15-494/694: Cognitive Robotics

Dave Touretzky

Lecture 12:
Backpropagation Learning

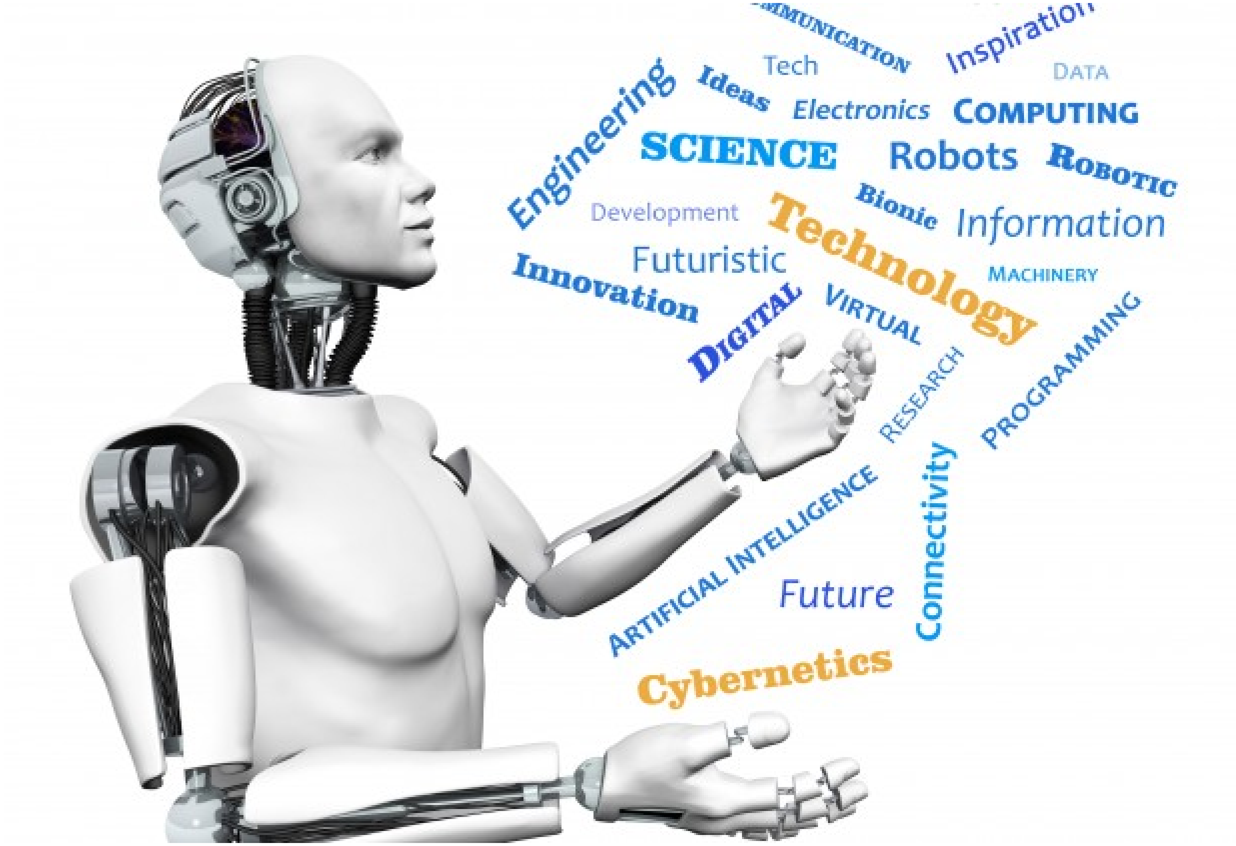
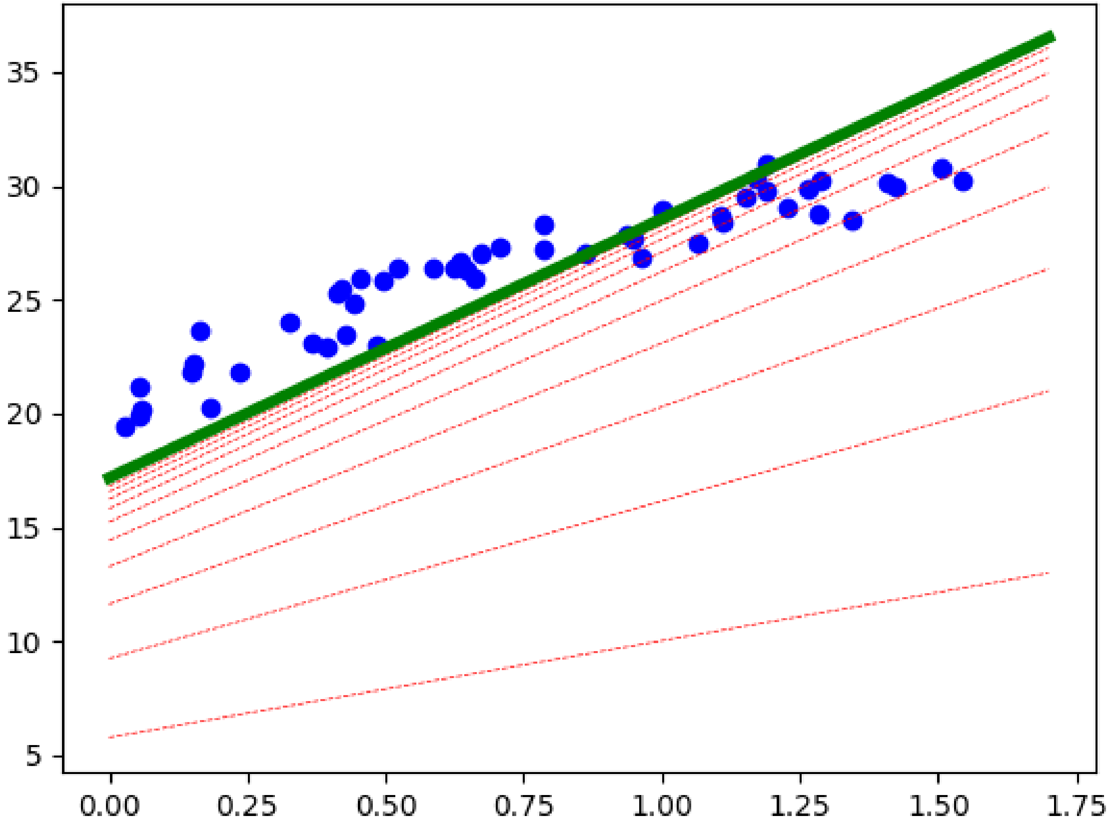
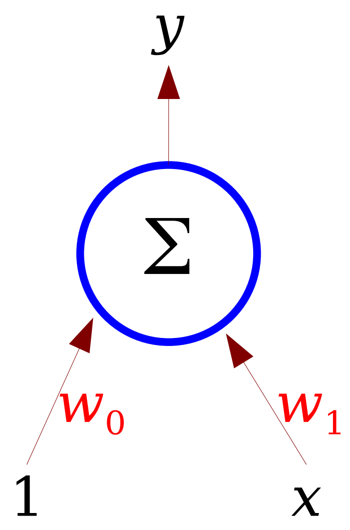


