Graph Traversals

Slides by Carl Kingsford

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Based on/Reading: Chapter 3 of Kleinberg & Tardos

Breadth-First Search

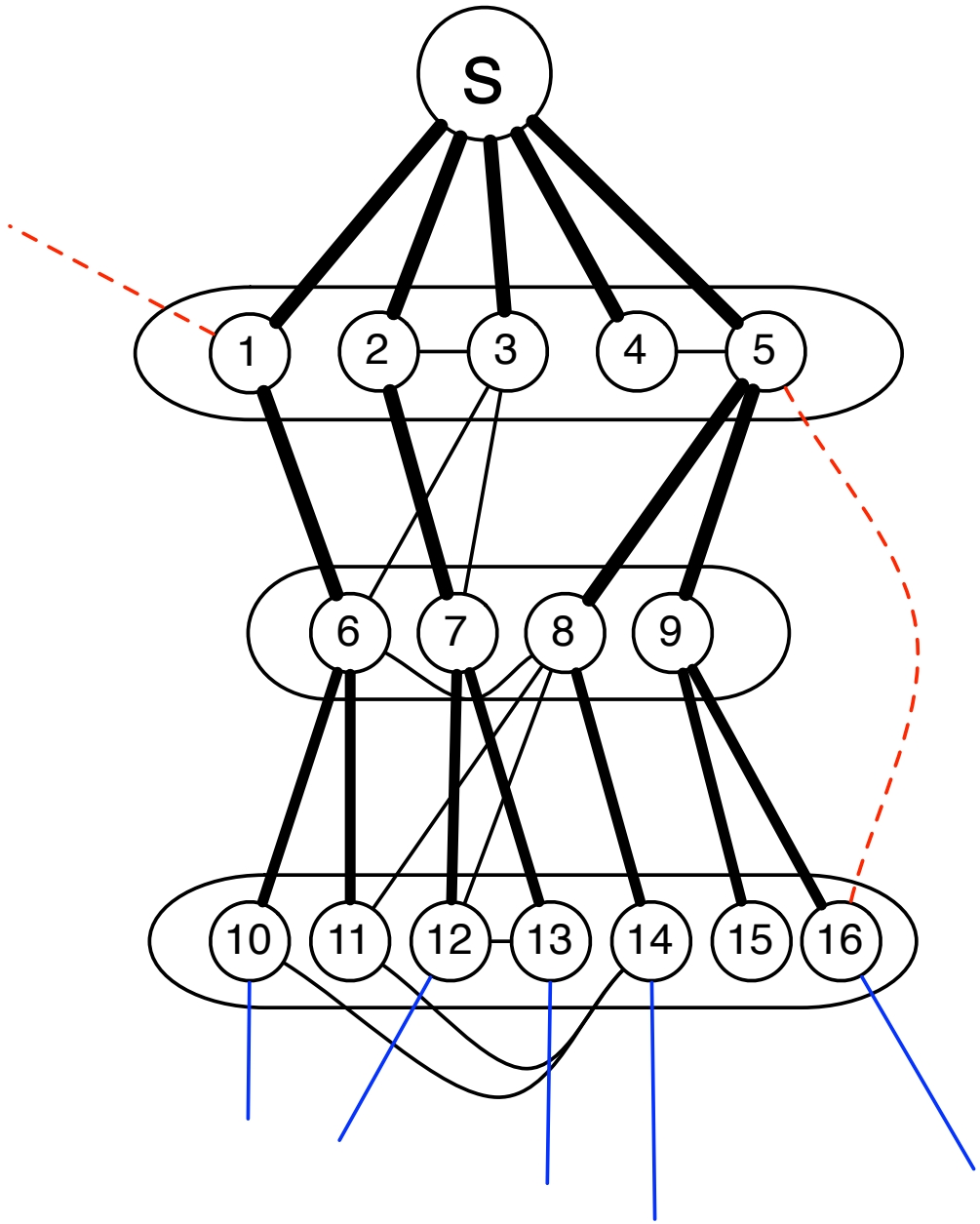
Breadth-first search explores the nodes of a graph in increasing distance away from some starting vertex s.

It decomposes the component into layers L_i such that the shortest path from s to each of nodes in L_i is of length i.

Breadth-First Search:

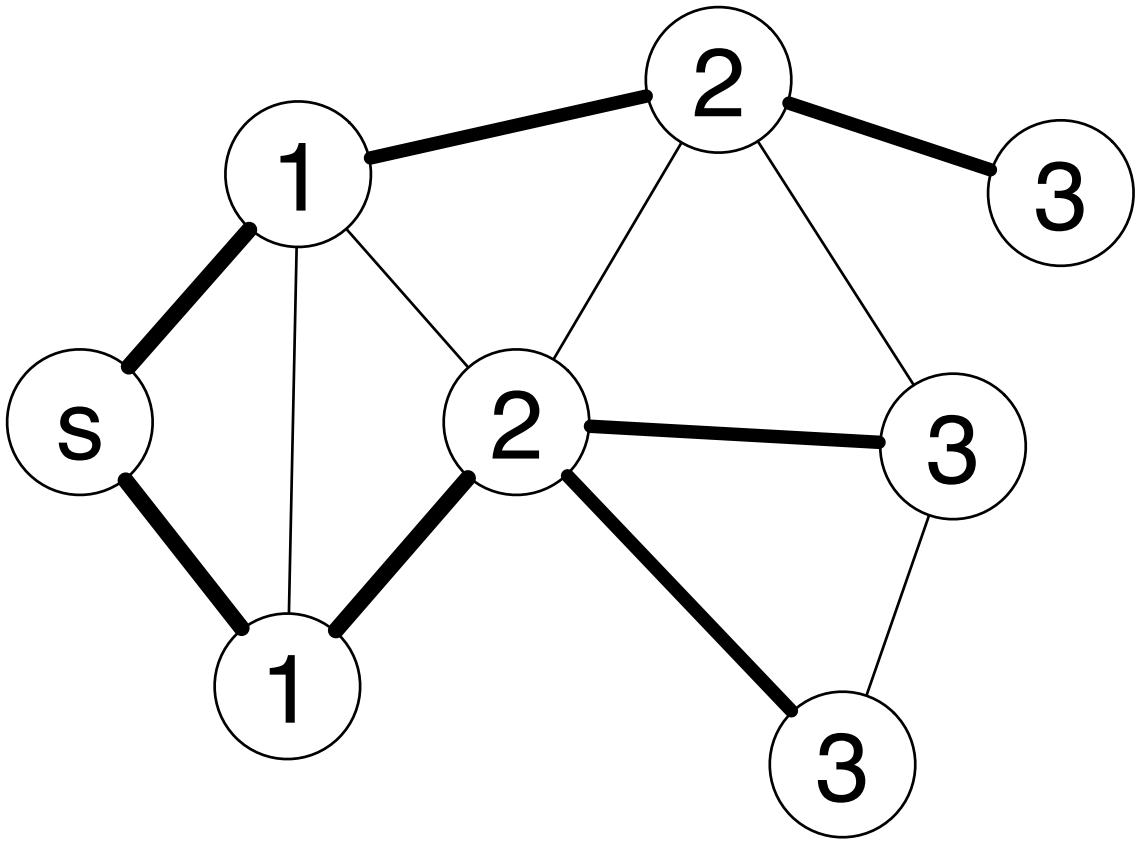
1. L_0 is the set $\{s\}$.
2. Given layers $L_0, L_1, \ldots, L_j$, then L_{j+1} is the set of nodes that are not in a previous layer and that have an edge to some node in layer L_j.

BFS Tree



BFS Tree Example

A BFS traversal of a graph results in a breadth-first search tree:



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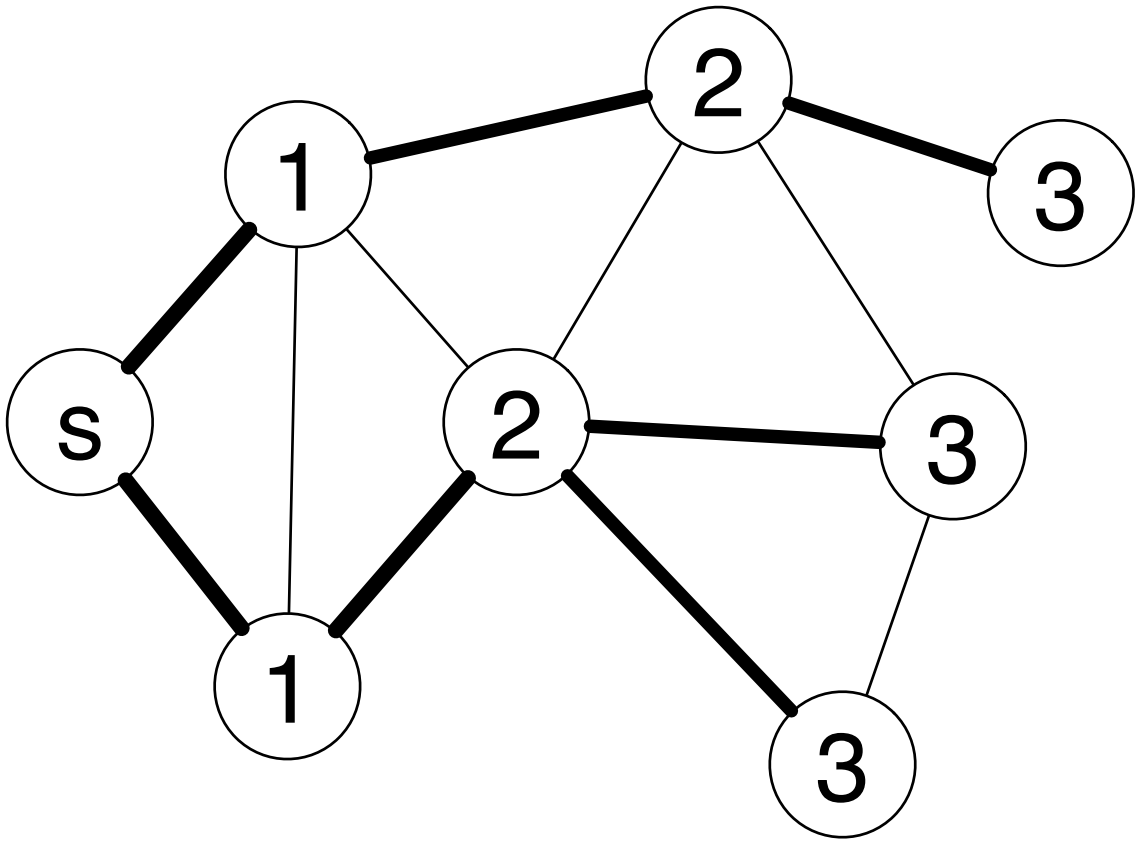
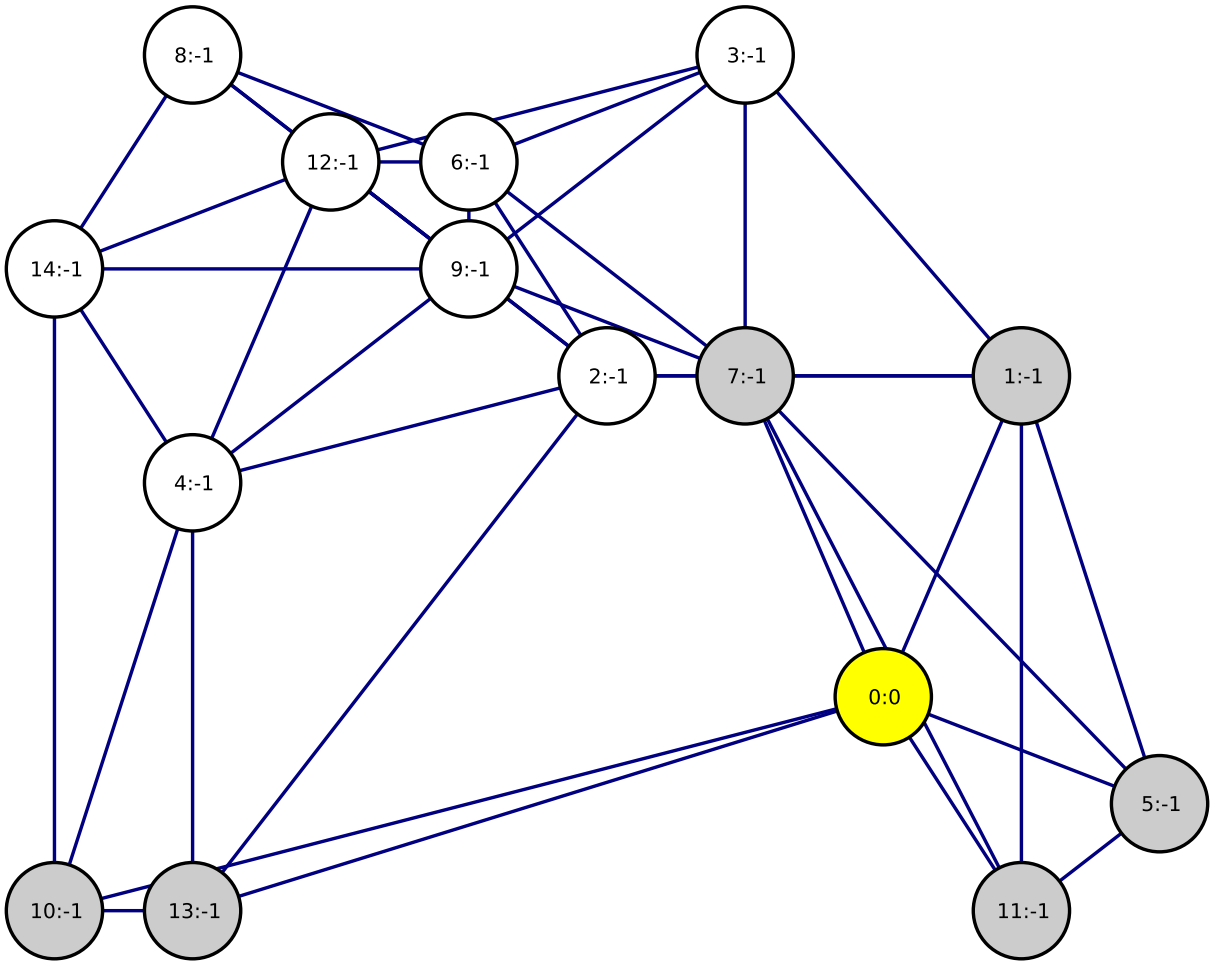


Can we say anything about the non-tree edges?

Example BFS



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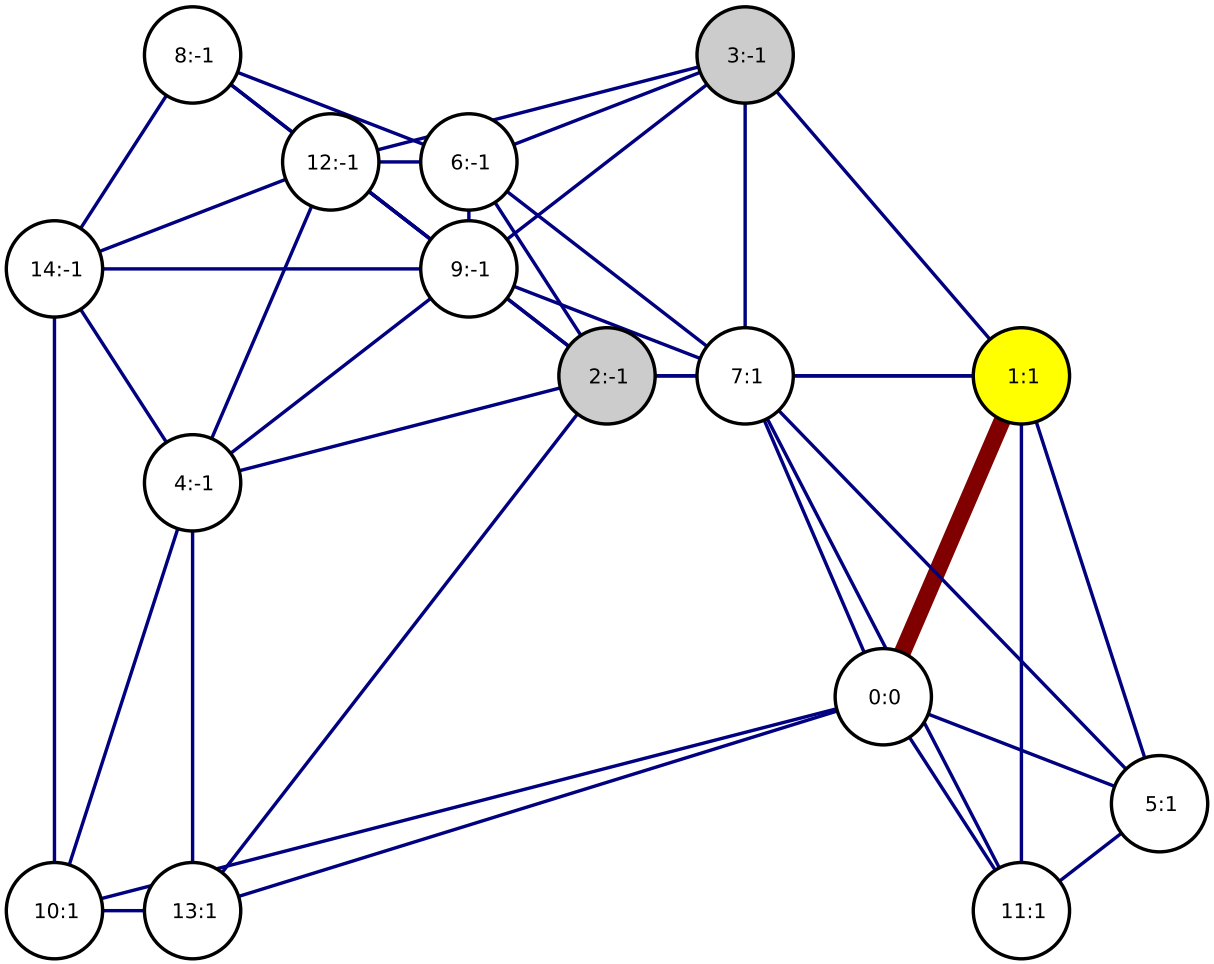

