Search-based Path Planning with Homotopy Class Constraints in 3D

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Abstract

Homotopy classes of trajectories, arising due to the presence of obstacles, are defined as sets of trajectories that can be transformed into each other by gradual bending and stretching without colliding with obstacles. The problem of exploring/finding the different homotopy classes in an environment and the problem of finding least-cost paths restricted to a specific homotopy class (or not belonging to certain homotopy classes) arises frequently in such applications as predicting paths for unpredictable entities and deployment of multiple agents for efficient exploration of an environment. In (Bhattacharya, Kumar, and Likhachev 2010) we have shown how homotopy classes of trajectories on a two-dimensional plane with obstacles can be classified and identified using the Cauchy Integral Theorem and the Residue Theorem from Complex Analysis. In more recent work (Bhattacharya, Likhachev, and Kumar 2011) we extended this representation to three-dimensional spaces by exploiting certain laws from the Theory of Electromagnetism (Biot-Savart law and Ampere's Law) for representing and identifying homotopy classes in three dimensions in an efficient way. Using such a representation, we showed that homotopy class constraints can be seamlessly weaved into graph search techniques for determining optimal path constrained to certain homotopy classes or forbidden from others, as well as for exploring different homotopy classes in an environment. [1]

Homotopy Classes and Homology Classes of Trajectories

Two trajectories τ_1 and τ_2 connecting the same start and end coordinates, $\mathbf{x}_s$ and $\mathbf{x}_g$ respectively, are called *homotopic* iff one can be continuously deformed into the other without intersecting any obstacle (Figure 1). Sets of homotopic trajectories form homotopy classes.

Give two trajectories, one may naively attempt to check if indeed one can be deformed into the other. However, such a process is highly non-trivial and may be extremely difficult to automate. Even if one is able to check, using such a method, whether of not two trajectories are homotopic, it is extremely difficult to incorporate the method in search-based planning algorithms to plan trajectories that are constrained to or avoid certain homotopy classes.

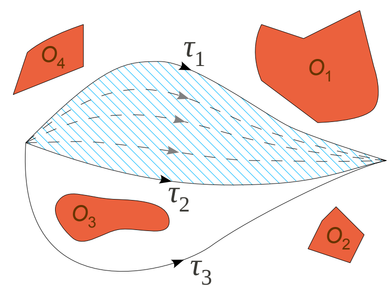
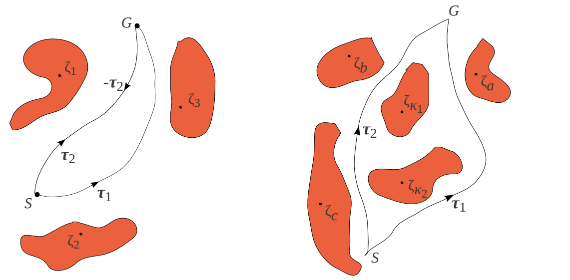


Figure 1: Illustration of homotopy and homology equivalences in 2 dimensions. In this example τ_1 and τ_2 are both homotopic (because of the existence of the sequence of trajectories shown by the dashed curves) as well as homologous (because of the presence of the area shown by blue hashing). But τ_3 is not homotopic nor homologous to either.

Thus, what one desires is to construct a functional of the trajectories, $\mathcal{H}(\tau)$ (which we will call the H-signature of τ), such that its value will uniquely identify the homotopy class of the trajectory (*i.e.* a complete invariant of homotopy classes of trajectories). We also desire that $\mathcal{H}$ be of the form of an integration, *i.e.*, $\mathcal{H}(\tau) = \int_\tau \mathrm{d}h$ (where $\mathrm{d}h$ is some differential 1-form – a quantity that can be integrated along a curve). This will let us compute least-cost paths in non trivial configuration spaces with topological constraints using graph search-based planning algorithms.

It is possible to find such desired 1-forms, $\mathrm{d}h$, as we did in our previous work for 2-dimensional configuration space (Bhattacharya, Kumar, and Likhachev 2010), where we exploited some theorems from complex analysis. However, it can be shown that by virtue of computing such integrals, what we end up obtaining from $\mathcal{H}(\tau)$ are complete invariants for *homology* classes rather than homotopy classes. Homology, although a close relative of homotopy and similar to it in many aspects, is subtly different from homotopy. Two trajectories τ_1 and τ_2 connecting the same start and end

[1]This is a condensed, non-technical overview of work previously published in the proceedings of Robotics: Science and Systems, 2011 conference (Bhattacharya, Likhachev, and Kumar 2011).