Image from http://www.futuristgerd.com/2015/09/10

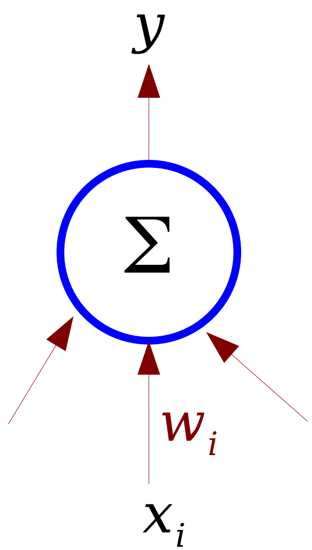
Training A Linear Unit

$$y = w_0 + w_1 \cdot x$$





LMS / Widrow-Hoff Rule



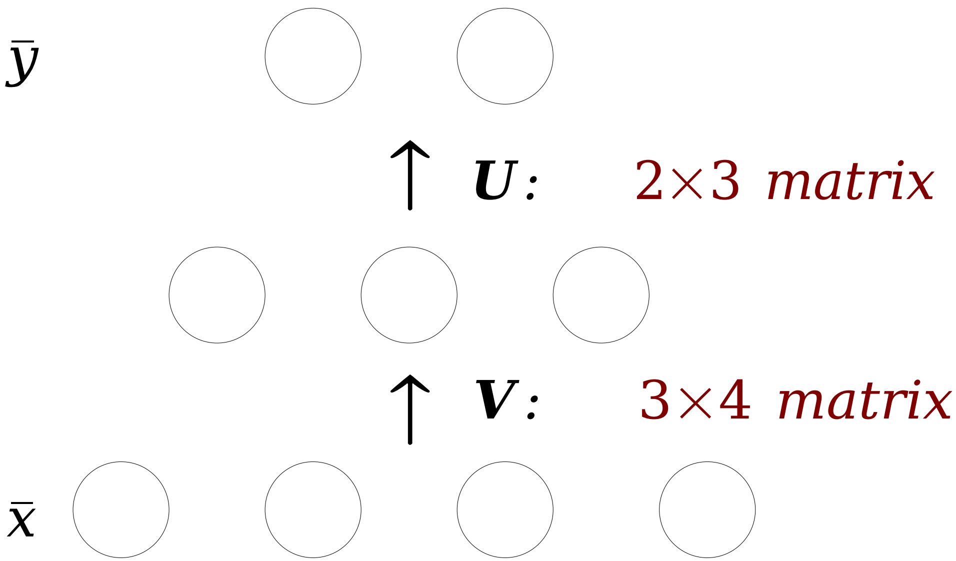
$$\Delta w_i = -\eta(y-d)x_i$$

η is a learning rate
($\mathit{could\ use}\ \eta=0.1$)

Works fine for a single layer of trainable weights.

What about multi-layer networks?

With Linear Units, Multiple Layers Don't Add Anything



$$\bar{y} = \boldsymbol{U} \times (\boldsymbol{V} \bar{x}) = \underbrace{(\boldsymbol{U} \times \boldsymbol{V})}_{2 \times 4} \bar{x}$$

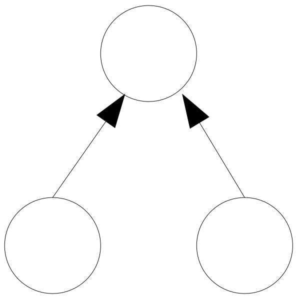
Linear operators are closed under composition.
Equivalent to a single layer of weights $\boldsymbol{W} = \boldsymbol{U} \times \boldsymbol{V}$

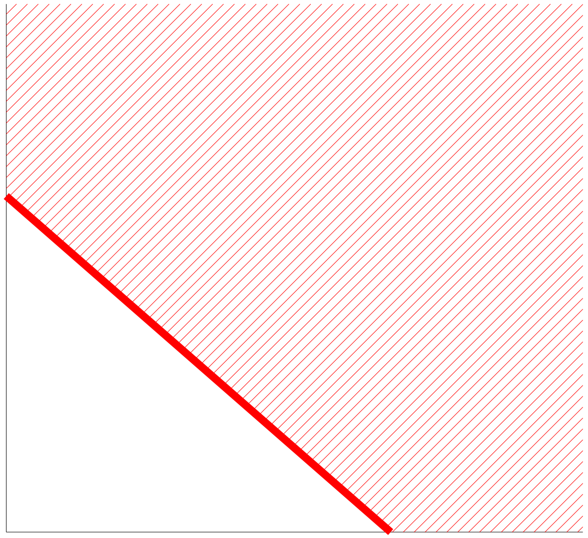
But with non-linear units, extra layers add computational power.

What Can be Done with Non-Linear (e.g., Threshold) Units?