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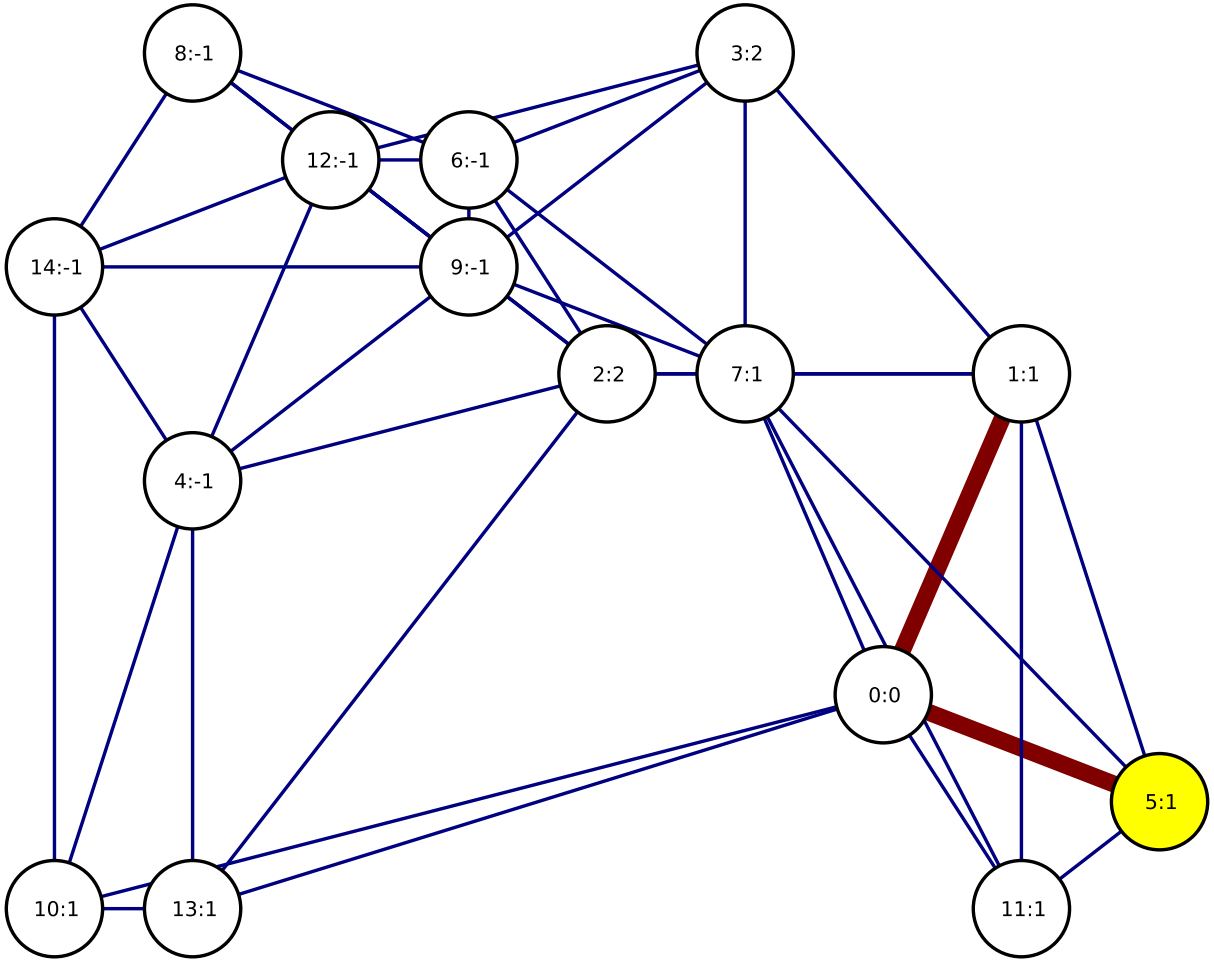

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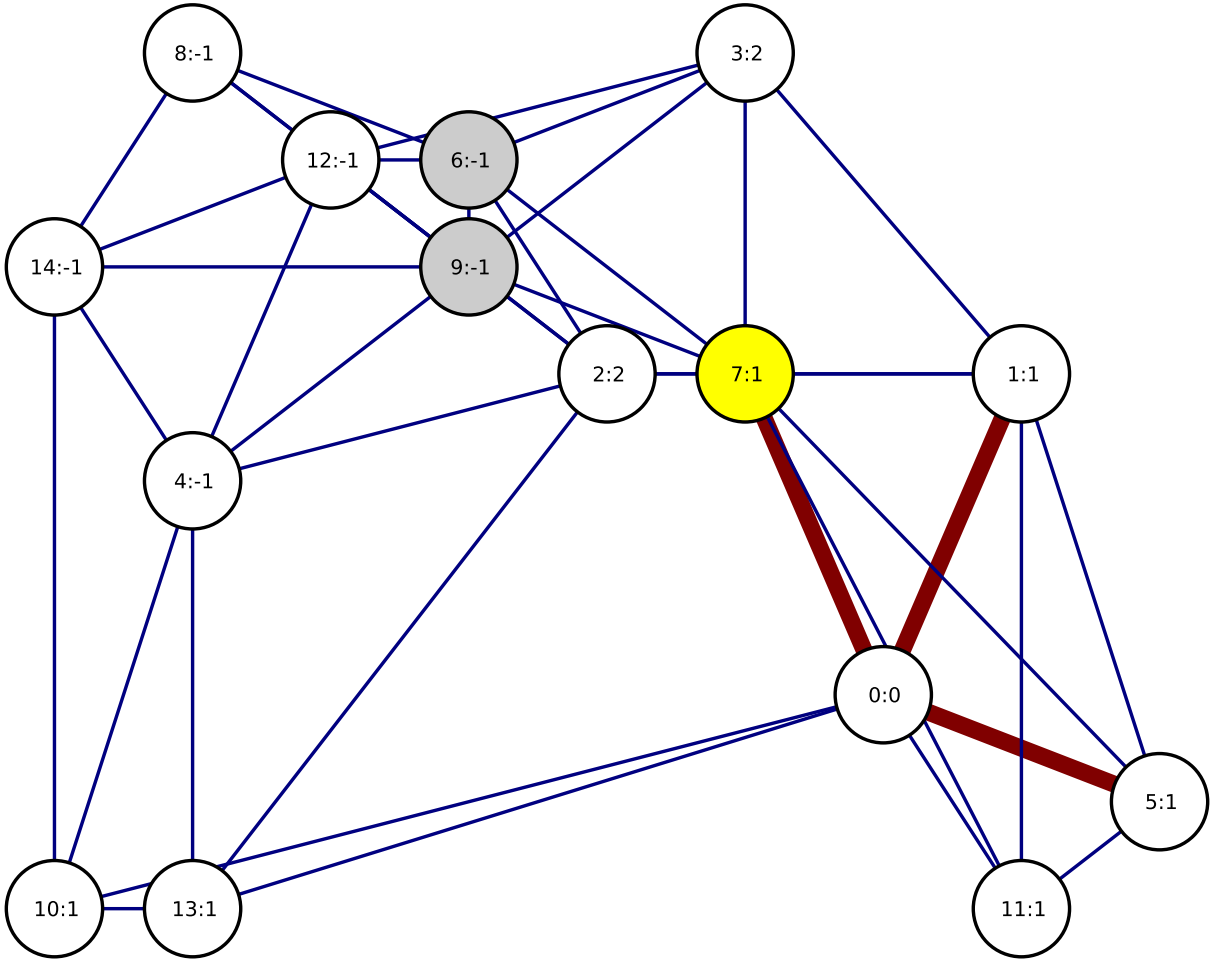

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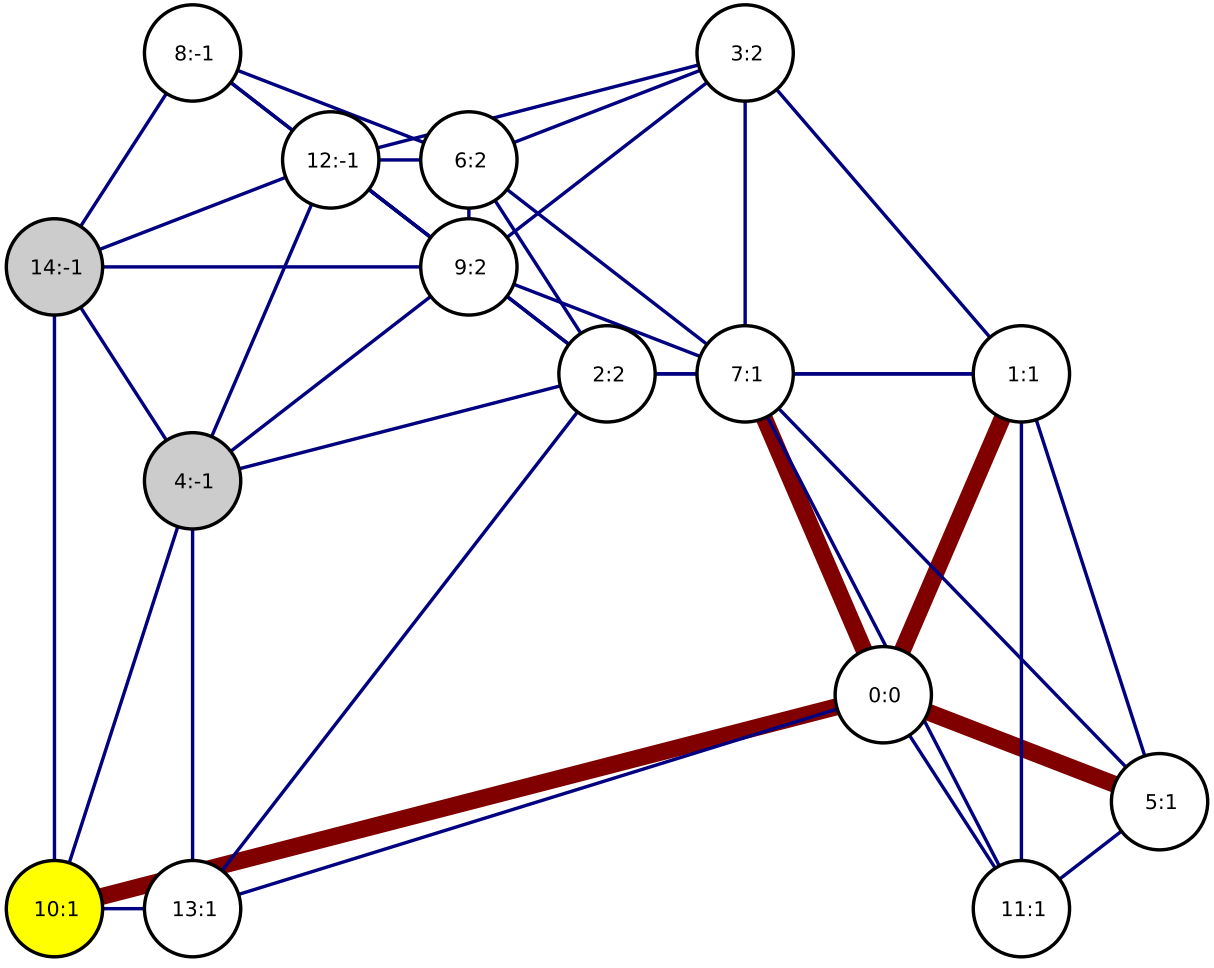

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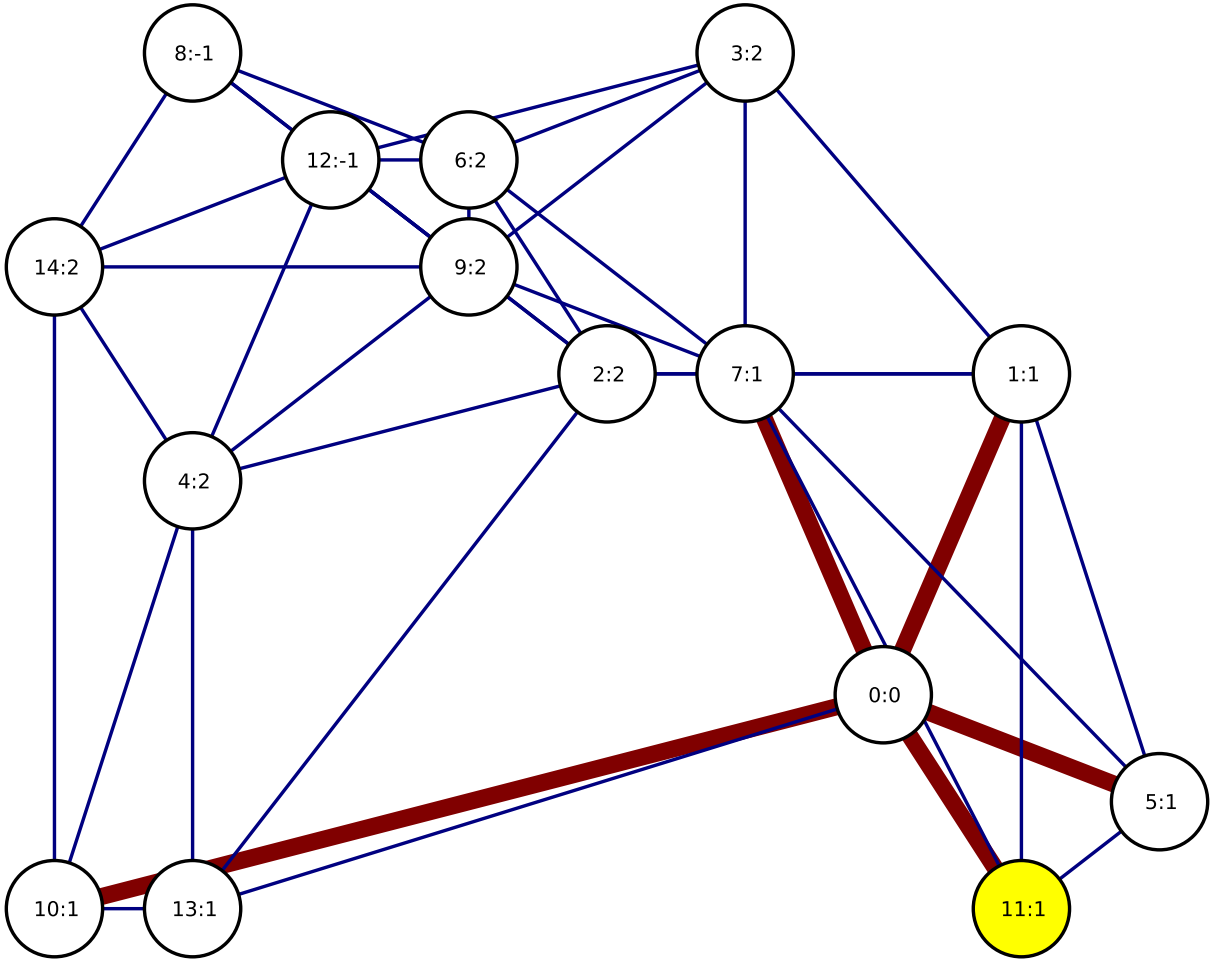

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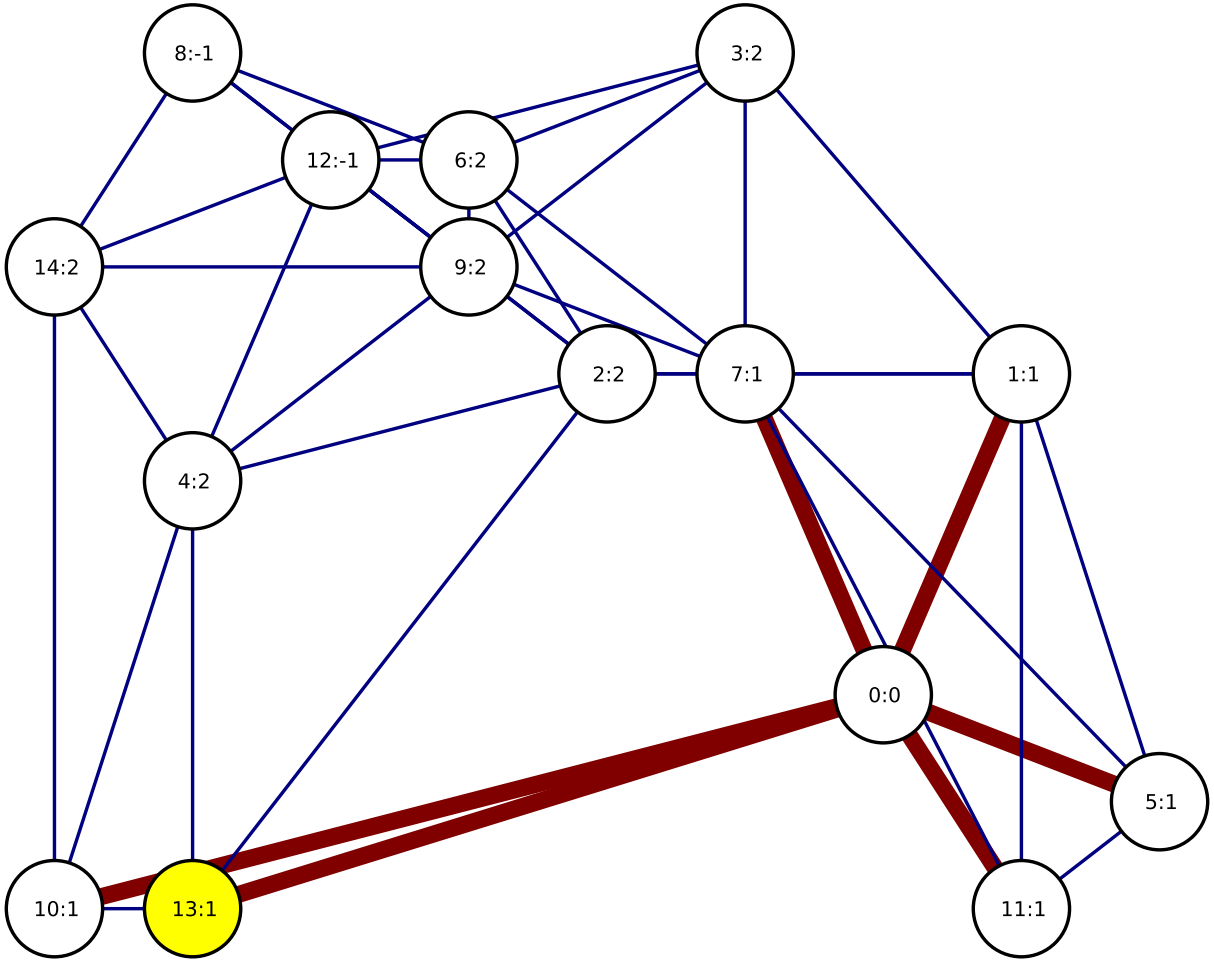

