Lecture 2

# Selection (deterministic & randomized): finding the median in linear time

## 2.1 Overview

Given an unsorted array, how quickly can one find the median element? Can one do it more quickly than by sorting? This was an open question for some time, solved affirmatively in 1972 by (Manuel) Blum, Floyd, Pratt, Rivest, and Tarjan. In this lecture we describe two linear-time algorithms for this problem: one randomized and one deterministic. More generally, we solve the problem of finding the $k$th smallest out of an unsorted array of $n$ elements.

## 2.2 The problem and a randomized solution

Consider the problem of finding the $k$th smallest element in an unsorted array of size $n$. (Let's say all elements are distinct to avoid the question of what we mean by the $k$th smallest when we have equalities). One way to solve this problem is to sort and then output the $k$th element. We can do this in time $O(n \log n)$ if we sort using Mergesort, Quicksort, or Heapsort. Is there something faster – a linear-time algorithm? The answer is yes. We will explore both a simple randomized solution and a more complicated deterministic one.

The idea for the randomized algorithm is to start with the Randomized-Quicksort algorithm (choose a random element as "pivot", partition the array into two sets LESS and GREATER consisting of those elements less than and greater than the pivot respectively, and then recursively sort LESS and GREATER). Then notice that there is a simple speedup we can make if we just need to find the $k$th smallest element. In particular, after the partitioning step we can tell which of LESS or GREATER has the item we are looking for, just by looking at their sizes. So, we only need to recursively examine one of them, not both. For instance, if we are looking for the 87th-smallest element in our array, and suppose that after choosing the pivot and partitioning we find that LESS has 200 elements, then we just need to find the 87th smallest element in LESS. On the other hand, if we find LESS has 40 elements, then we just need to find the $87 - 40 - 1 = 46$th smallest element in GREATER. (And if LESS has size exactly 86 then we can just return the pivot). One might at first think that allowing the algorithm to only recurse on one subset rather than both would just cut down time by a factor of 2. However, since this is occuring recursively, it compounds the savings and we end up with $\Theta(n)$ rather than $\Theta(n \log n)$ time. This algorithm is often called Randomized-Select, or QuickSelect.

**QuickSelect:** Given array $A$ of size $n$ and integer $k \leq n$,

1. Pick a pivot element $p$ at random from $A$.
2. Split $A$ into subarrays LESS and GREATER by comparing each element to $p$ as in Quicksort. While we are at it, count the number $L$ of elements going in to LESS.
3. (a) If $L = k-1$, then output $p$.
   (b) If $L > k-1$, output QuickSelect(LESS, $k$).
   (c) If $L < k-1$, output QuickSelect(GREATER, $k-L-1$)

**Theorem 2.1** *The expected number of comparisons for QuickSelect is $O(n)$.*

Before giving a formal proof, let's first get some intuition. If we split a candy bar at random into two pieces, then the expected size of the larger piece is 3/4 of the bar. If the size of the larger subarray after our partition was always 3/4 of the array, then we would have a recurrence $T(n) \leq (n-1) + T(3n/4)$ which solves to $T(n) < 4n$. Now, this is not quite the case for our algorithm because $3n/4$ is only the *expected* size of the larger piece. That is, if $i$ is the size of the larger piece, our expected cost to go is really $\mathbf{E}[T(i)]$ rather than $T(\mathbf{E}[i])$. However, because the answer is linear in $n$, the average of the $T(i)$'s turns out to be the same as $T$(average of the $i$'s). Let's now see this a bit more formally.

**Proof (Theorem 2.1):** Let $T(n,k)$ denote the expected time to find the $k$th smallest in an array of size $n$, and let $T(n) = \max_k T(n,k)$. We will show that $T(n) < 4n$.

First of all, it takes $n-1$ comparisons to split into the array into two pieces in Step 2. These pieces are equally likely to have size 0 and $n-1$, or 1 and $n-2$, or 2 and $n-3$, and so on up to $n-1$ and 0. The piece we recurse on will depend on $k$, but since we are only giving an upper bound, we can imagine that we always recurse on the larger piece. Therefore we have:

$$\begin{aligned} T(n) &\leq (n-1) + \frac{2}{n}\sum_{i=n/2}^{n-1} T(i) \\ &= (n-1) + \text{avg}\left[T(n/2), \ldots, T(n-1)\right]. \end{aligned}$$

We can solve this using the "guess and check" method based on our intuition above. Assume inductively that $T(i) \leq 4i$ for $i < n$. Then,

$$\begin{aligned} T(n) &\leq (n-1) + \text{avg}\left[4(n/2), 4(n/2+1), \ldots, 4(n-1)\right] \\ &\leq (n-1) + 4(3n/4) \\ &< 4n, \end{aligned}$$

and we have verified our guess. ■

## 2.3 A deterministic linear-time algorithm

What about a deterministic linear-time algorithm? For a long time it was thought this was impossible – that there was no method faster than first sorting the array. In the process of trying to prove this claim it was discovered that this thinking was incorrect, and in 1972 a deterministic linear time algorithm was developed.