$$y = h(w_0 + w_1 \cdot x_1 + w_2 \cdot x_2)$$

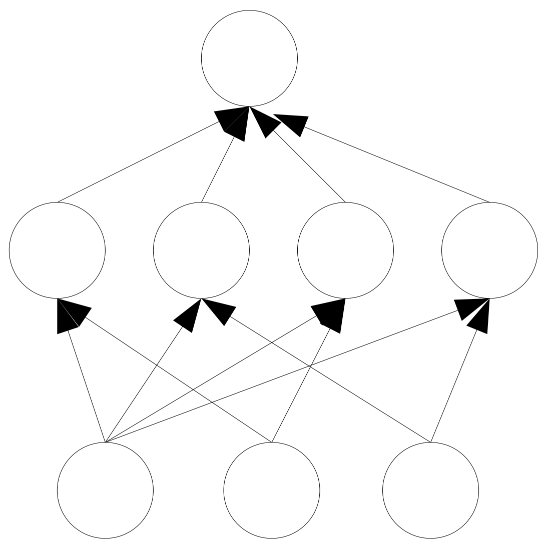
1 layer of trainable weights

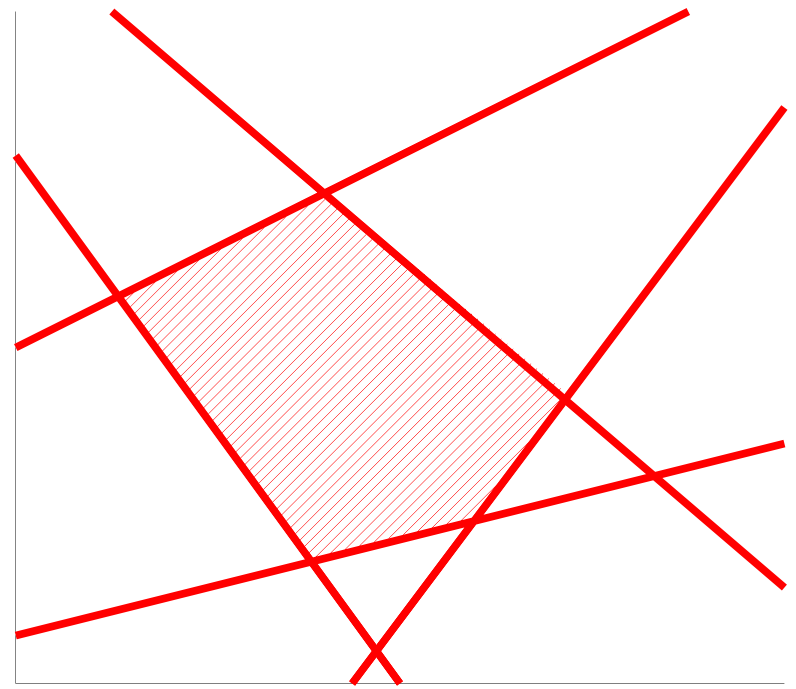




separating hyperplane

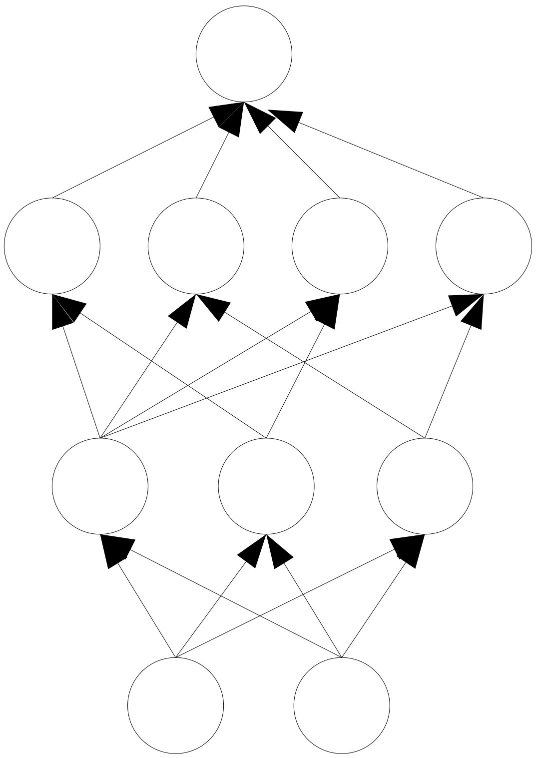
2 layers of trainable weights

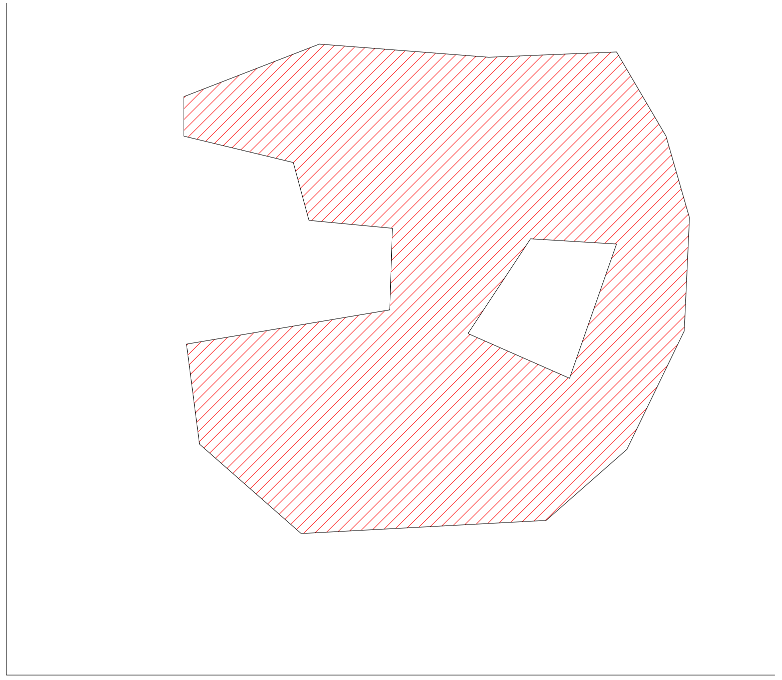




convex polygon region

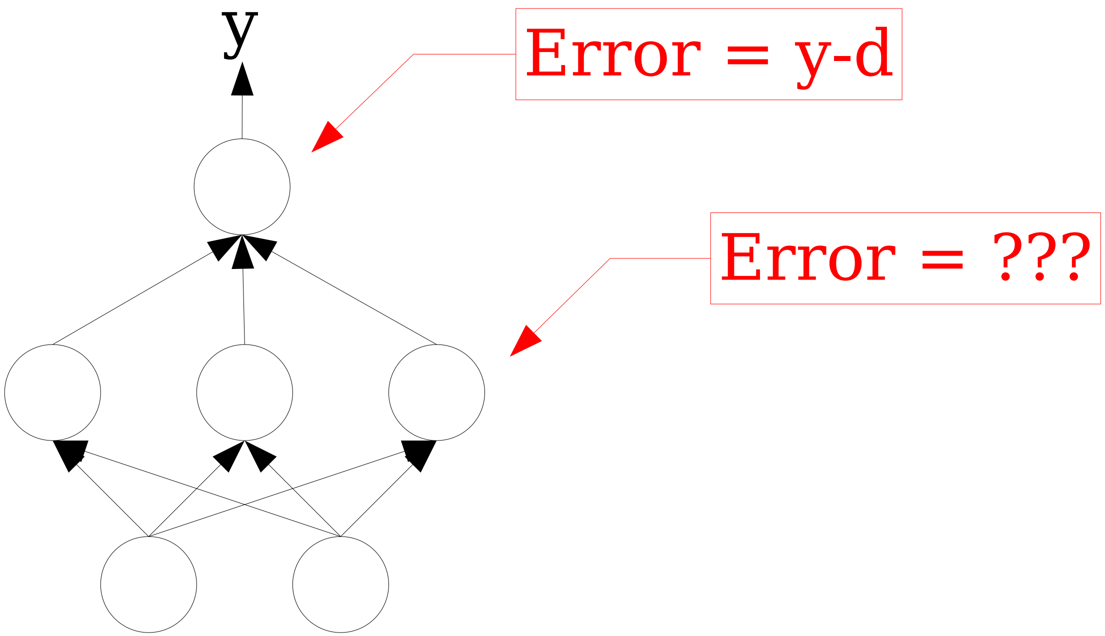
3 layers of trainable weights





composition of polygons: arbitrary regions

How Do We Train A Multi-Layer Network?

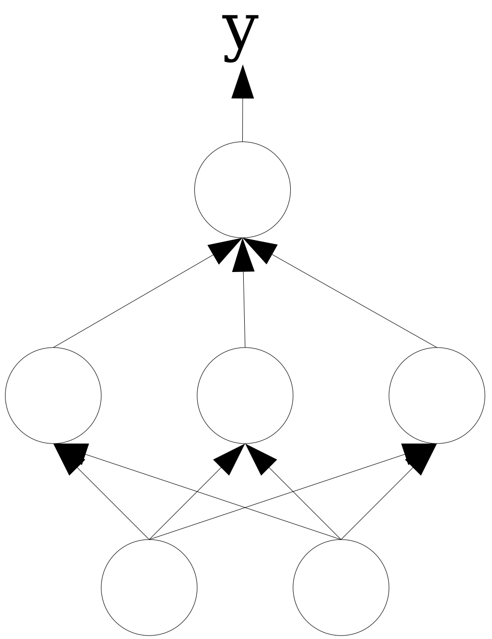


Can't use perceptron training algorithm because we don't know the 'correct' outputs for hidden units.