(a) In same Homotopy class, forming a closed contour

(b) In different Homotopy classes, enclosing obstacles

Figure 2: Two trajectories in same and different homotopy classes in 2 dimensions

coordinates, $\mathbf{x}_s$ and $\mathbf{x}_g$ respectively, are *homologous* iff τ_1 together with τ_2 (the later with opposite orientation) forms the complete boundary of a 2-dimensional region embedded in the configuration space not containing/intersecting any of the obstacles (Figure 1).

It can in fact be shown that homotopic trajectories are always homologous (equivalently, trajectories that are not homologous are not homotopic either), but the converse may not always be true. However, the difference between the two appear infrequently in practical robot configuration spaces, and as we demonstrate through our results, homology serves as a fair analog of homotopy in most practical robotics problems.

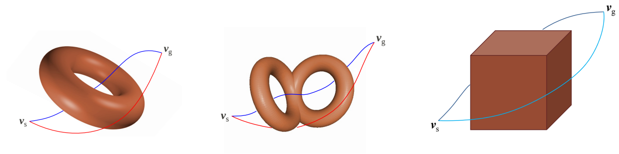
H-signature as Topological Invariants

Background: H-signature in 2-D

The basic principle (Bhattacharya, Kumar, and Likhachev 2010) in solving the problem in 2-dimensions was based on the *Residue Theorem* from *Complex Analysis*. We represented the two dimensional plane in which the robot's path is to be planned by the complex plane (*i.e.* a point (x, y) on it is represented as $z = x + iy$). We hence defined the H-signature (which we previously called the L-value in (Bhattacharya, Kumar, and Likhachev 2010)) of a trajectory, τ, as a complex path integral of a complex vector function as follows,

$$\mathcal{H}_2(\tau) = \int_\tau \begin{bmatrix} \frac{f_1(z)}{z-\zeta_1} \\ \frac{f_2(z)}{z-\zeta_2} \\ \vdots \\ \frac{f_M(z)}{z-\zeta_M} \end{bmatrix} \mathrm{d}z \qquad (1)$$

The quantity inside the integration (a complex vector) is an analytic function everywhere in the complex plane, except for distinct points, ζ_i, which we called *representative points*, placed on the obstacles (Figure 2), where the function has *poles*. As a consequence of this it could be shown using *Residue Theorem* that for two trajectories τ_1 and τ_2 connecting the same start and goal points in a 2-dimensional configuration space, $\mathcal{H}_2(\tau_1) = \mathcal{H}_2(\tau_2)$ if the trajectories are in the same homotopy class, but the values are different if they are in different homology classes.

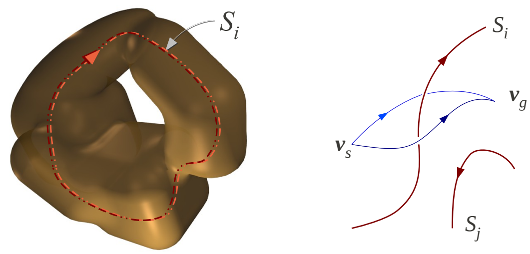


(a) A torus-shaped genus 1 obstacle.

(b) A genus 2 obstacle.

(c) A solid cube **does not** induce homotopy classes.

Figure 3: Examples of obstacles in 3-D. (a-b) induce homotopy classes, (c) does not.



(a) Skeleton of a generic genus 1 obstacle is modeled as a current-carrying conductor, S_i.

(b) Theorems from electromagnetism then gives us homotopy class invariants for trajectories.

Figure 4: Skeletons of obstacles in 3-D are modeled as current carrying conductor.

H-signature in 3-D

Just as we exploited theorems from complex analysis in 2 dimensions for constructing the homotopy class invariants, we can exploit certain theorems from Electromagnetism to achieve the same in 3 dimensions. In 3 dimensions multiple homotopy classes can only be induced by obstacles with *genus* [2] one or more, or with obstacles stretching to infinity. Figure 3 shows some examples of obstacles that can or cannot induce such classes for trajectories. A sphere or a solid cube, for example, cannot induce multiple homotopy classes in an environment.

Analogous to the *representative points* in the 2-dimensional case, in 3 dimensions we need to consider closed curves that represent the obstacles of genus 1 or higher (Figure 4(a)). These curves are the *skeletons* of the obstacles – 1-dimensional curves that are *homotopy equivalents* (Hatcher 2001) of the obstacles. We represent these skeletons by S_i, where $i = 1, 2, \cdots, M$.