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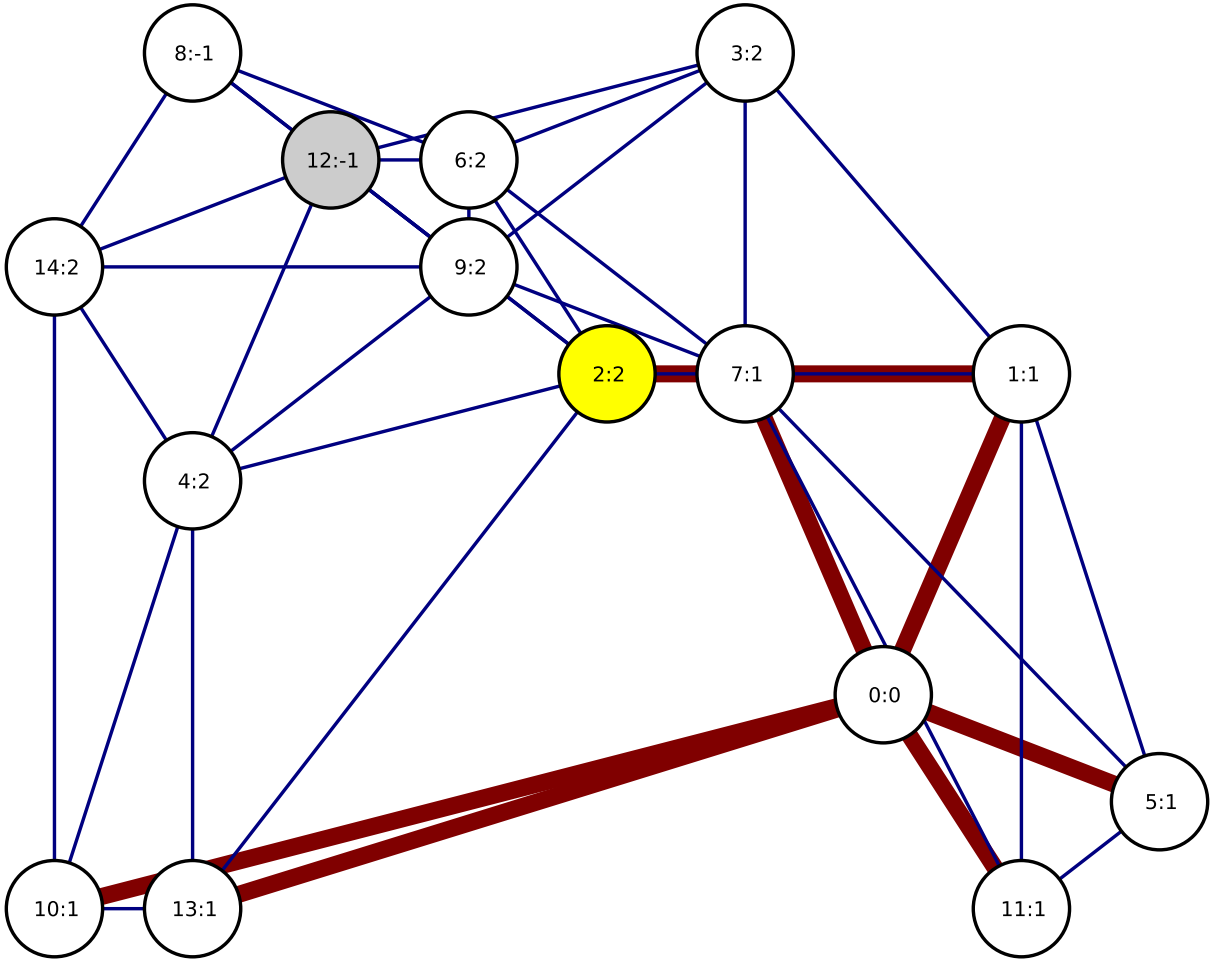

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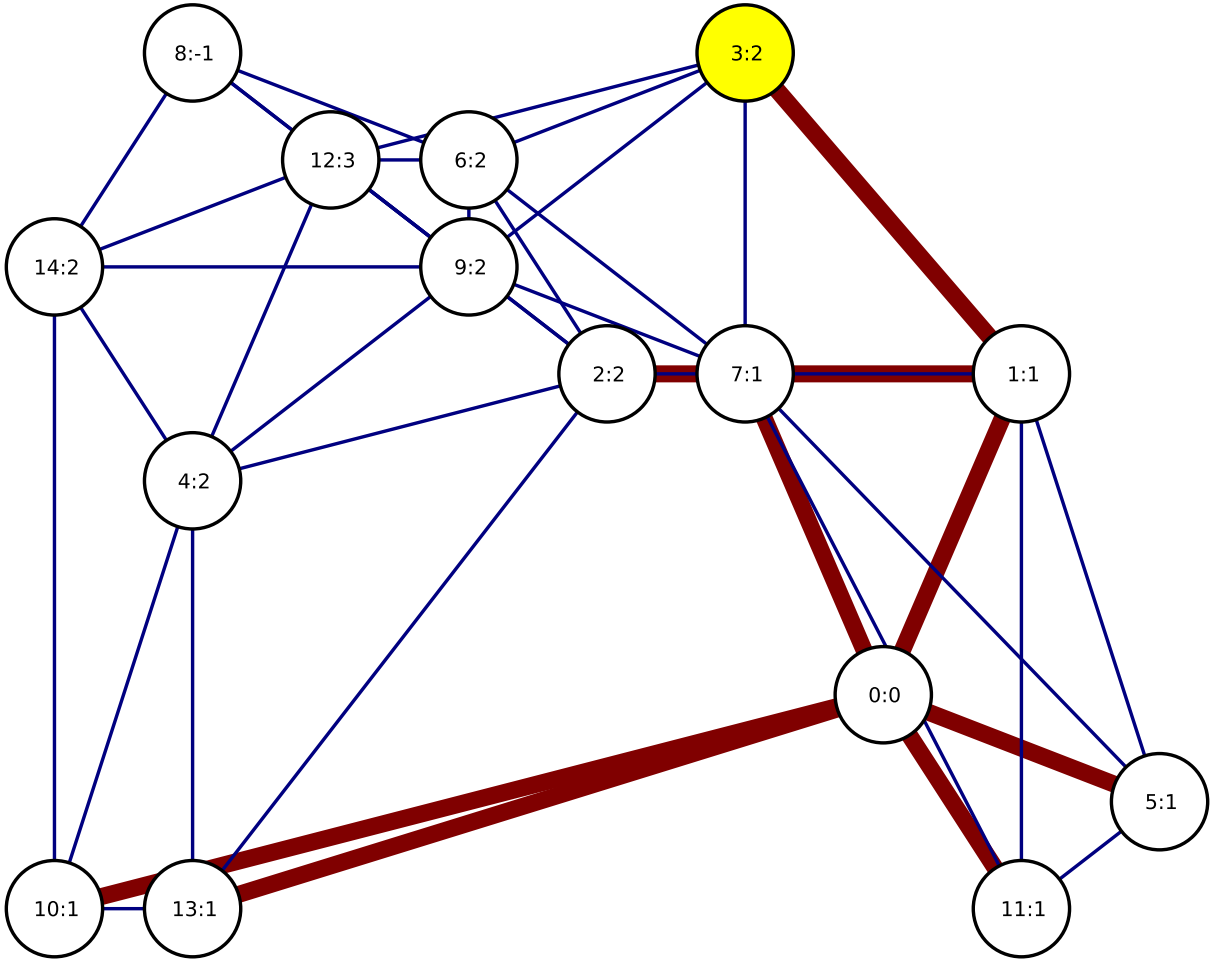

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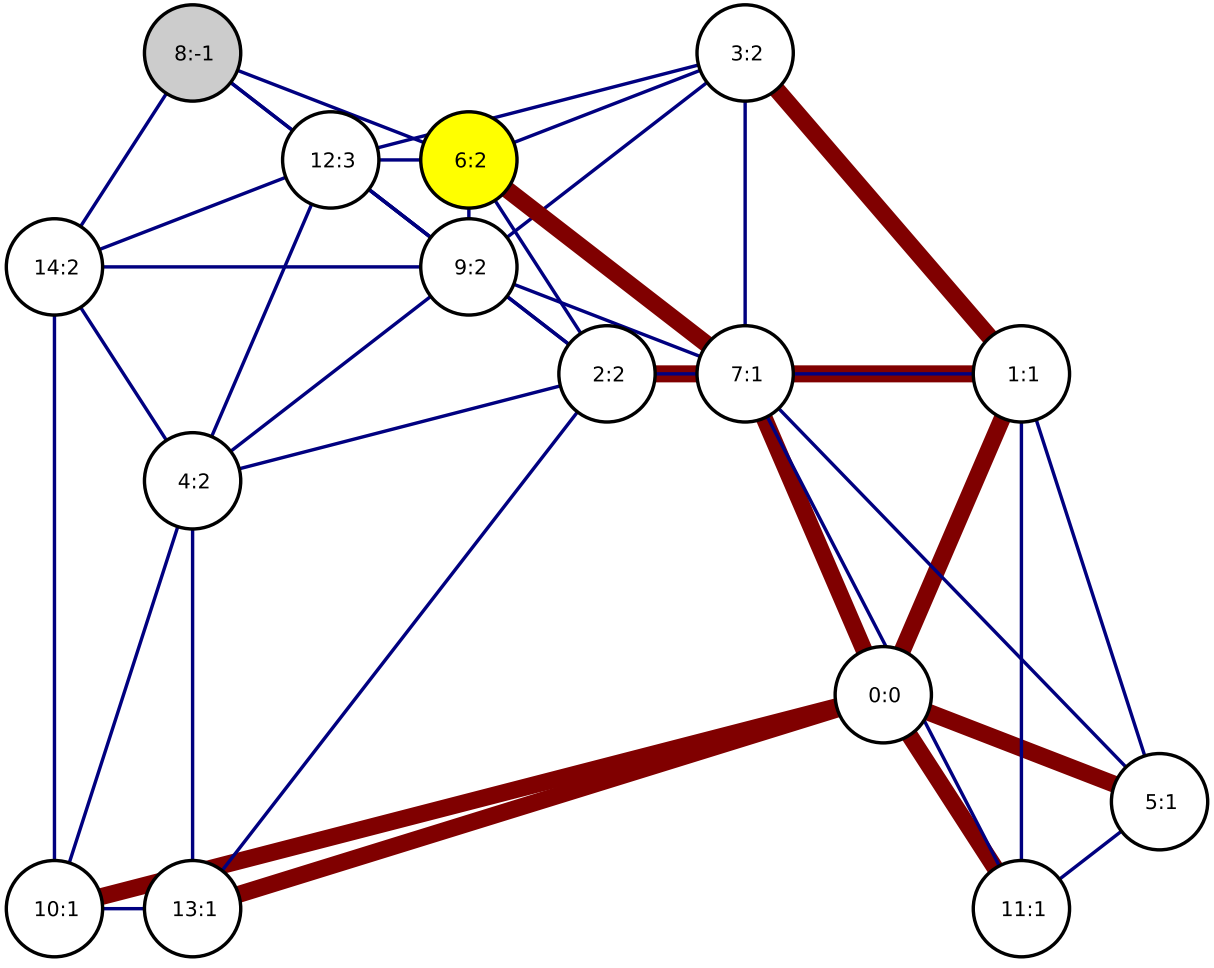

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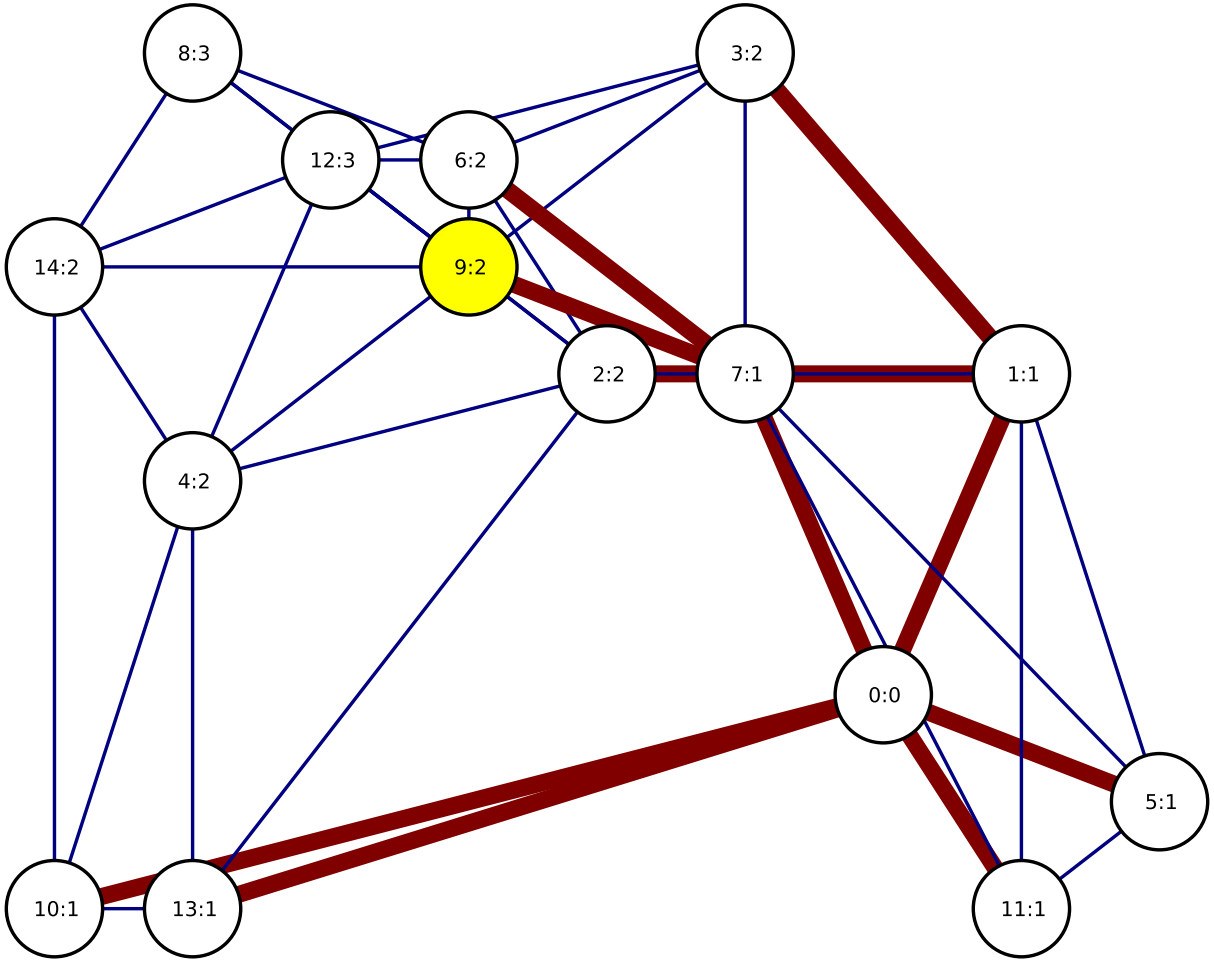