How Do We Train A Multi-Layer Network?

Define sum-squared error:

$$E = \frac{1}{2}\sum_{p}(d^p - y^p)^2$$



Use gradient descent error minimization:

$$\Delta w_{ij} = -\eta \frac{\partial E}{\partial w_{ij}}$$

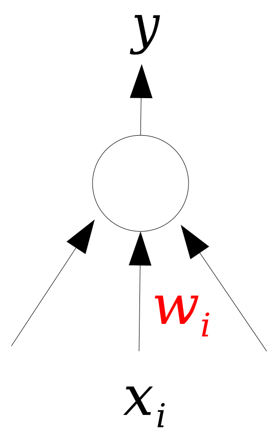
Works if the nonlinear transfer function is differentiable.

Deriving the LMS or "Delta" Rule As Gradient Descent Learning

$$y = \sum_i w_i x_i$$

$$E = \frac{1}{2} \sum_p \left(d^p - y^p\right)^2$$

$$\frac{dE}{dy} = y - d$$



$$\frac{\partial E}{\partial w_i} = \frac{dE}{dy} \cdot \frac{\partial y}{\partial w_i} = (y - d) x_i$$

$$\Delta w_i = -\eta \frac{\partial E}{\partial w_i} = -\eta (y - d) x_i$$

How do we extend this to two layers?

Switch to Smooth Nonlinear Units

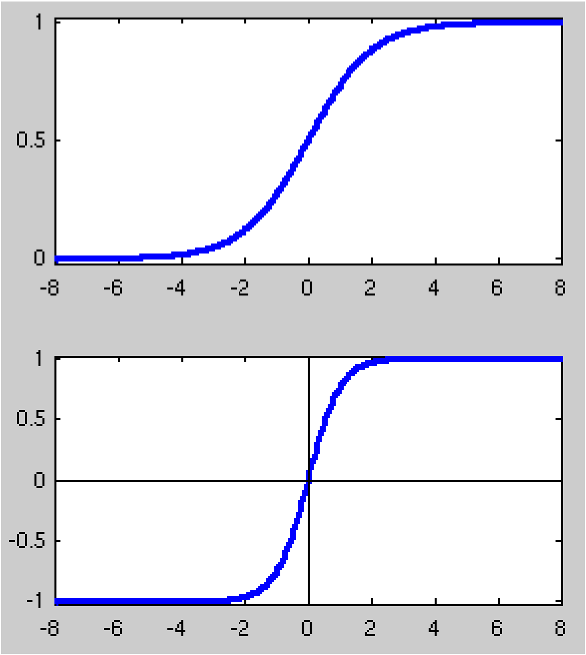
$$\text{net}_j = \sum_i w_{ij} y_i$$

$$y_j = \text{g}(\text{net}_j)$$ *g must be differentiable*

Common choices for g:

$$g(x) = \frac{1}{1+e^{-x}}$$

$$g'(x) = g(x) \cdot (1 - g(x))$$

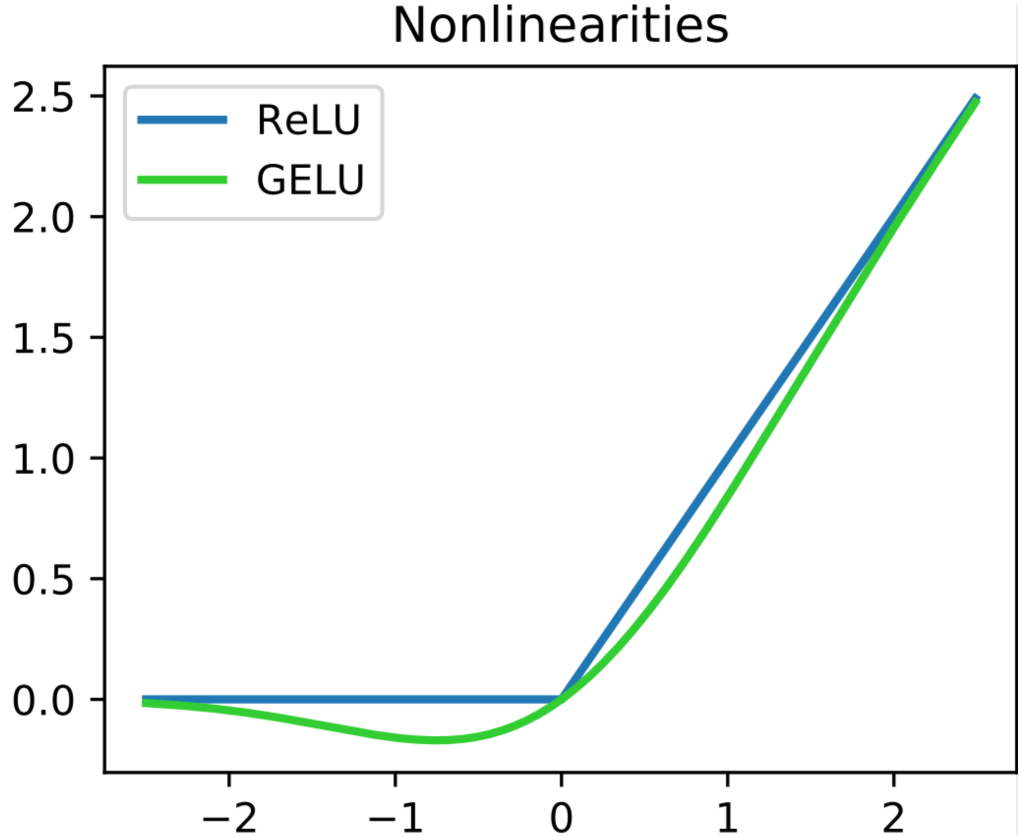


$$g(x) = \tanh(x)$$

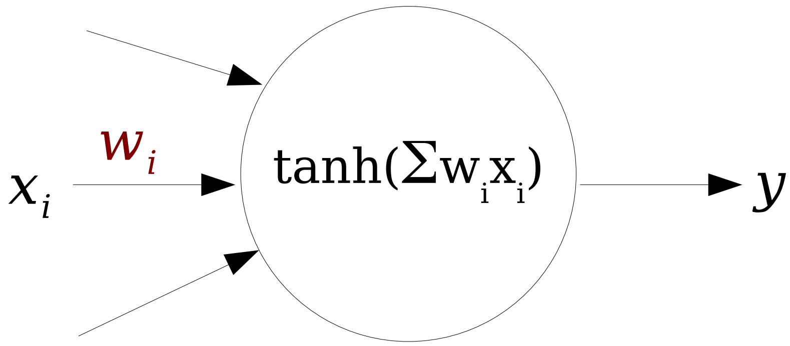
$$g'(x) = 1/\cosh^2(x)$$

ReLU and GELU Functions

- ReLU: Rectified Linear Unit
- GELU: Gaussian Error Linear Unit



Gradient Descent with Nonlinear Units

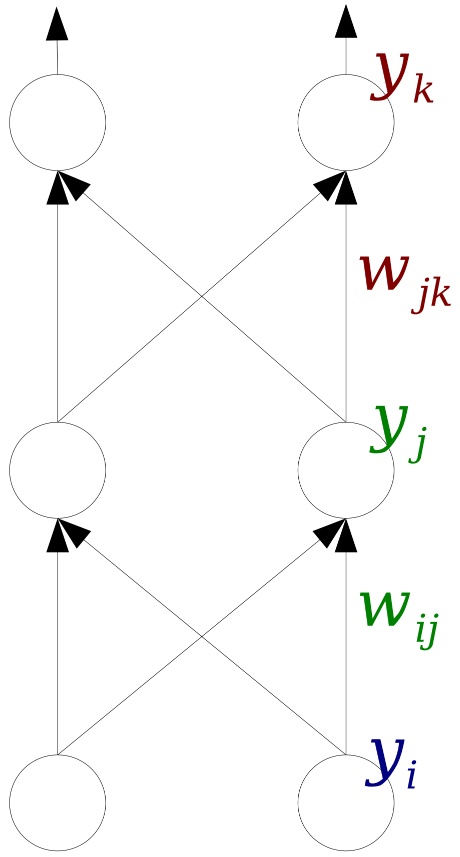


$$y = g(net) = \tanh\left(\sum_i w_i x_i\right)$$

$$\frac{dE}{dy} = (y-d), \qquad \frac{dy}{d\,net} = 1/\cosh^2(net), \qquad \frac{\partial\,net}{\partial w_i} = x_i$$

$$\begin{aligned}\frac{\partial E}{\partial w_i} &= \frac{dE}{dy}\cdot\frac{dy}{d\,net}\cdot\frac{\partial\,net}{\partial w_i} \\ &= (y-d)/\cosh^2\left(\sum_i w_i x_i\right)\cdot x_i\end{aligned}$$

Now We Can Use The Chain Rule



$$\frac{\partial E}{\partial y_k} = (y_k - d_k)$$

$$\delta_k = \frac{\partial E}{\partial net_k} = (y_k - d_k) \cdot g'(net_k)$$

$$\frac{\partial E}{\partial w_{jk}} = \frac{\partial E}{\partial net_k} \cdot \frac{\partial net_k}{\partial w_{jk}} = \frac{\partial E}{\partial net_k} \cdot y_j$$

$$\frac{\partial E}{\partial y_j} = \sum_k \left(\frac{\partial E}{\partial net_k} \cdot \frac{\partial net_k}{\partial y_j} \right)$$

$$\delta_j = \frac{\partial E}{\partial net_j} = \frac{\partial E}{\partial y_j} \cdot g'(net_j)$$

$$\frac{\partial E}{\partial w_{ij}} = \frac{\partial E}{\partial net_j} \cdot y_i$$

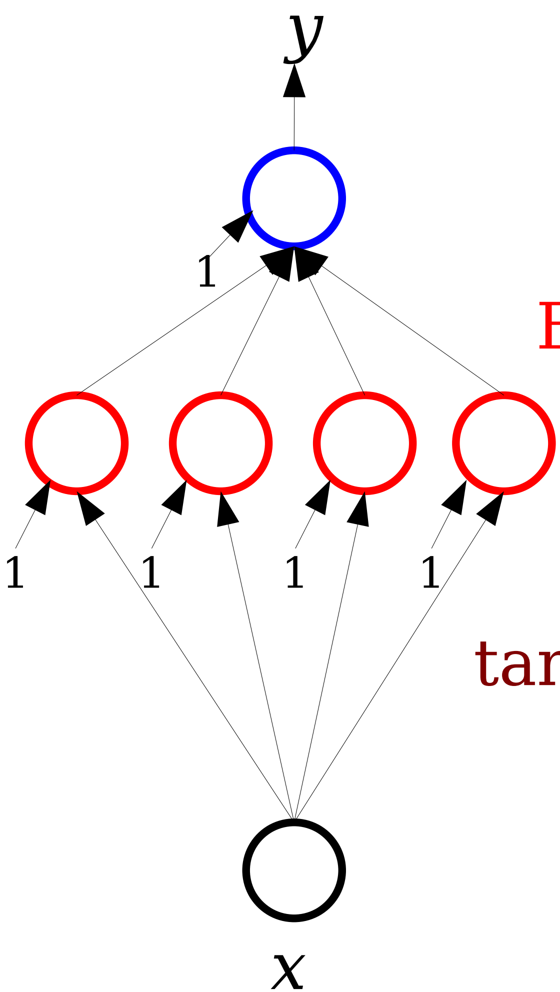
Weight Updates

$$\frac{\partial E}{\partial w_{jk}} = \frac{\partial E}{\partial net_k} \cdot \frac{\partial net_k}{\partial w_{jk}} = \delta_k \cdot y_j$$

$$\frac{\partial E}{\partial w_{ij}} = \frac{\partial E}{\partial net_j} \cdot \frac{\partial net_j}{\partial w_{ij}} = \delta_j \cdot y_i$$

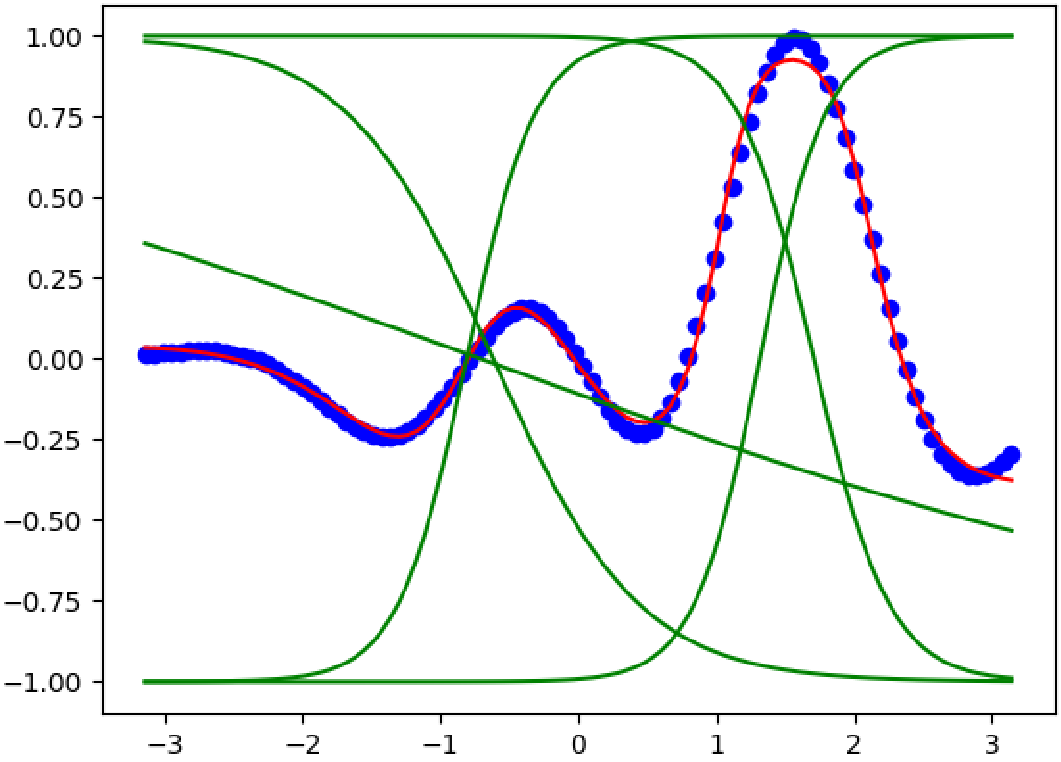
$$\Delta w_{jk} = -\eta \cdot \frac{\partial E}{\partial w_{jk}} \qquad \Delta w_{ij} = -\eta \cdot \frac{\partial E}{\partial w_{ij}}$$