The key idea in designing a H-signature for solid obstacles in 3-dimensions is to model these skeletons of the obstacles as "virtual conductors" carrying unit current. Upon doing so, using the Biot-Severts Law and Ampere's Law (Griffiths 1998), one can design the H-signature for trajectories τ in 3-dimensions as follows,

$$\mathcal{H}_3(\tau) = \int_\tau \begin{bmatrix} \mathbf{B}_1(\mathbf{l}) \\ \mathbf{B}_2(\mathbf{l}) \\ \vdots \\ \mathbf{B}_M(\mathbf{l}) \end{bmatrix} \mathrm{d}\mathbf{l} \qquad (2)$$

[2] The *genus* of an obstacle refers to the number of *holes* or *handles* (Munkres 1999).

where,
$$\mathbf{B}_i(\mathbf{r}) = \frac{1}{4\pi} \int_{S_i} \frac{(\mathbf{x} - \mathbf{r}) \times \mathrm{d}\mathbf{x}}{\|\mathbf{x} - \mathbf{r}\|^3} \tag{3}$$

is a “virtual magnetic field vector” due to the virtual current flowing through the skeletons. Using Biot-Severts Law and Ampere’s Law, it can be shown that for two trajectories τ_1 and τ_2 connecting the same start and goal points in a 3-dimensional configuration space, $\mathcal{H}_3(\tau_1) = \mathcal{H}_3(\tau_2)$ if the trajectories are in the same homotopy class, but the values are different if they are in different homology classes (Figure 4(b)).

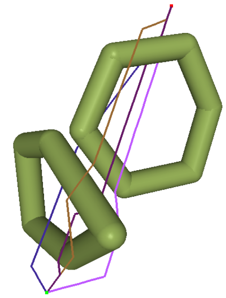
Incorporating H-signature in Graph-search Based Algorithms

Having designed a H-signature for 3-dimensional configuration spaces in form of an integral, the approach in graph search-based planning is very similar to the one we adopted in (Bhattacharya, Kumar, and Likhachev 2010). We construct an *augmented graph* from the given discrete graph representation of the environment, such that each vertex in this new graph has the H-signature of a path leading up to the coordinate of the vertex from $\mathbf{v}_s$ (start vertex) appended to it. The consequence of augmenting each vertex of original graph, $\mathcal{G}$, with a H-signature is that now vertices are distinguished not only by their coordinates, but also the H-signature of the trajectory followed to reach it. Since the H-signature is in form of an integration, during expansion of the vertices in the search algorithm, its value for newly expanded vertices can be easily computed by adding to the value of its parent the H-signature of the edge connecting them. Planning trajectories in this augmented graph thus allows us impose constraints on the H-signature of the desired trajectories as well as find optimal trajectories in different homotopy classes in the environment. For more details on the graph construction the reader may refer to (Bhattacharya, Likhachev, and Kumar 2011).

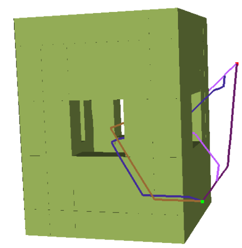
Results

Figure 5(a)-(c) shows examples where we find least cost trajectories in different homotopy classes in a few environments by searching in the the augmented graphs. In each of these simulations there were certain pre-computations involved, where we computed the H-signature for every edge in the original graph. This pre-computation, that needs to be performed only once for a given environment, takes about 15 minutes. The searches in the augmented graphs for finding trajectories in different homotopy classes itself took less than a minute.

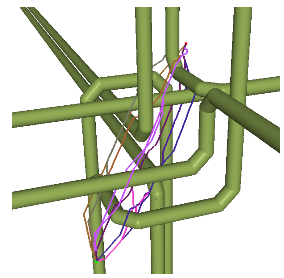
Figure 5(d) demonstrates a planning problem with H-signature constraint. The darker trajectory is the global least cost path found from a search in the original graph for the given start and goal coordinates. The H-signature for that trajectory was computed, and hence we computed the signature of the *complementary class* (*i.e* the class corresponding to the trajectory that passes on the *other side* of every obstacle). The lighter trajectory is the one planned with that H-signature as constraint. Thus, this trajectory goes on the *opposite side* of each and every pipe in the environment as compared to the darker trajectory.



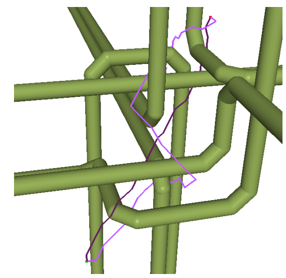
(a) Two hoops.



(b) A room with windows.



(c) Exploring 10 distinct homotopy classes.



(d) Plan in the *complementary homotopy class* of the least cost path.

Figure 5: Exploring homotopy classes and planning with H-signature constraints (Bhattacharya 2012).

For more details, results in other interesting configuration spaces (including one in $X - Y - Time$) as well as for higher dimensional extensions, please see (Bhattacharya, Likhachev, and Kumar 2011) and (Bhattacharya 2012).

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