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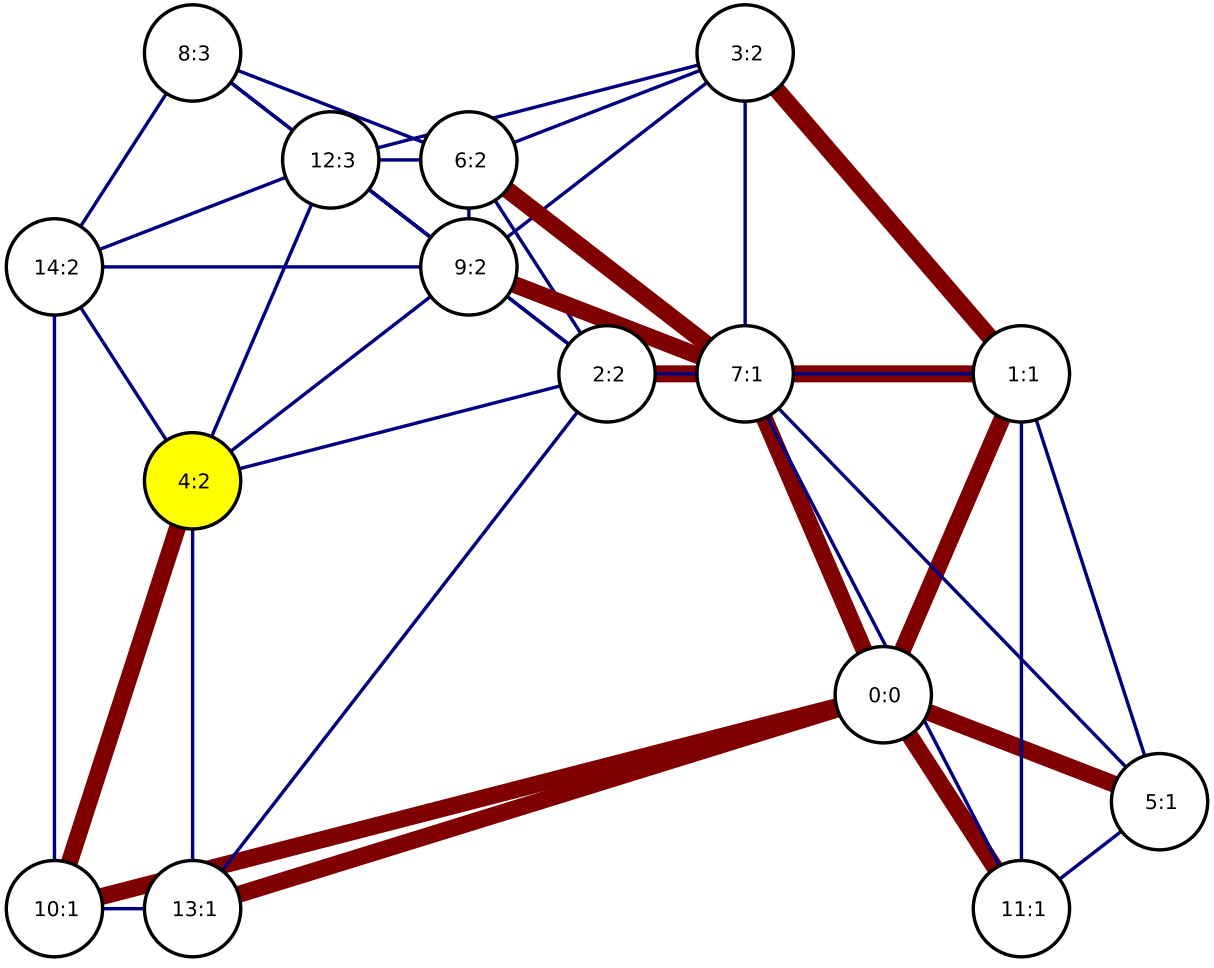

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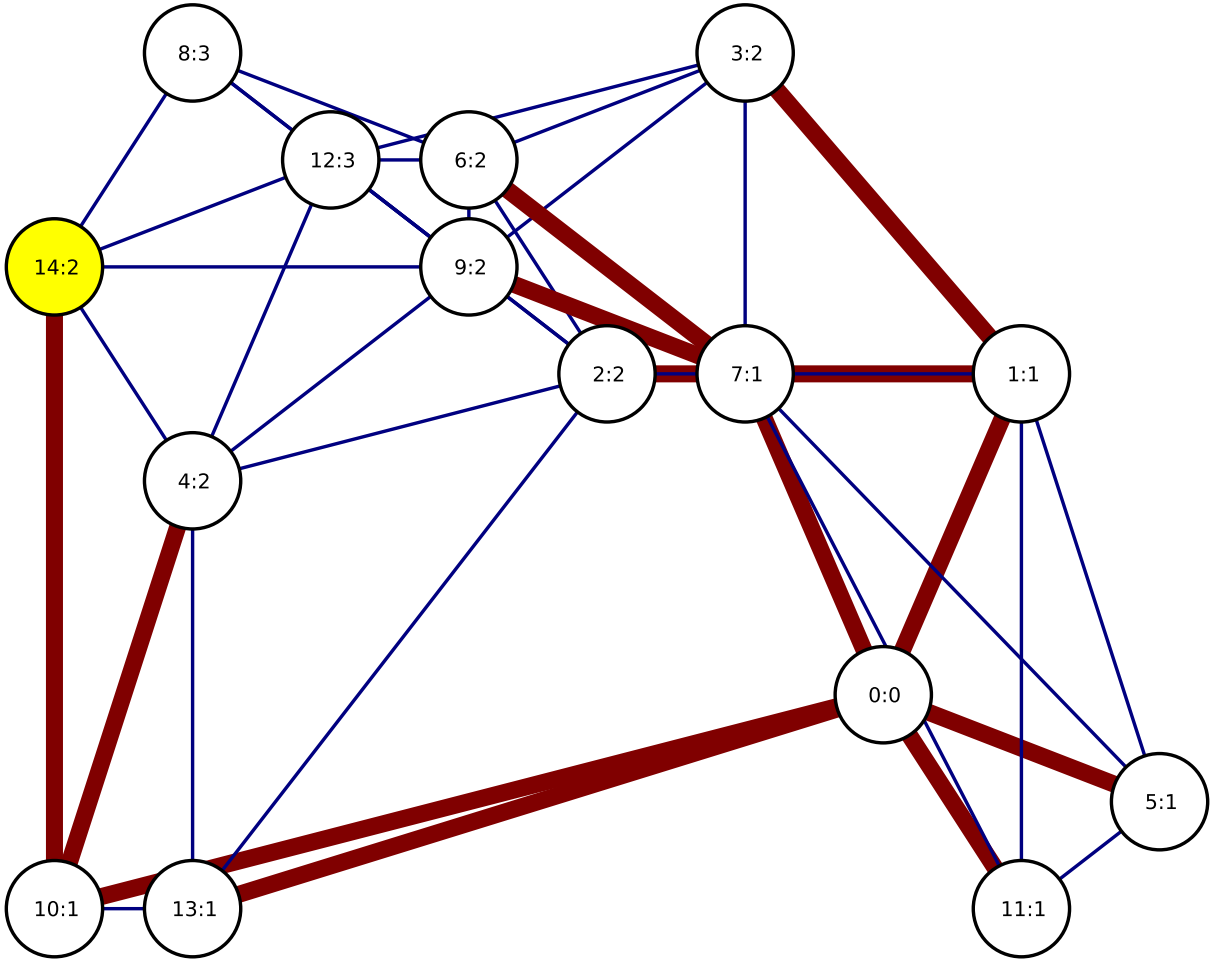

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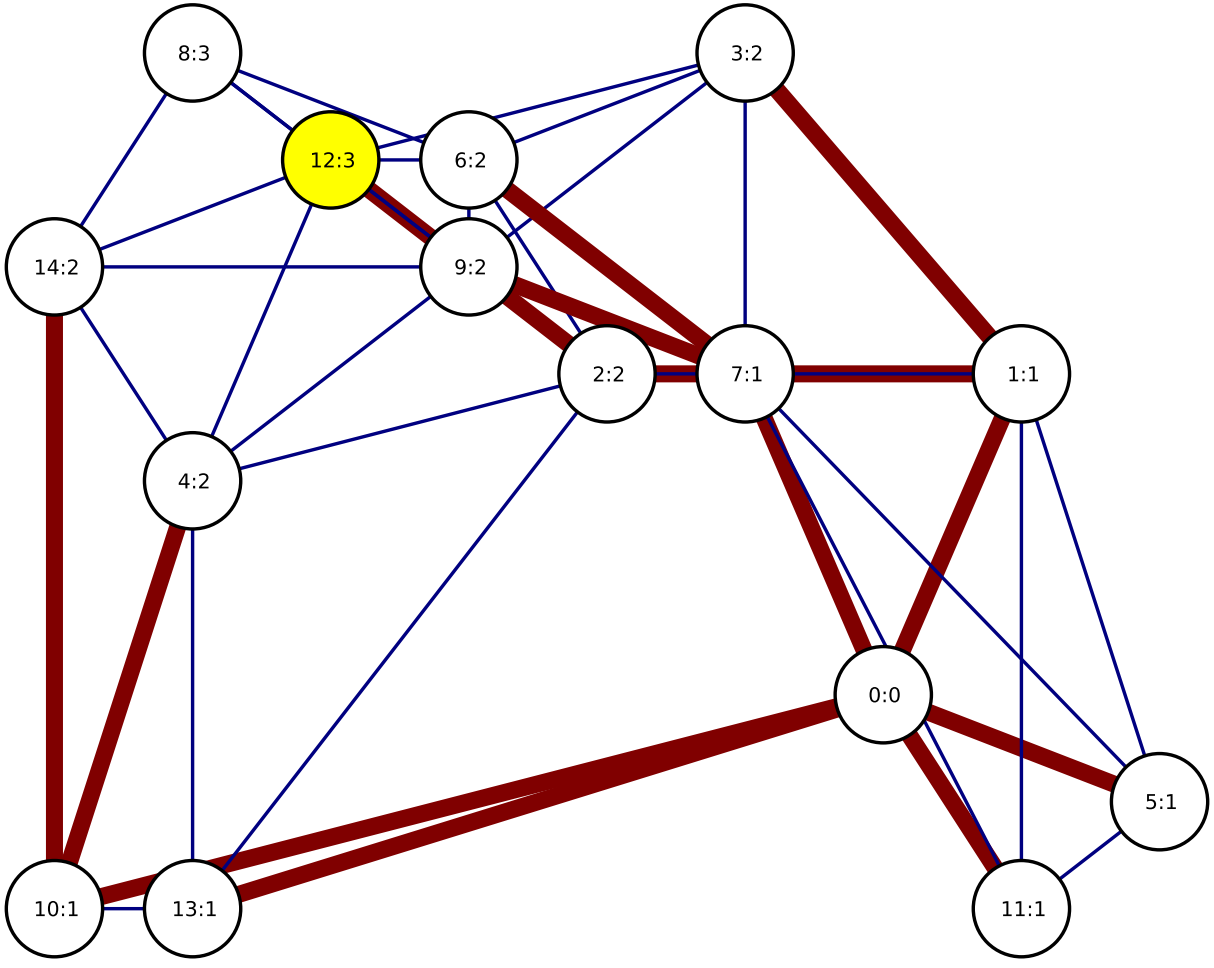

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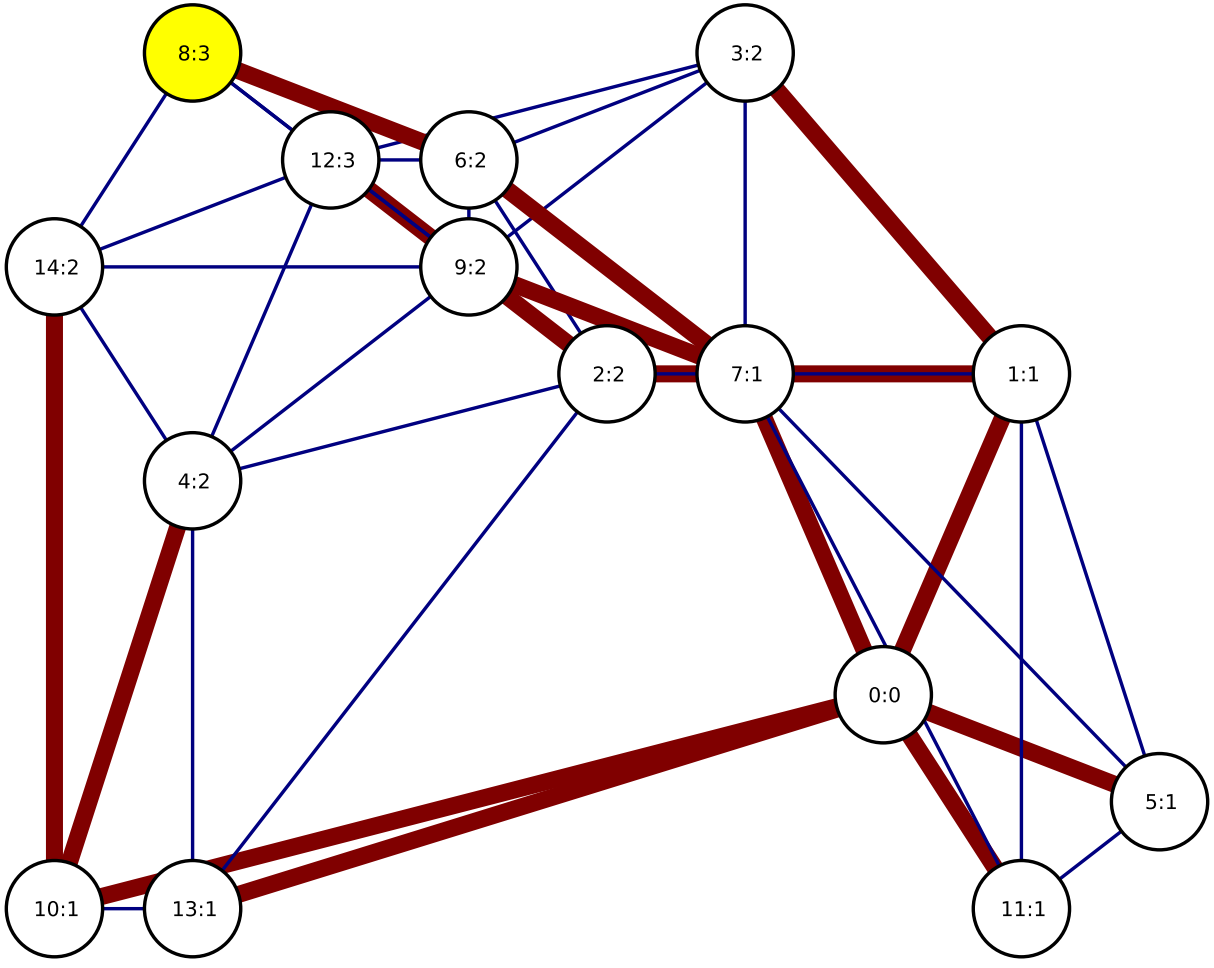


Depth-First Search

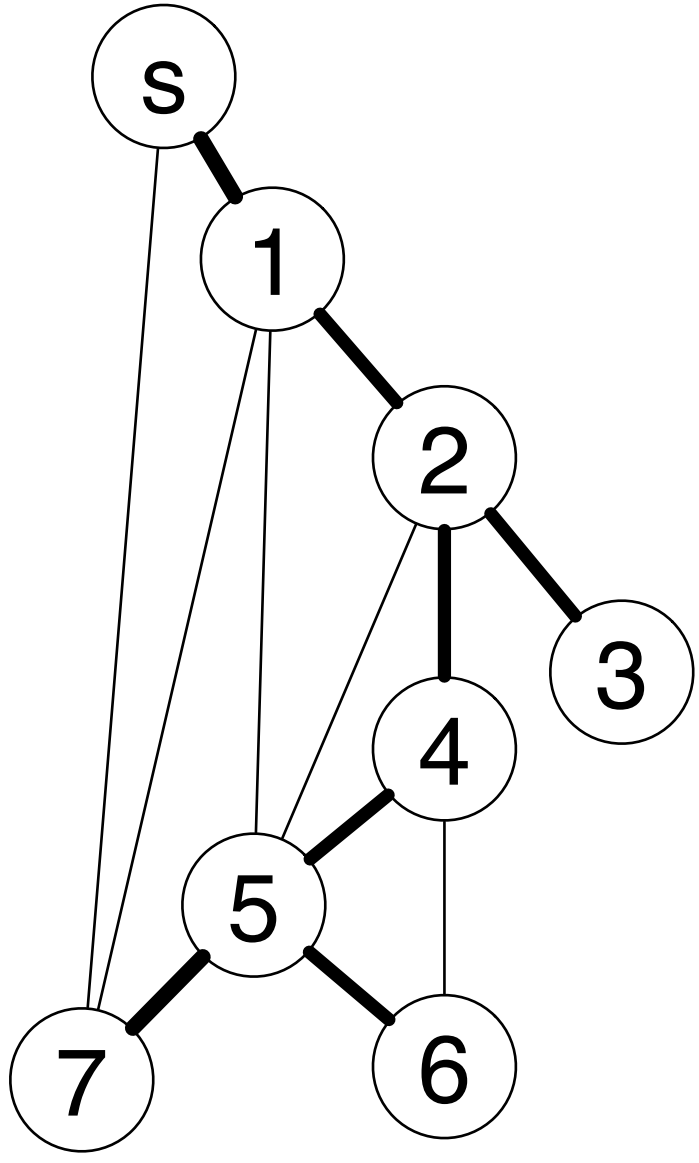
DFS keeps walking down a path until it is forced to backtrack.

It backtracks until it finds a new path to go down.

Think: Solving a maze.

It results in a search tree, called the depth-first search tree.

In general, the DFS tree will be very different than the BFS tree.