Function Approximation



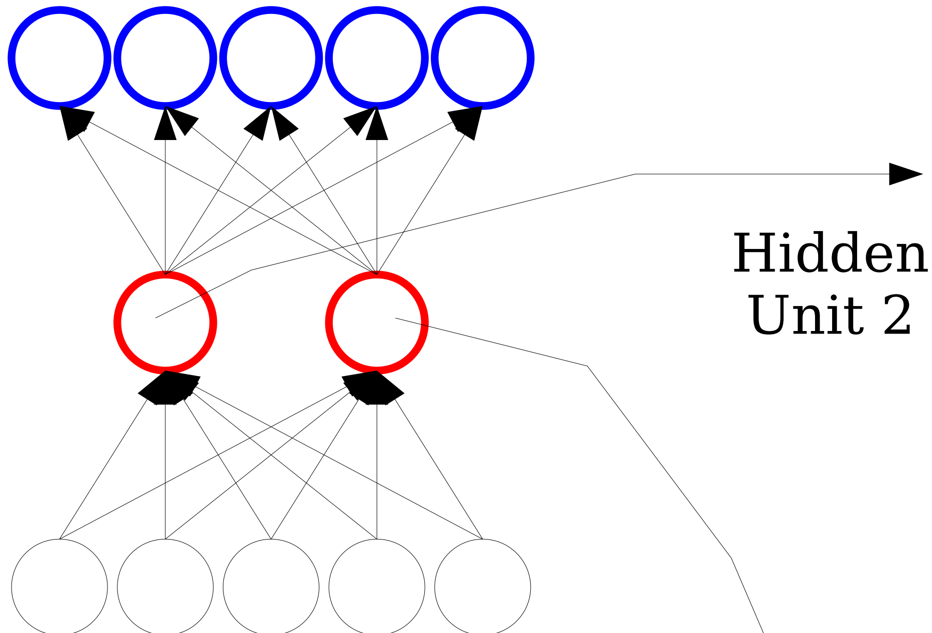
Bumps from which we compose f(x)