The idea of the algorithm is that one would like to pick a pivot deterministically in a way that produces a good split. Ideally, we would like the pivot to be the median element so that the two sides are the same size. But, this is the same problem we are trying to solve in the first place! So, instead, we will give ourselves leeway by allowing the pivot to be any element that is "roughly" in the middle: at least 3/10 of the array below the pivot and at least 3/10 of the array above. The algorithm is as follows:

**DeterministicSelect:** Given array $A$ of size $n$ and integer $k \leq n$,

1. Group the array into $n/5$ groups of size 5 and find the median of each group. (For simplicity, we will ignore integrality issues.)
2. Recursively, find the true median of the medians. Call this $p$.
3. Use $p$ as a pivot to split the array into subarrays LESS and GREATER.
4. Recurse on the appropriate piece.

**Theorem 2.2** *DeterministicSelect makes $O(n)$ comparisons to find the $k$th smallest in an array of size $n$.*

**Proof:** Let $T(n, k)$ denote the worst-case time to find the $k$th smallest out of $n$, and $T(n) = \max_k T(n, k)$ as before.

Step 1 takes time $O(n)$, since it takes just constant time to find the median of 5 elements. Step 2 takes time at most $T(n/5)$. Step 3 again takes time $O(n)$. Now, we claim that at least 3/10 of the array is $\leq p$, and at least 3/10 of the array is $\geq p$. Assuming for the moment that this claim is true, Step 4 takes time at most $T(7n/10)$, and we have the recurrence:

$$T(n) \quad \leq \quad cn + T(n/5) + T(7n/10), \tag{2.1}$$

for some constant $c$. Before solving this recurrence, lets prove the claim we made that the pivot will be roughly near the middle of the array. So, the question is: how bad can the median of medians be?

Let's first do an example. Suppose the array has 15 elements and breaks down into three groups of 5 like this:

$$\{1, 2, 3, 10, 11\}, \quad \{4, 5, 6, 12, 13\}, \quad \{7, 8, 9, 14, 15\}.$$

In this case, the medians are 3, 6, and 9, and the median of the medians $p$ is 6. There are five elements less than $p$ and nine elements greater.

In general, what is the worst case? If there are $g = n/5$ groups, then we know that in at least $\lceil g/2 \rceil$ of them (those groups whose median is $\leq p$) at least three of the five elements are $\leq p$. Therefore, the total number of elements $\leq p$ is at least $3\lceil g/2 \rceil \geq 3n/10$. Similarly, the total number of elements $\geq p$ is also at least $3\lceil g/2 \rceil \geq 3n/10$.

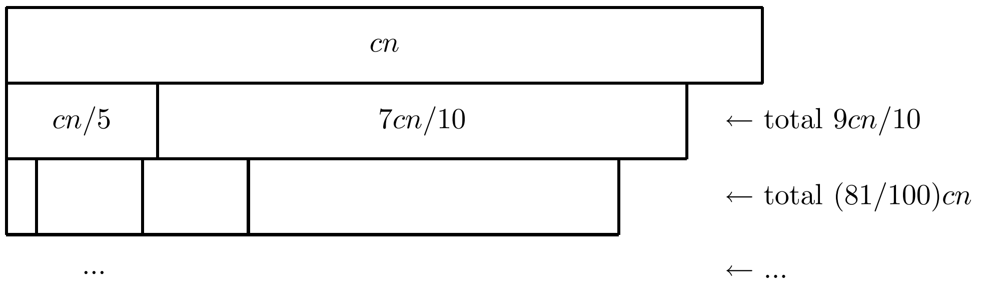

Figure 2.1: Stack of bricks view of recursions tree for recurrence 2.1.

Now, finally, let's solve the recurrence. We have been solving a lot of recurrences by the "guess and check" method, which works here too, but how could we just stare at this and *know* that the answer is linear in $n$? One way to do that is to consider the "stack of bricks" view of the recursion tree discussed in Appendix A.

In particular, let's build the recursion tree for the recurrence (2.1), making each node as wide as the quantity inside it:

Notice that even if this stack-of-bricks continues downward forever, the total sum is at most

$$cn(1 + (9/10) + (9/10)^2 + (9/10)^3 + \ldots),$$

which is at most $10cn$. This proves the theorem. ■

Notice that in our analysis of the recurrence (2.1) the key property we used was that $n/5+7n/10 < n$. More generally, we see here that if we have a problem of size $n$ that we can solve by performing recursive calls on pieces whose total size is at most $(1-\epsilon)n$ for some constant $\epsilon > 0$ (plus some additional $O(n)$ work), then the total time spent will be just linear in $n$. This gives us a nice extension to our "Master theorem" from Appendix A.

**Theorem 2.3** *For constants $c$ and $a_1, \ldots, a_k$ such that $a_1 + \ldots a_k < 1$, the recurrence*

$$T(n) \quad \leq \quad T(a_1 n) + T(a_2 n) + \ldots T(a_k n) + cn$$

*solves to $T(n) = O(n)$.*