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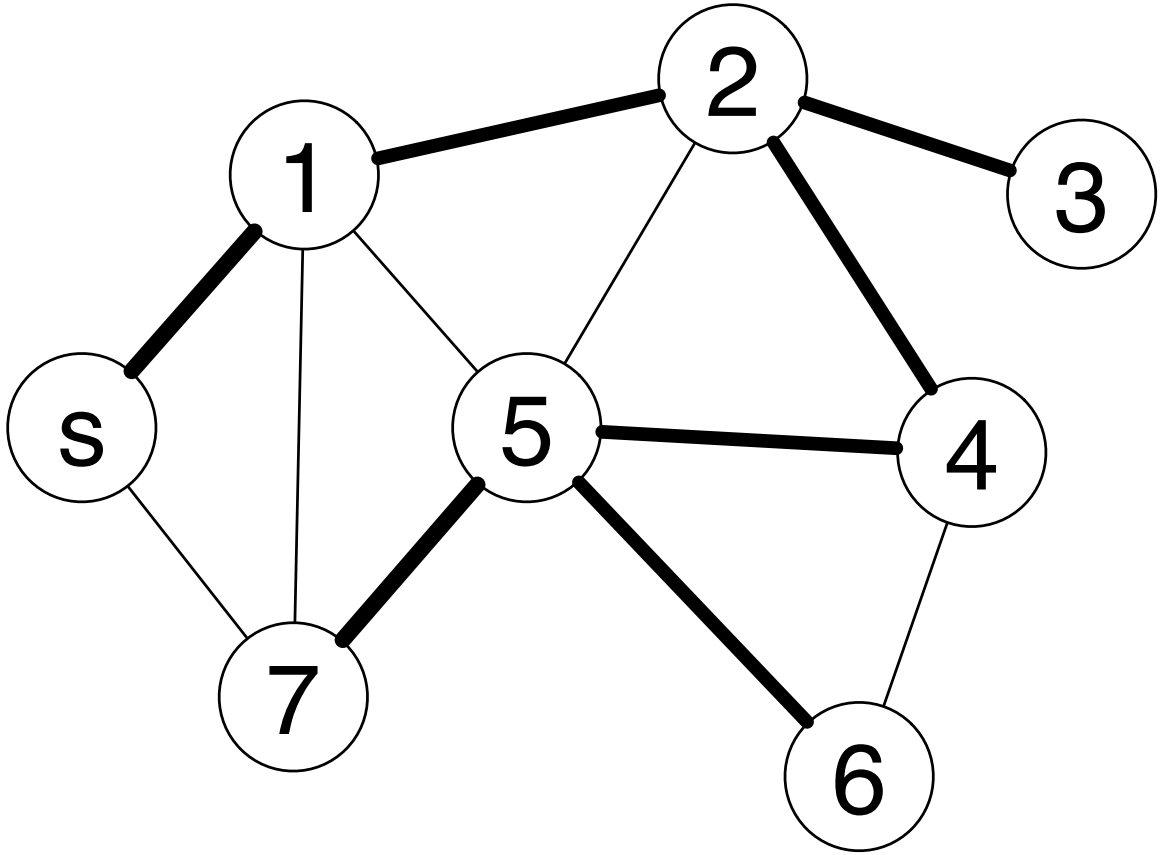
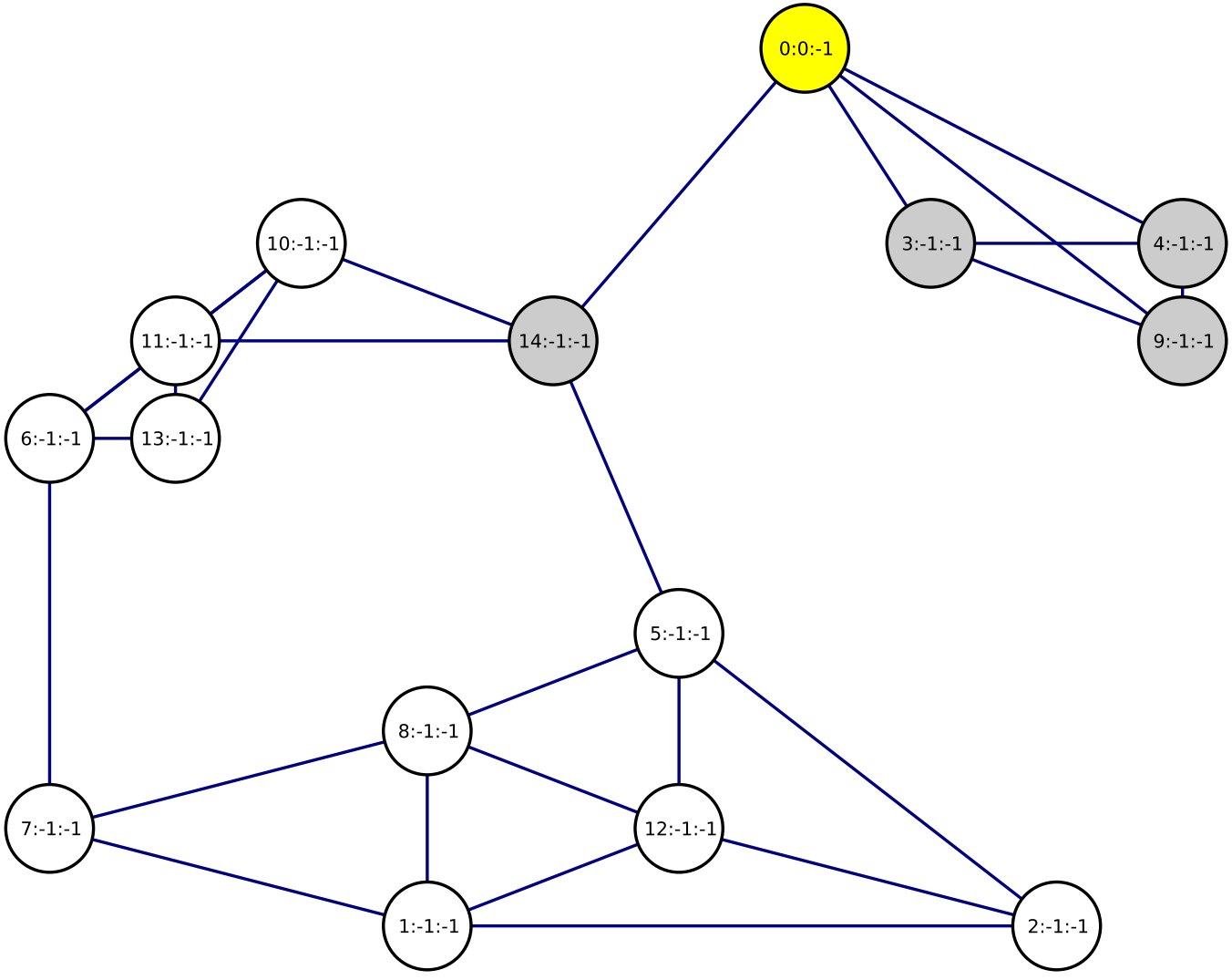