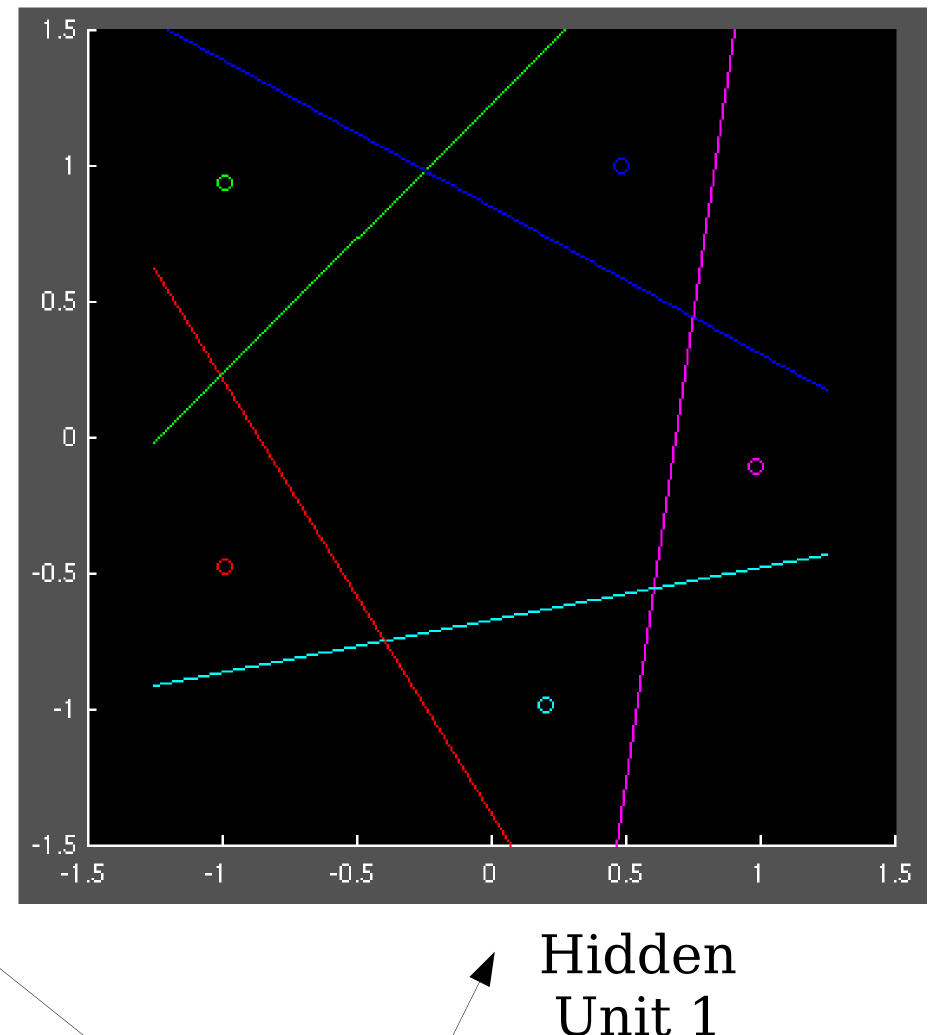
$\tanh(w_0 + w_1 x)$



3n+1 free parameters for *n* hidden units

Encoder Problem

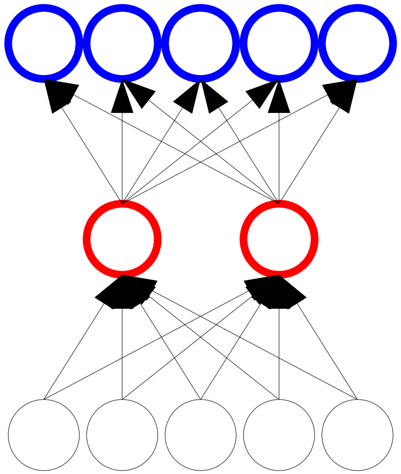




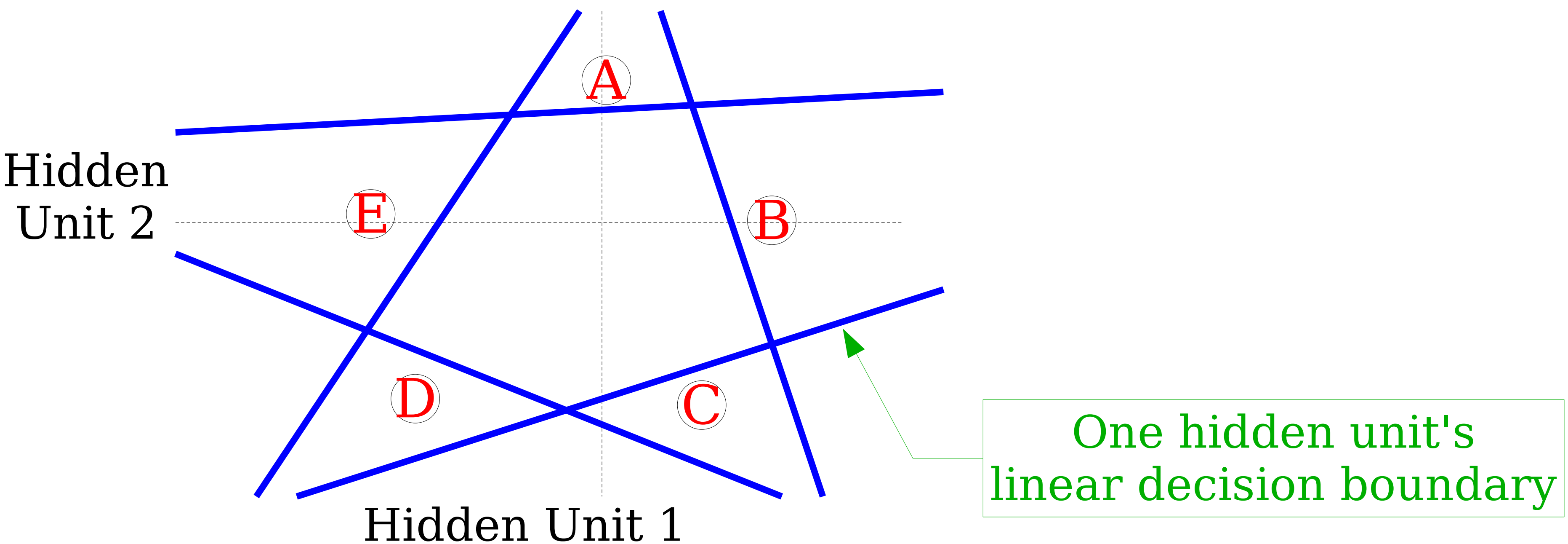
Input patterns: 1 bit on out of N.
Output pattern: same as input.

Only 2 hidden units: bottleneck!

5-2-5 Encoder Problem

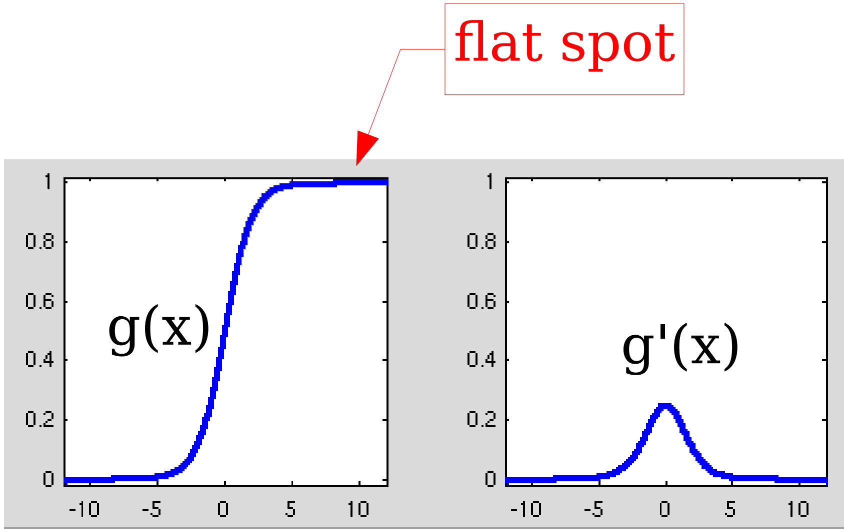


Training patterns:						*Hidden code:*
A:	0	0	0	0	1	0,2
B:	0	0	0	1	0	2,0
C:	0	0	1	0	0	1,−1
D:	0	1	0	0	0	−1,1
E:	1	0	0	0	0	−1,0



Flat Spots

If weights become large, net_j becomes large, derivative of g() goes to zero.



Fahlman's trick: add a small constant to g'(x) to keep the derivative from going to zero. Typical value is 0.1.

Momentum

Learning is slow if the learning rate is set too low.

Gradient may be steep in some directions but shallow in others.

Solution: add a momentum term α.

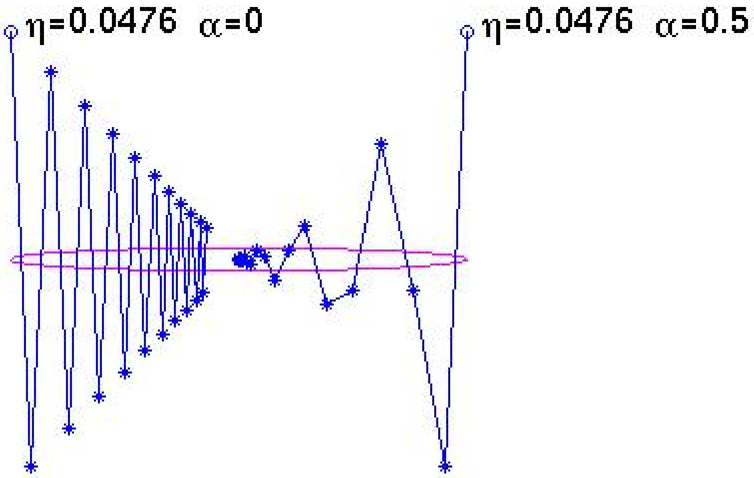
$$\Delta w_{ij}(t) = -\eta \frac{\partial E}{\partial w_{ij}(t)} + \alpha \cdot \Delta w_{ij}(t-1)$$

Typical value for α is 0.5.

If the direction of the gradient remains constant, the algorithm will take increasingly large steps.

