_{i \in A} S_i| \quad \text{for} \quad A \subseteq [n].$$

   Prove that f is monotone and submodular.

   More generally, consider any non-negative function $w : U \to R_+$, and for $M \subseteq U$, define $w(M) = \sum_{x \in M} w(x)$. Prove that the function f defined by

   $$f(A) = w\left(\cup_{i \in A} S_i\right) \quad \text{for} \quad A \subseteq [n]$$

   is monotone and submodular.

   (b) Prove that the minimizers of any submodular function are closed under union and intersection.

2. **Matroid congestion games:** Specify which of the following games are matroid congestion games: fair cost sharing games, routing games with linear latency functions, consensus games.

Extra Credit:

1. **General minimax theorems:** Let $f(x, y)$ denote a bounded real-valued function defined over $X \times Y$, where X and Y are convex sets and X is compact. Suppose $f(\cdot, y)$ is convex and continuous for each fixed y and $f(x, \cdot)$ is concave for each fixed $x \in X$. Derive a learning style argument showing that:

   $$\inf_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \inf_{x \in X} f(x, y).$$

2. **Boolean submodular functions:** Prove that the class of Boolean monotone submodular functions is PAC learnable.