Example DFS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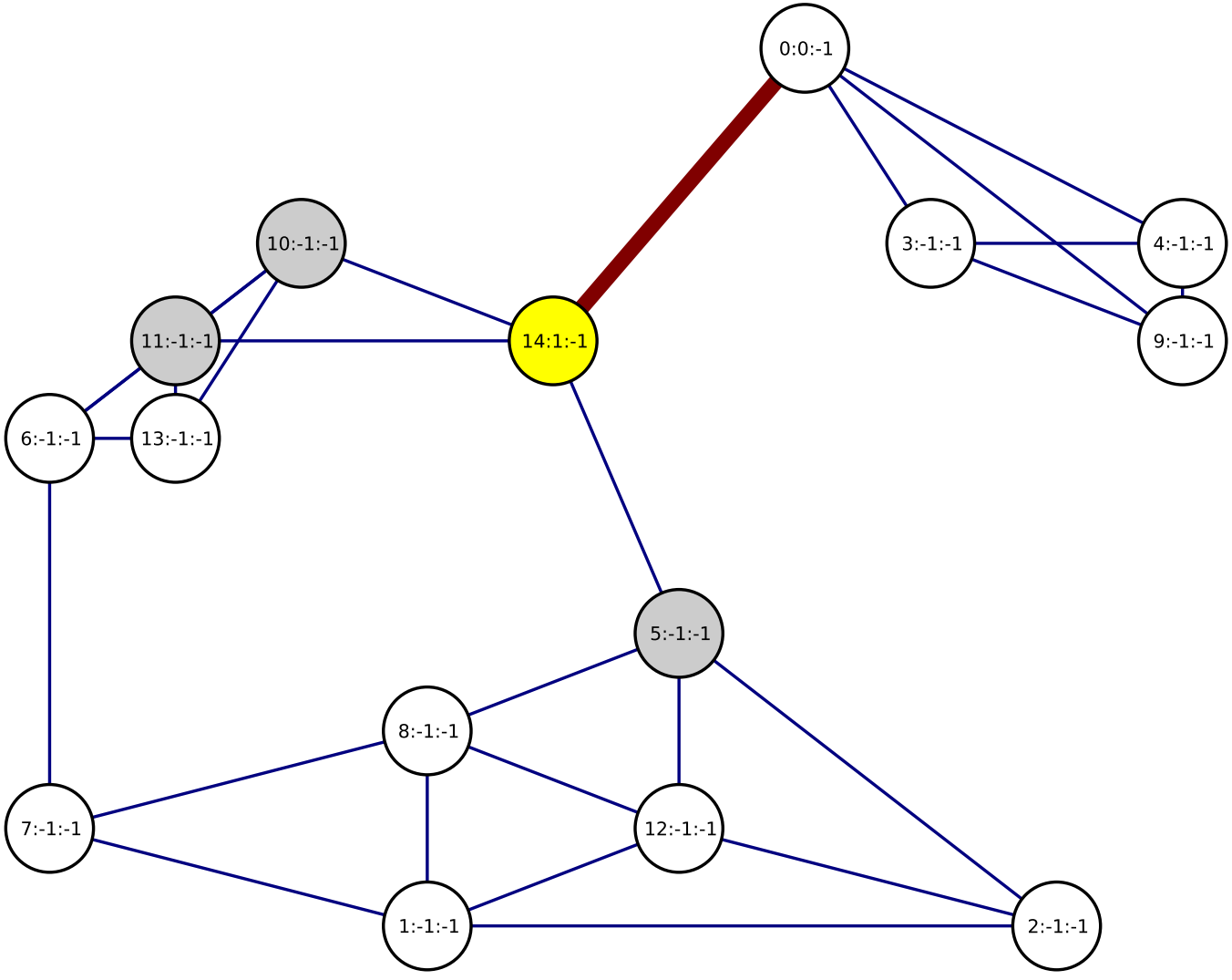
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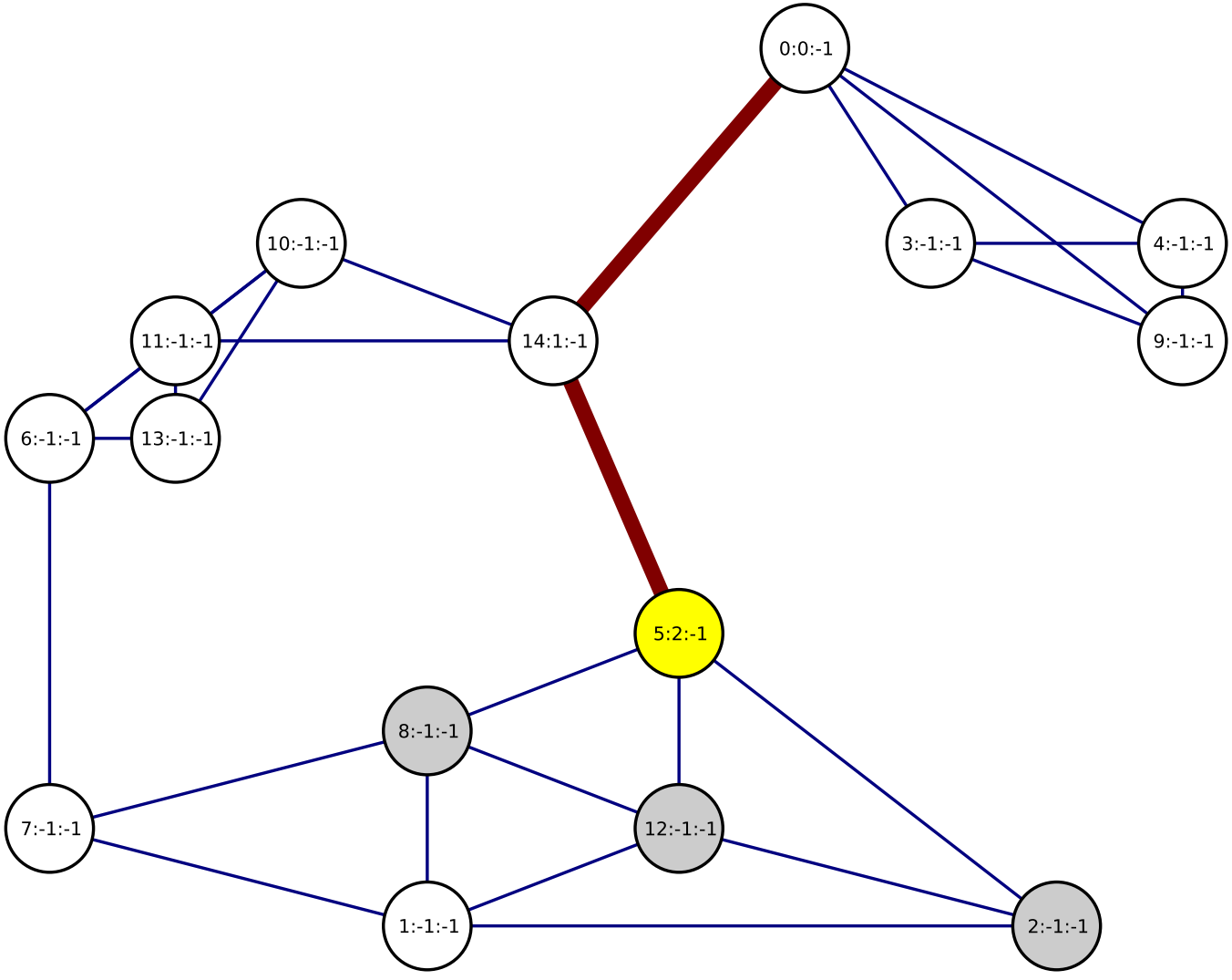
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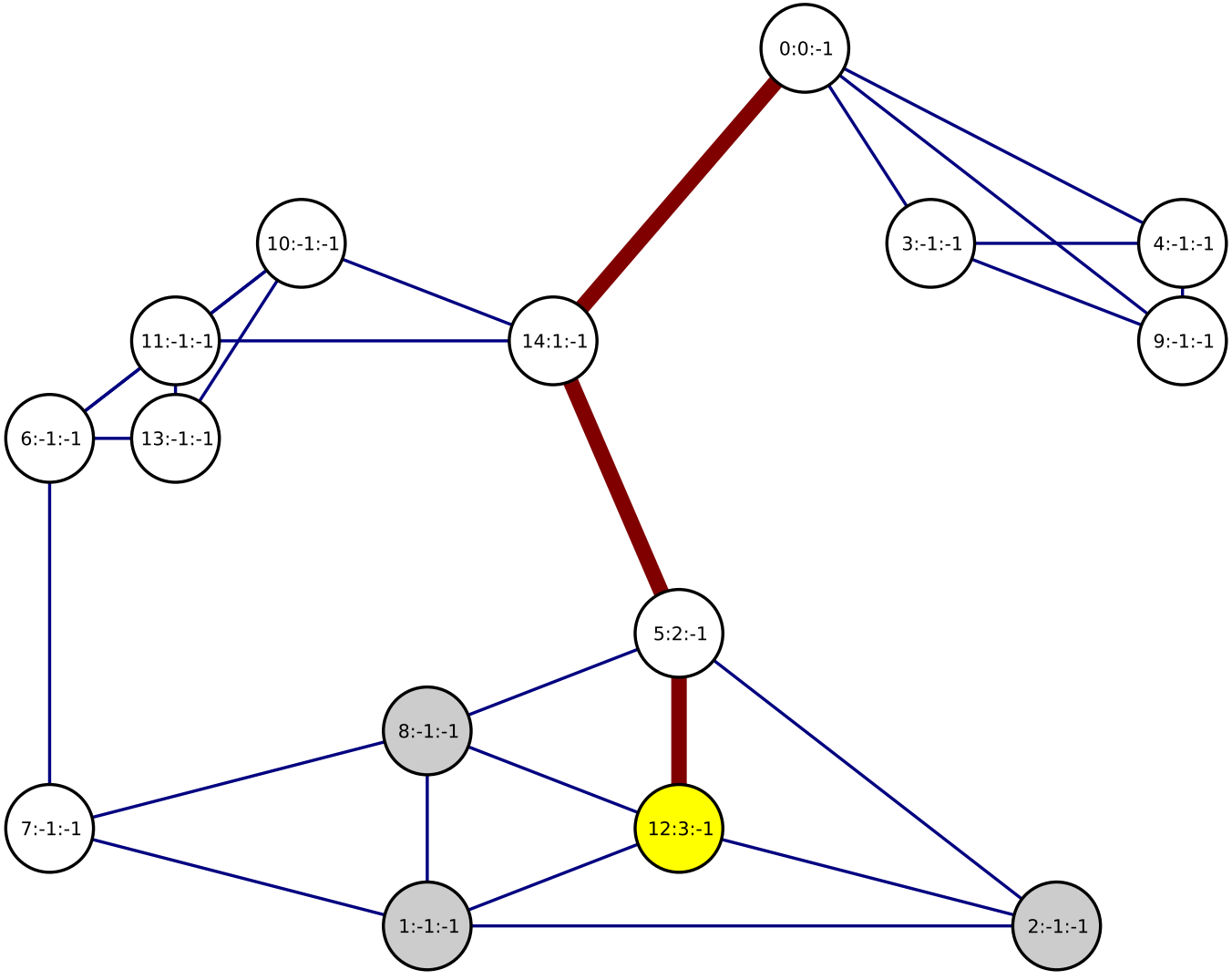
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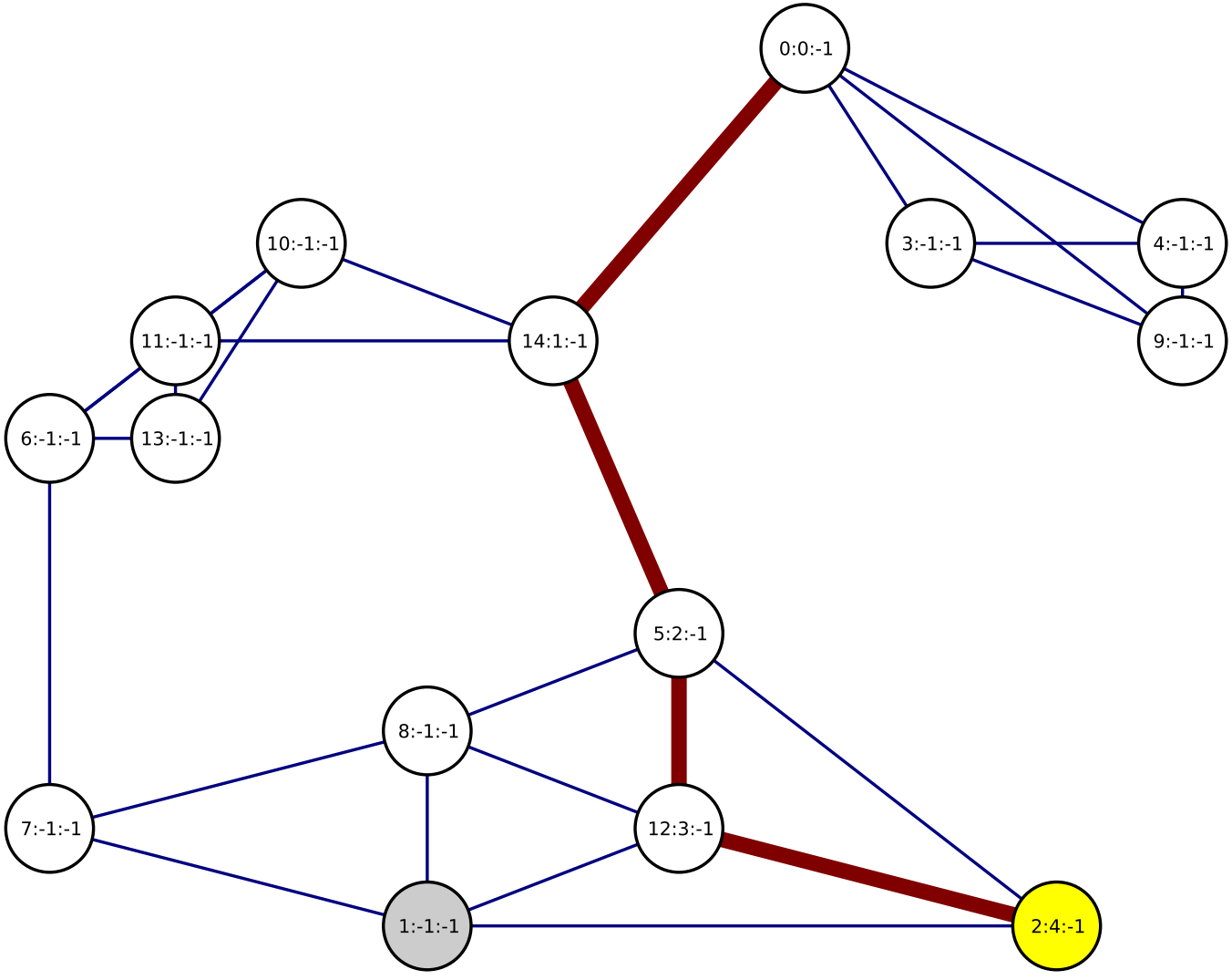
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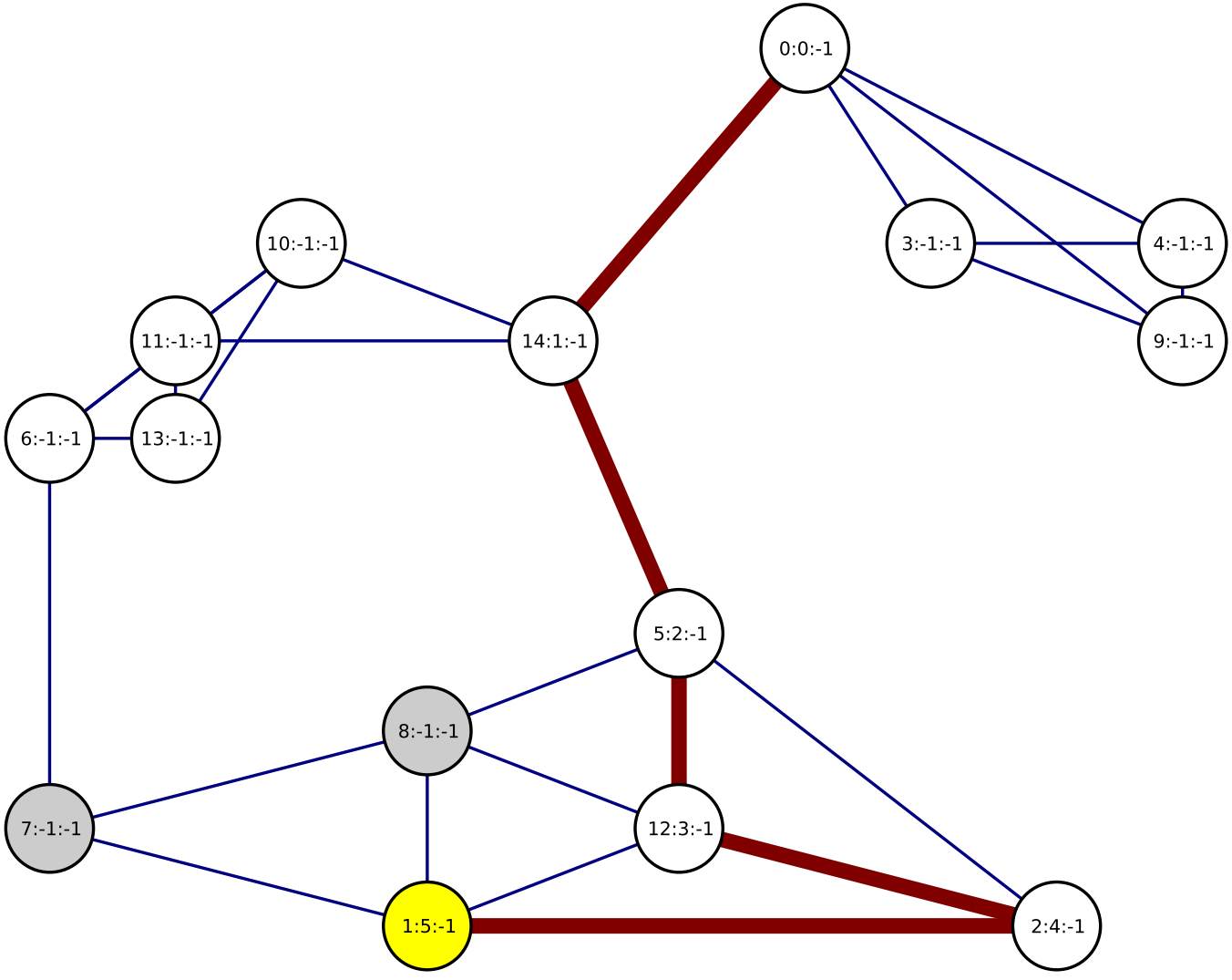
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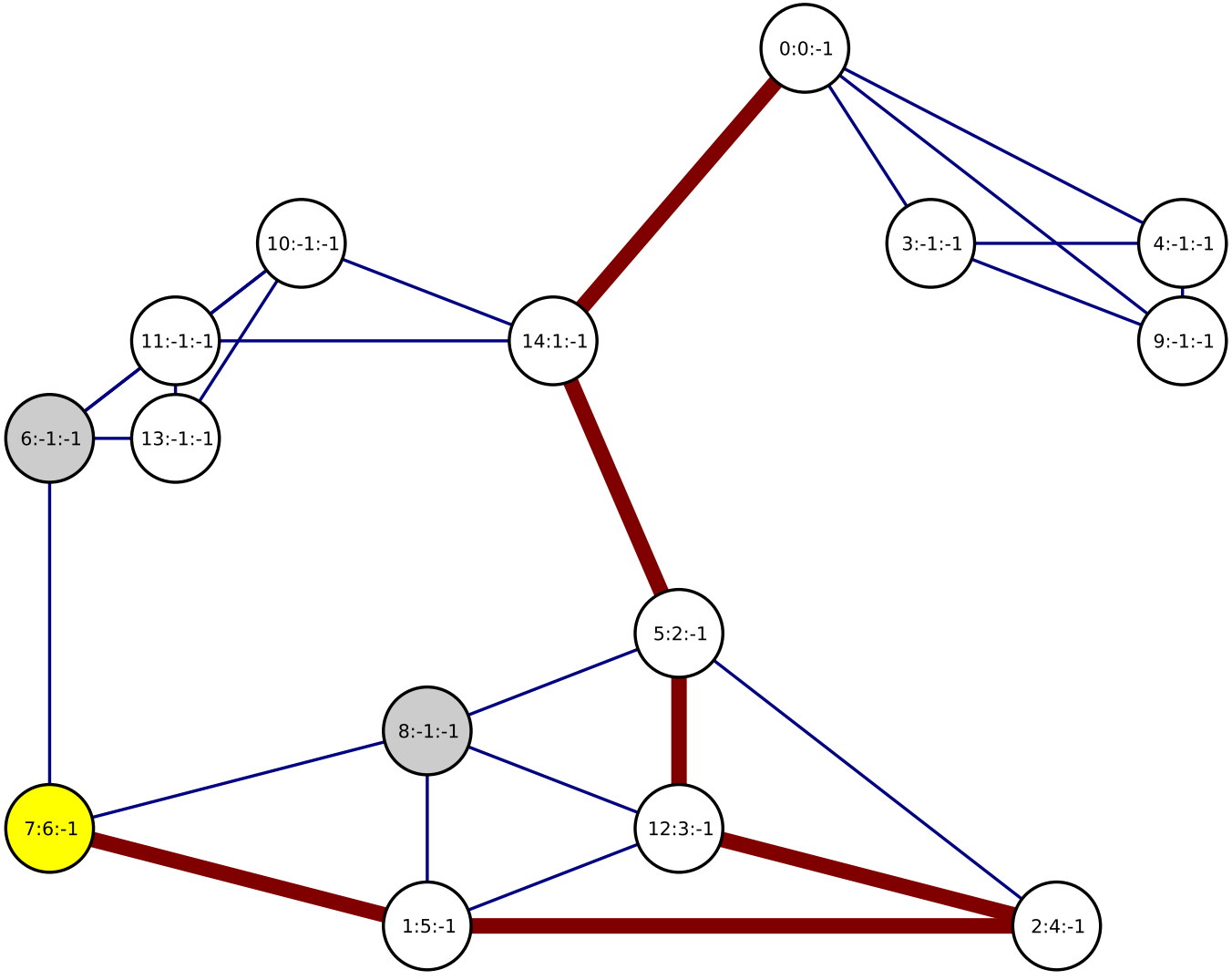
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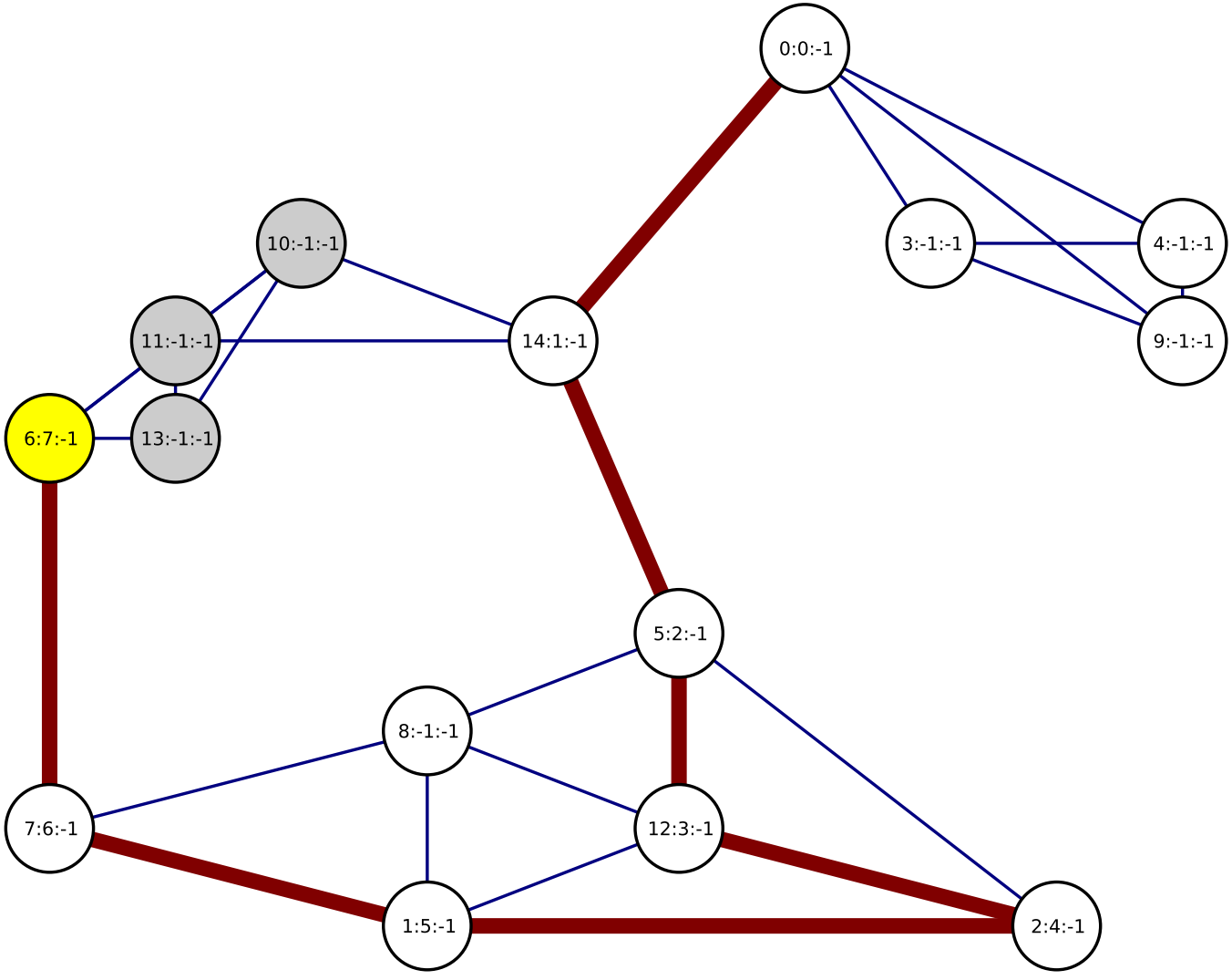
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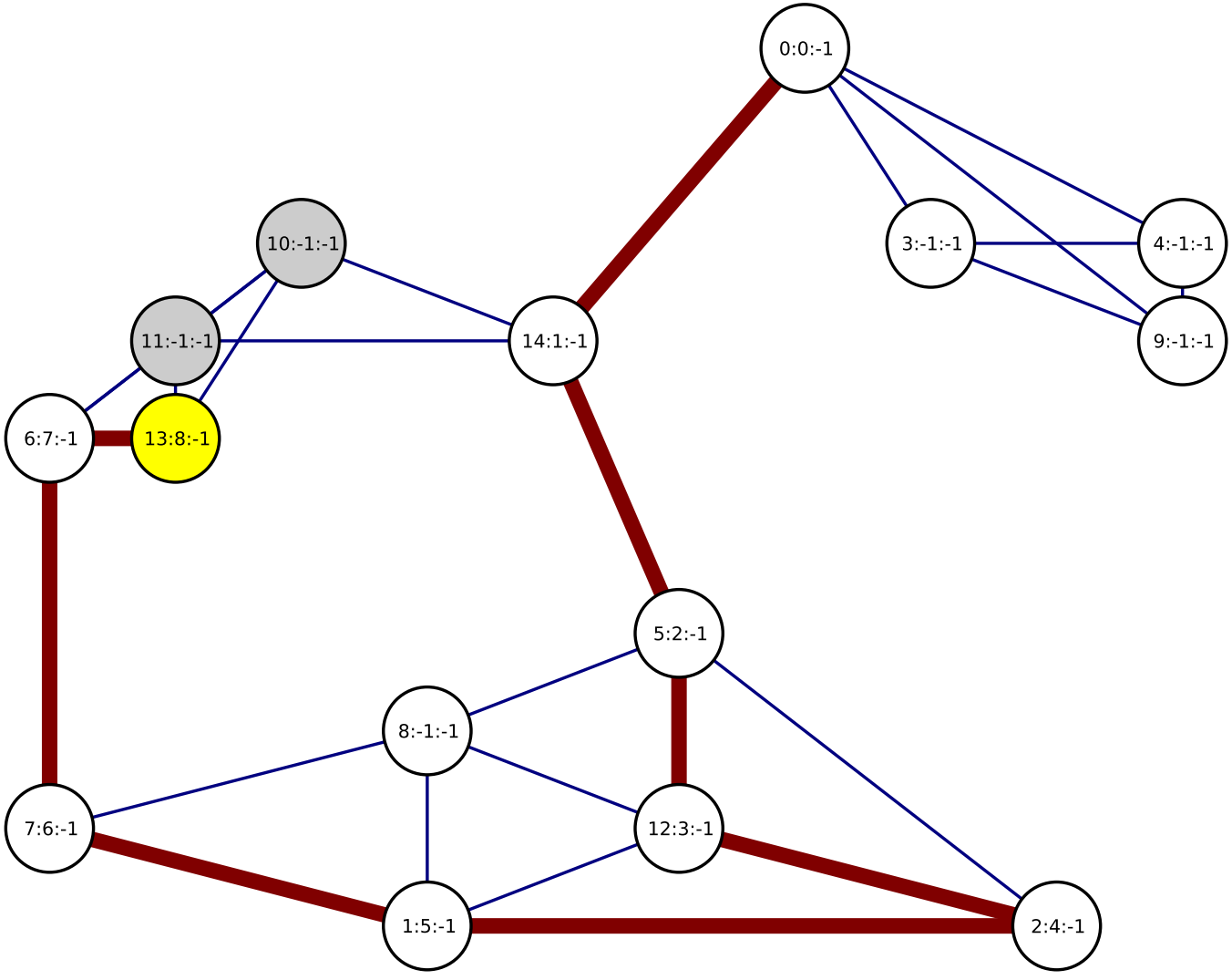