Momentum Illustration

Hertz, Krogh & Palmer figs. 5.10 and 6.3: gradient descent on a quadratic error surface E (no neural net) involved:



$$E = x^2 + 20y^2$$

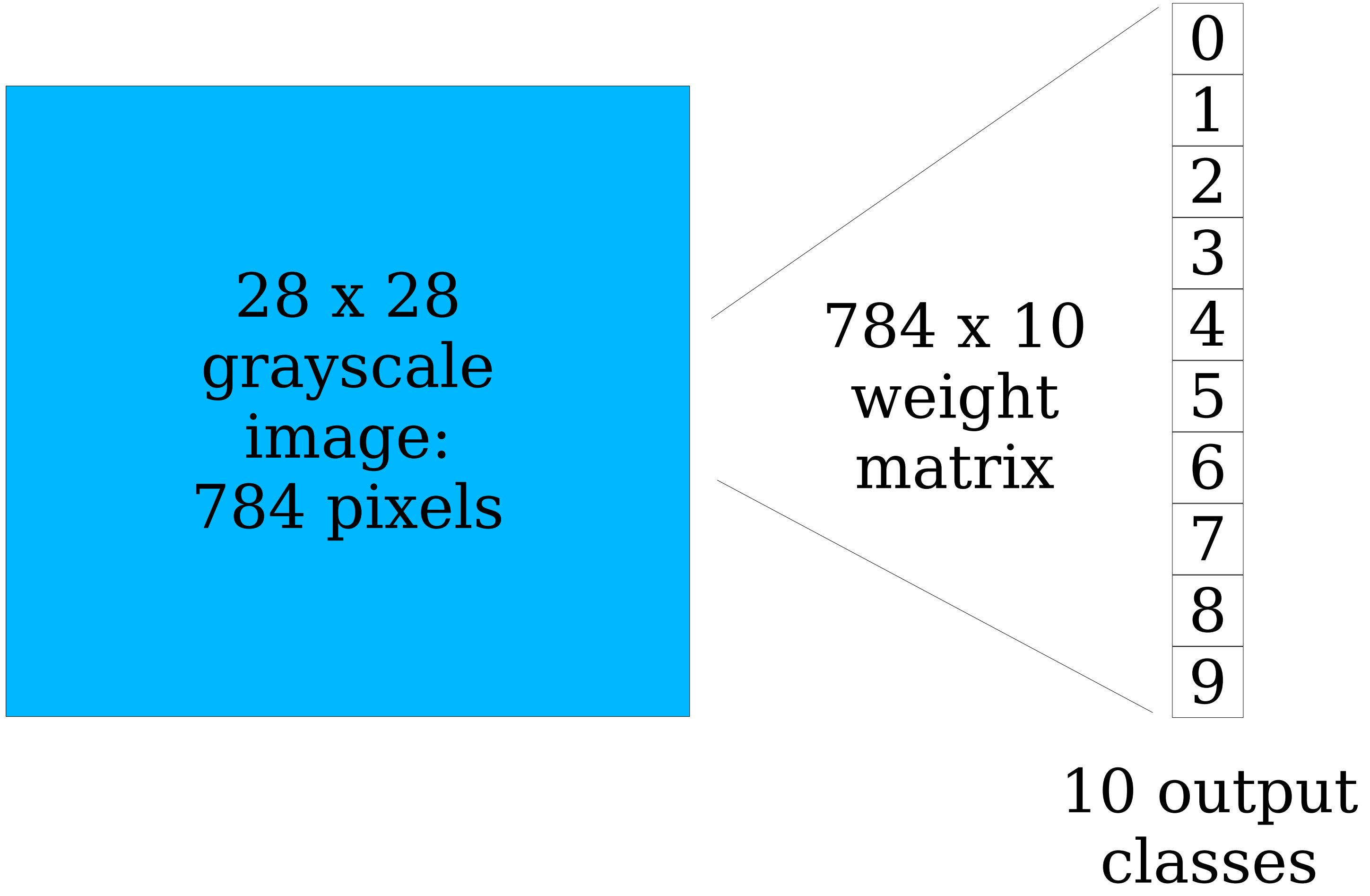
$$\frac{\partial E}{\partial x} = 2\text{x}, \quad \frac{\partial E}{\partial y} = 40\text{y}$$

Initial $[x, y] = [-1, 1]$ or $[1, 1]$

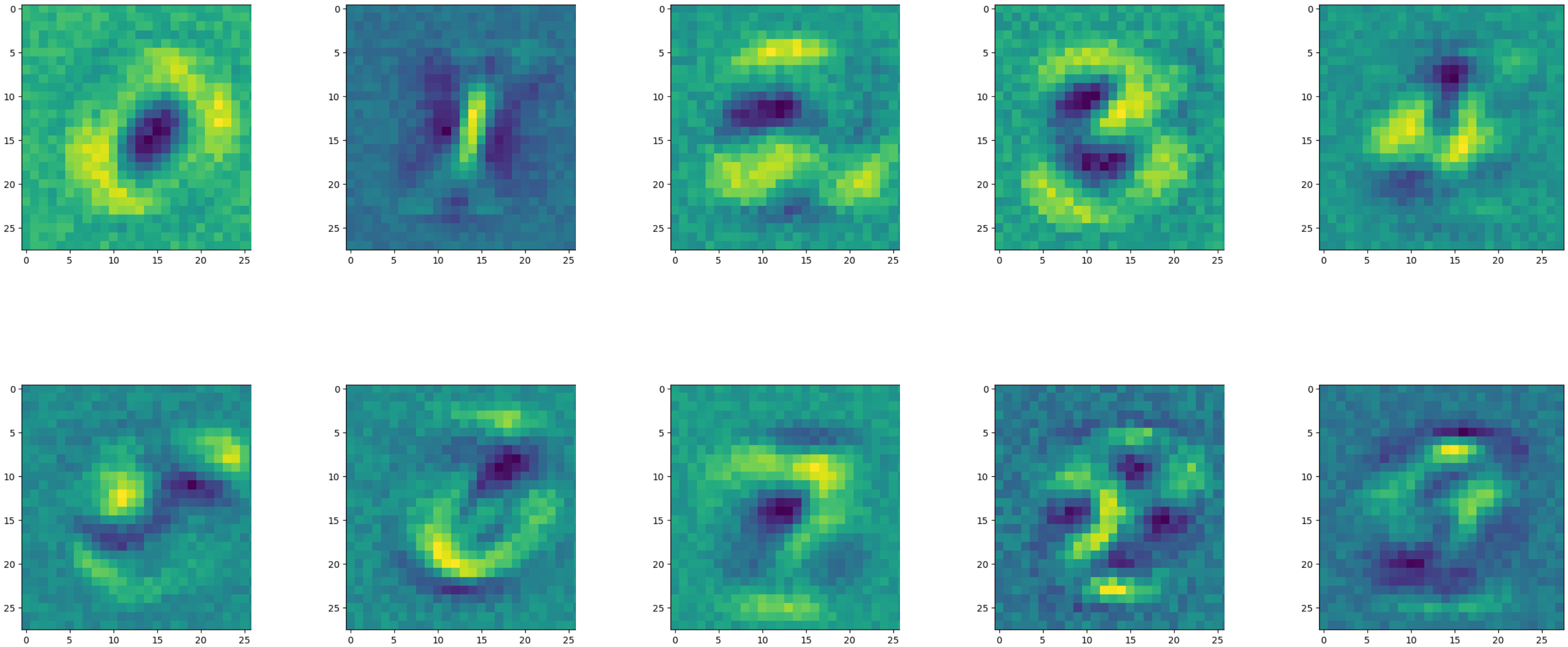
MNIST Dataset

- 60,000 labeled handwritten digits
- 28 x 28 pixel grayscale images

Recognition With a Linear Network



Learned Weights to Output Units



Training set performance: 89% correct.

TensorFlow Playground

Google's interactive backprop simulator.
https://playground.tensorflow.org