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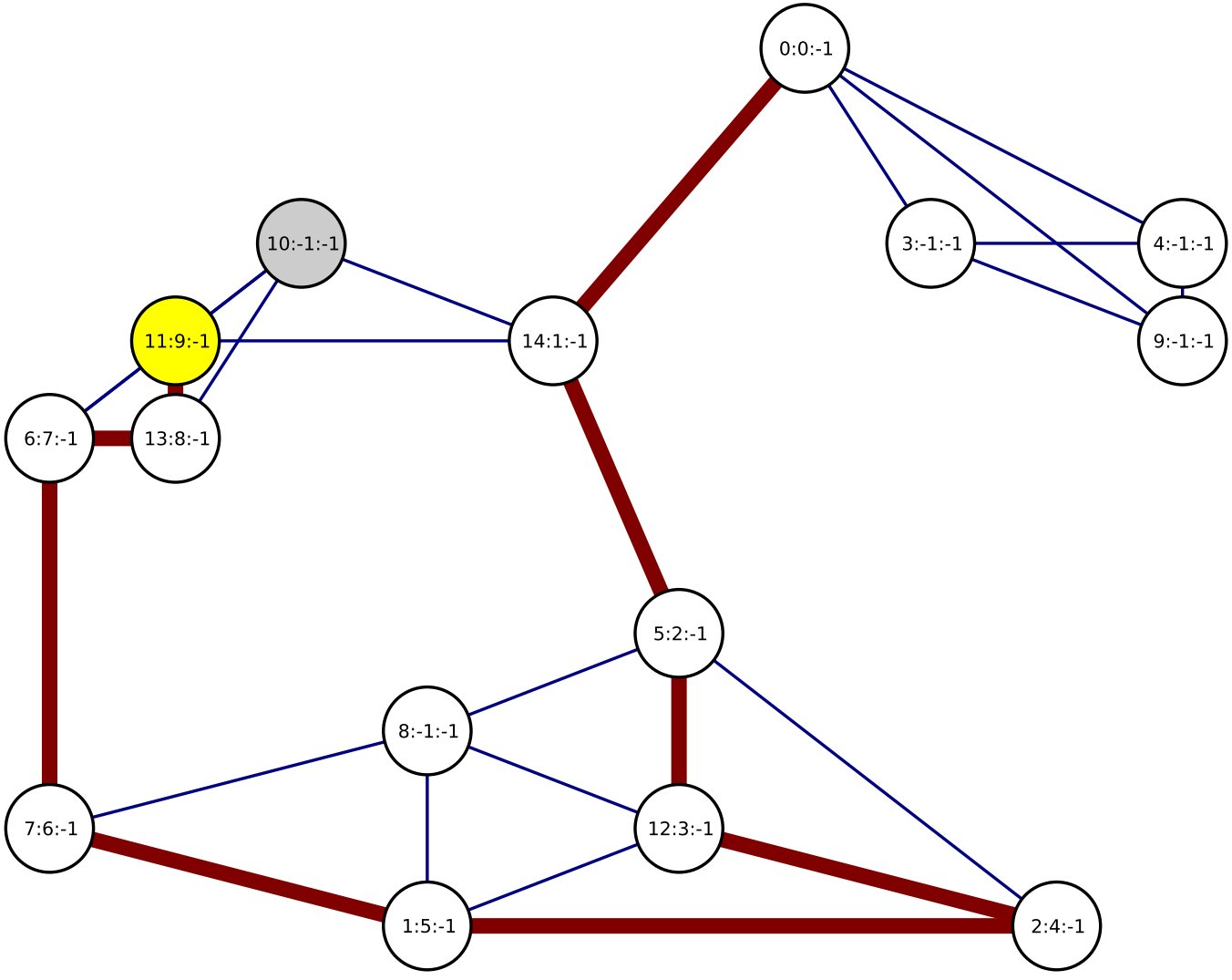
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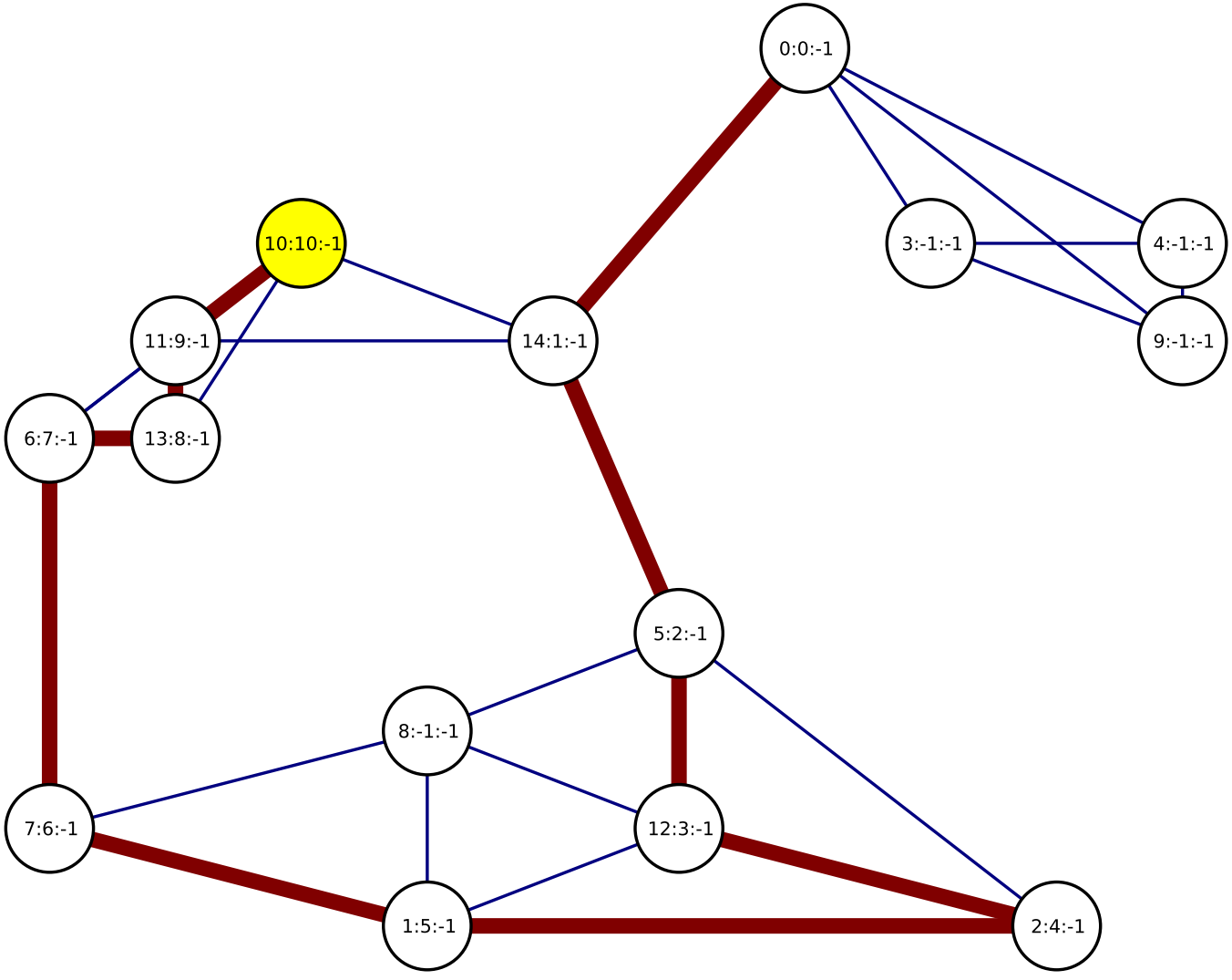
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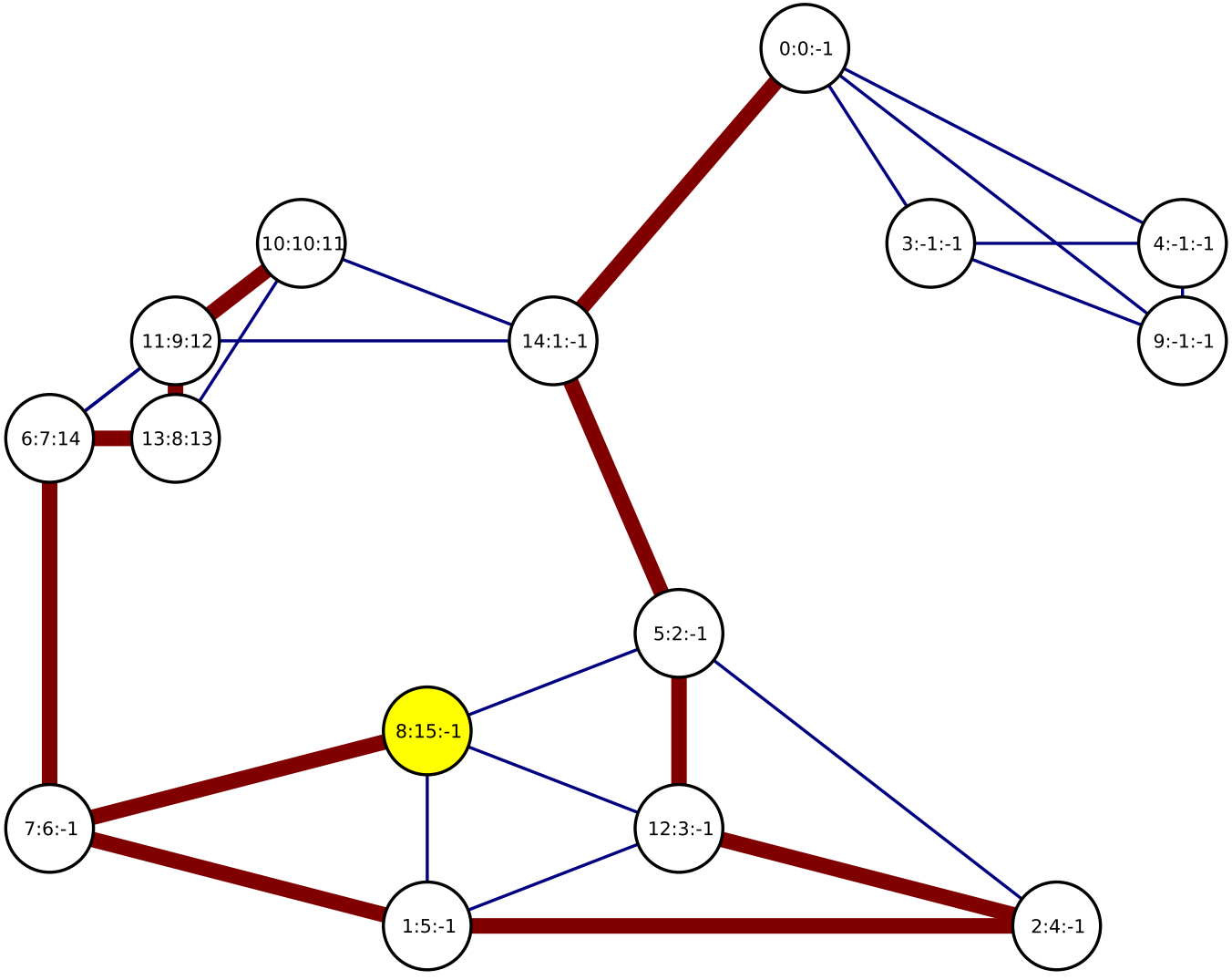
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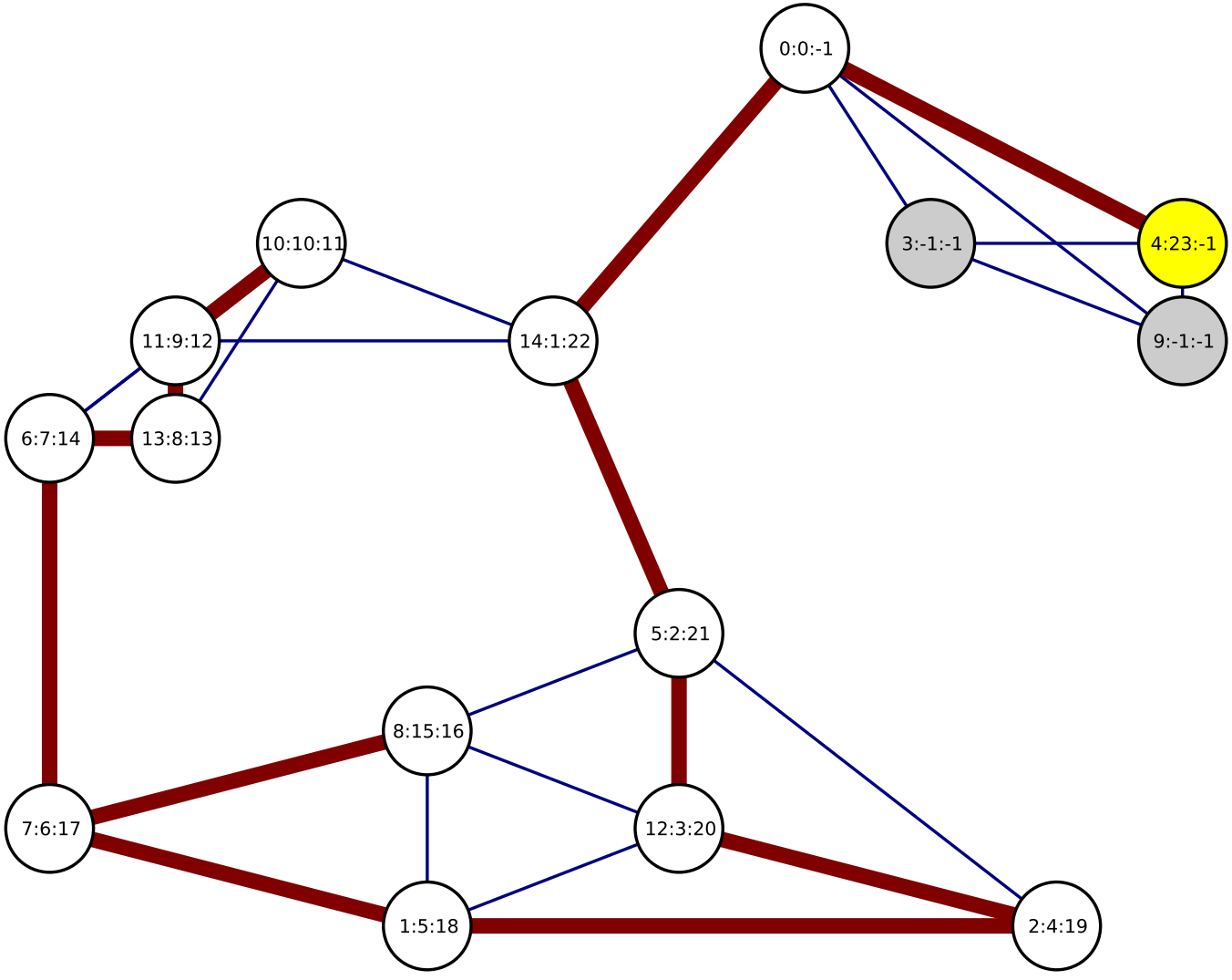
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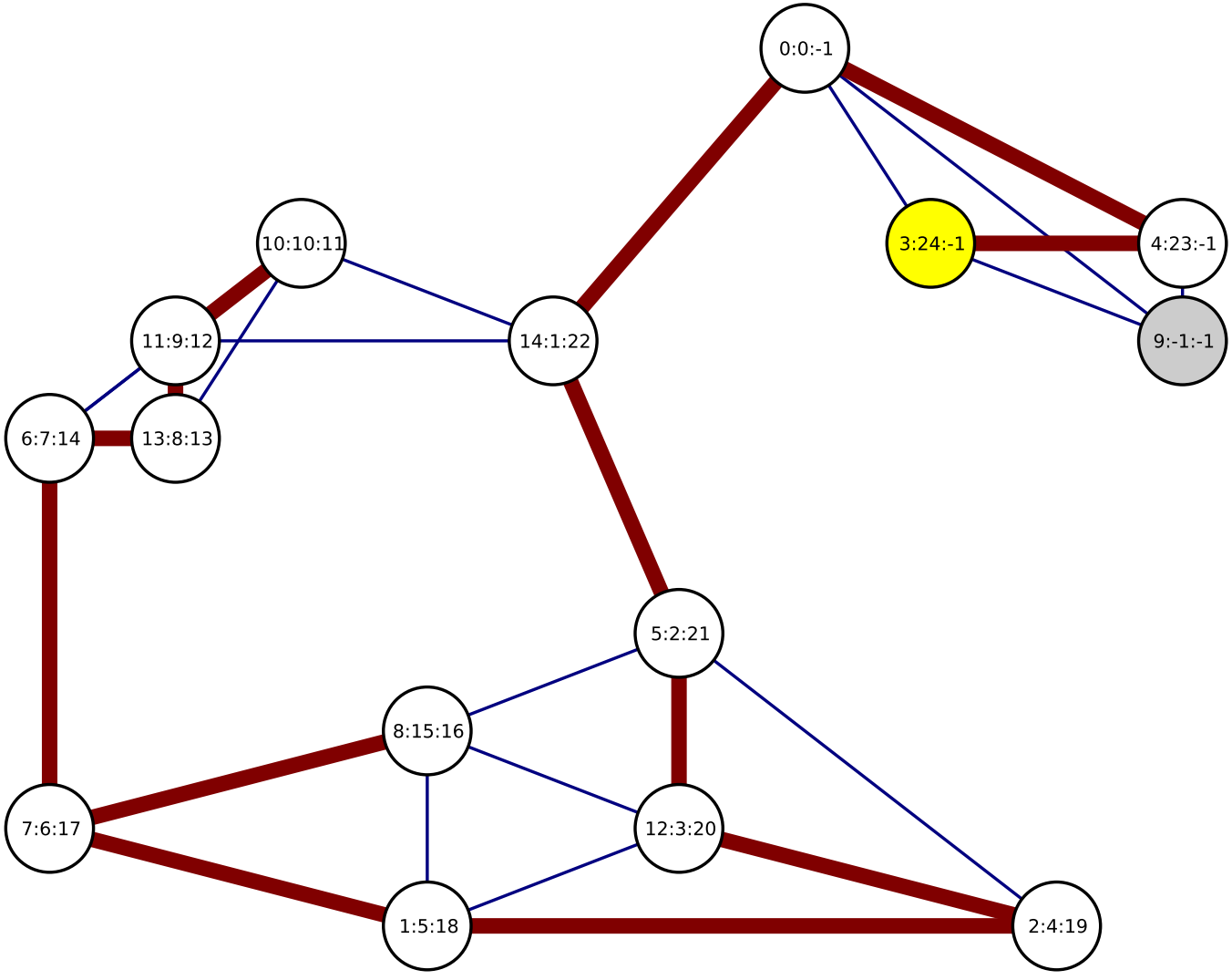
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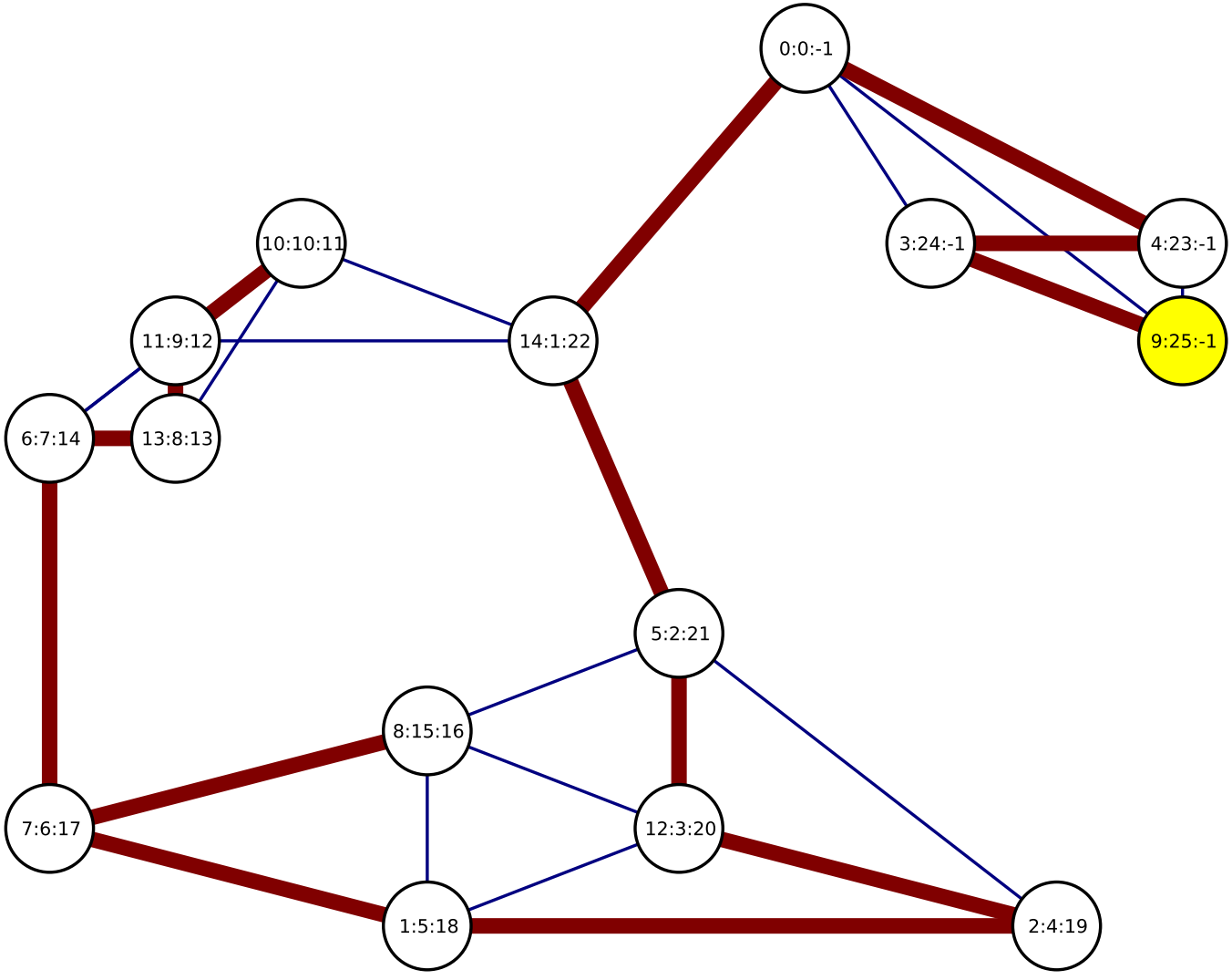
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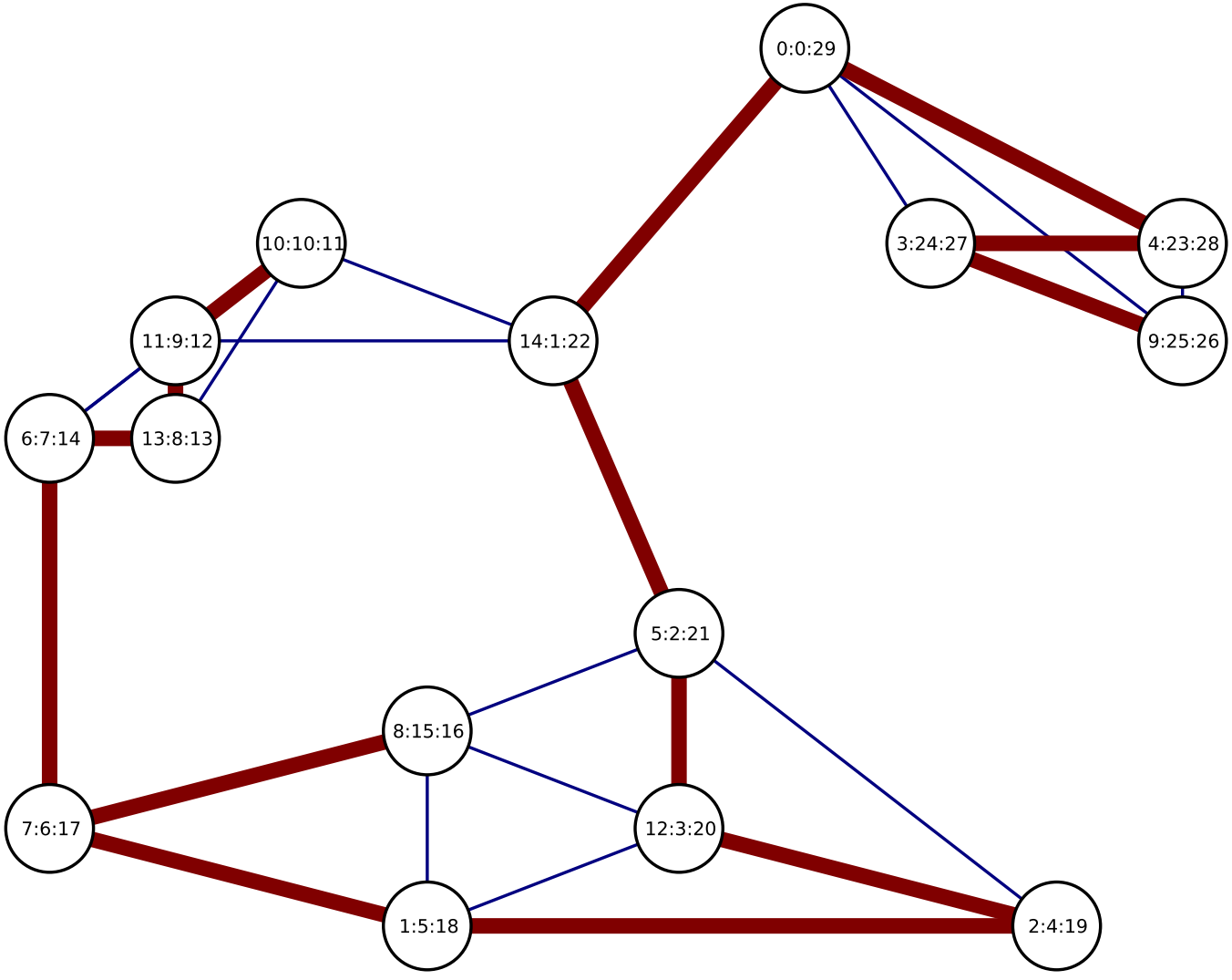
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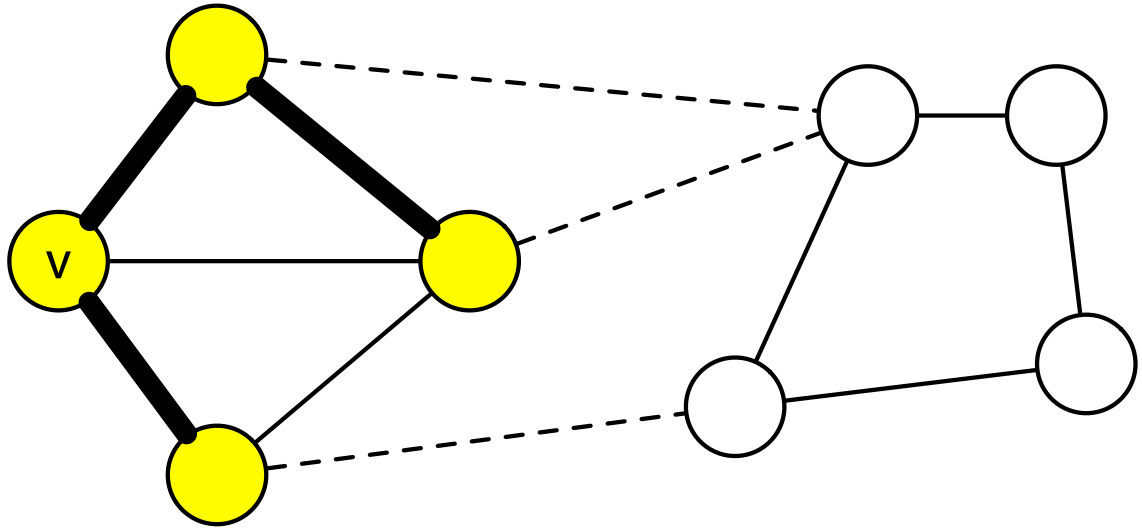
General Tree Growing (following Gross & Yellen)

We can think of BFS and DFS (and several other algorithms) as special cases of tree growing:

- Let T be the current tree T, and
- Maintain a list of frontier edges: the set of edges of G that have one endpoint in T and one endpoint not in T:



- Repeatedly choose a frontier edge (somehow) and add it to T.

Tree Growing

```
TreeGrowing(graph G, vertex v, func nextEdge):
    T = (v,∅)
    S = set of edges incident to v
    While S is not empty:
        e = nextEdge(G, S)
        T = T + e                    // add edge e to T
        S = updateFrontier(G, S, e)
    return T
```

- The function `nextEdge(G, S)` returns a frontier edge from S.
- `updateFrontier(G, S, e)` returns the new frontier after we add edge e to T.

Tree Growing

These algorithms are all special cases / variants of Tree Growing, with different versions of `nextEdge`:

1. Depth-first search
2. Breadth-first search
3. Prim's minimum spanning tree algorithm
4. Dijkstra's shortest path
5. A*

BFS & DFS as Tree Growing

What's `nextEdge` for DFS?

What's `nextEdge` for BFS?

BFS & DFS as Tree Growing

What's `nextEdge` for DFS?

> Select a frontier edge whose tree endpoint was discovered most recently.

Why? We can use a stack to implement DFS.
Runtime: $O(|\text{Edges}|)$

What's `nextEdge` for BFS?

BFS & DFS as Tree Growing

What's `nextEdge` for DFS?

> Select a frontier edge whose tree endpoint was discovered most recently.

Why? We can use a stack to implement DFS.
Runtime: $O(|\text{Edges}|)$

What's `nextEdge` for BFS?

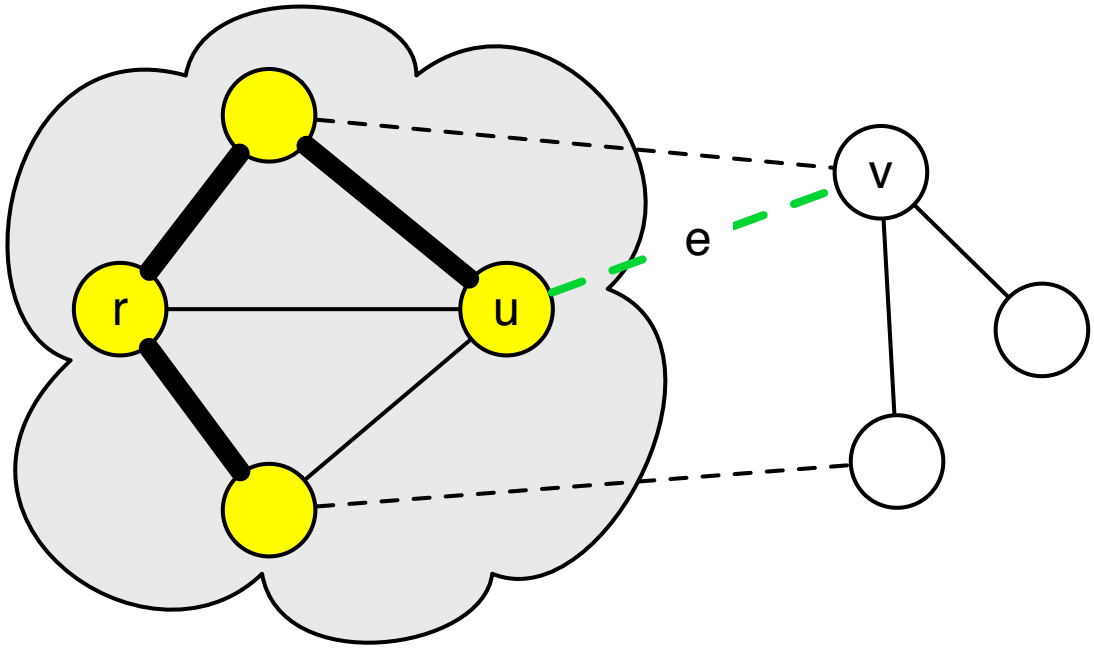
> Select a frontier edge whose tree endpoint was discovered earliest.

Why? We can use a queue to implement BFS.
Runtime: $O(|\text{Edges}|)$

Prim's Algorithm

Prim's Algorithm: Run `TreeGrowing` starting with any root node, adding the frontier edge with the smallest weight.

Theorem. *Prim's algorithm produces a minimum spanning tree.*



S = set of nodes already in the tree when e is added

Implementations of BFS and DFS

BFS implementation

```
procedure bfs(G, s):
    Q := queue containing only s
    while Q not empty
        v := Q.front(); Q.remove_front()
        for w ∈ G.neighbors(v):
            if w not seen:
                mark w seen
                Q.enqueue(w)
```

Recursive implementation of DFS

```
procedure dfs(G, u):
    while u has an unvisited neighbor in G
        v := an unvisited neighbor of u
        mark v visited
        dfs(G, v)
```

Stack-based implementation of DFS

```
procedure dfs(G, s):
    S := stack containing only s
    while S not empty
        v := S.pop()
        if v not visited:
            mark v visited
            for w ∈ G.neighbors(v): S.push(w)
```

Properties of BFS and DFS

Property of Non-BFS-Tree Edges

Theorem. *Choose $x \in L_i$ and $y \in L_j$ such that $\{x, y\}$ is an edge in undirected graph G. Then i and j differ by at most 1.*

In other words, edges of G that do not appear in the tree connect nodes either in the same layer or adjacent layer.

Proof. Suppose not, and that $i < j - 1$.

All the neighbors of x will be found by layer $i + 1$.

Therefore, the layer of y is less than $i + 1$, so $j \leq i + 1$, which contradicts $i < j - 1$. □

A property of Non-DFS-Tree Edges

Theorem. *Let x and y be nodes in the DFS tree T_G such that $\{x, y\}$ is an edge in undirected graph G. Then one of x or y is an ancestor of the other in T_G.*

Proof. Suppose, wlog, x is reached first in the DFS.

All the nodes that are marked explored between first encountering x and leaving x for the last time are descendants of x in T_G.

When we reach x, node y must not yet have been explored.

It must become explored before leaving x for the last time (otherwise, we should add $\{x, y\}$ to T_G). Hence, y is a descendent of x in T_G. □