15-213

"The course that gives CMU its Zip!"

Integers
Sep 2, 2003

Topics

- Numeric Encodings
  - Unsigned & Two's complement
- Programming Implications
  - C promotion rules
- Basic operations
  - Addition, negation, multiplication
- Programming Implications
  - Consequences of overflow
  - Using shifts to perform power-of-2 multiply/divide

`class03.ppt`

C Puzzles

- Taken from old exams
- Assume machine with 32 bit word size, two's complement integers
- For each of the following C expressions, either:
  - Argue that is true for all argument values
  - Give example where not true

Initialization

```
int x = foo();
int y = bar();
unsigned ux = x;
unsigned uy = y;
```

- `x < 0` $\Rightarrow$ `((x*2) < 0)`
- `ux >= 0`
- `x & 7 == 7` $\Rightarrow$ `(x<<30) < 0`
- `ux > -1`
- `x > y` $\Rightarrow$ `-x < -y`
- `x * x >= 0`
- `x > 0 && y > 0` $\Rightarrow$ `x + y > 0`
- `x >= 0` $\Rightarrow$ `-x <= 0`
- `x <= 0` $\Rightarrow$ `-x >= 0`

Encoding Integers

Unsigned

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

Two's Complement

$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

Sign Bit

```
short int x =  15213;
short int y = -15213;
```

- C short 2 bytes long

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
y	-15213	C4 93	11000100 10010011

Sign Bit

- For 2's complement, most significant bit indicates sign
  - 0 for nonnegative
  - 1 for negative

Encoding Example (Cont.)

```
x =      15213: 00111011 01101101
y =     -15213: 11000100 10010011
```

Weight	15213		-15213	
1	1	1	1	1
2	0	0	1	2
4	1	4	0	0
8	1	8	0	0
16	0	0	1	16
32	1	32	0	0
64	1	64	0	0
128	0	0	1	128
256	1	256	0	0
512	1	512	0	0
1024	0	0	1	1024
2048	1	2048	0	0
4096	1	4096	0	0
8192	1	8192	0	0
16384	0	0	1	16384
-32768	0	0	1	-32768
Sum		**15213**		**-15213**

Numeric Ranges

Unsigned Values

- *UMin* = 0
  - 000…0
- *UMax* = $2^w - 1$
  - 111…1

Two's Complement Values

- *TMin* = -2^{w-1}
  - 100…0
- *TMax* = $2^{w-1} - 1$
  - 011…1

Other Values

- Minus 1
  - 111…1

Values for *W* = 16

	Decimal	Hex	Binary
UMax	65535	FF FF	11111111 11111111
TMax	32767	7F FF	01111111 11111111
TMin	-32768	80 00	10000000 00000000
-1	-1	FF FF	11111111 11111111
0	0	00 00	00000000 00000000

Values for Different Word Sizes

	W			
	8	16	32	64
UMax	255	65,535	4,294,967,295	18,446,744,073,709,551,615
TMax	127	32,767	2,147,483,647	9,223,372,036,854,775,807
TMin	-128	-32,768	-2,147,483,648	-9,223,372,036,854,775,808

Observations

- |*TMin*| = *TMax* + 1
  - Asymmetric range
- *UMax* = 2 * *TMax* + 1

C Programming

- #include <limits.h>
  - K&R App. B11
- Declares constants, e.g.,
  - ULONG_MAX
  - LONG_MAX
  - LONG_MIN
- Values platform-specific

Unsigned & Signed Numeric Values

X	B2U(*X*)	B2T(*X*)
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	–8
1001	9	–7
1010	10	–6
1011	11	–5
1100	12	–4
1101	13	–3
1110	14	–2
1111	15	–1

Equivalence

- Same encodings for nonnegative values

Uniqueness

- Every bit pattern represents unique integer value
- Each representable integer has unique bit encoding

⇒ Can Invert Mappings

- U2B(*x*) = $\mathrm{B2U}^{-1}(x)$
  - Bit pattern for unsigned integer
- T2B(*x*) = $\mathrm{B2T}^{-1}(x)$
  - Bit pattern for two's comp integer

Casting Signed to Unsigned

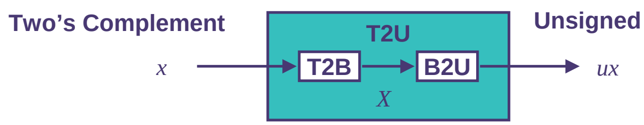
C Allows Conversions from Signed to Unsigned

```
short int          x =  15213;
unsigned short int ux = (unsigned short) x;
short int          y  = -15213;
unsigned short int uy = (unsigned short) y;
```

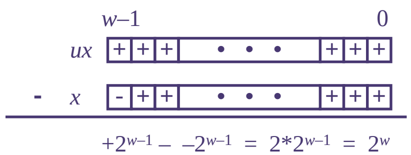
Resulting Value

- No change in bit representation
- Nonnegative values unchanged
  - *ux* = 15213
- Negative values change into (large) positive values
  - *uy* = 50323

Relation between Signed & Unsigned



Maintain Same Bit Pattern



$$ux = \begin{cases} x & x \ge 0 \\ x + 2^w & x < 0 \end{cases}$$

Relation Between Signed & Unsigned

Weight	-15213		50323	
1	1	1	1	1
2	1	2	1	2
4	0	0	0	0
8	0	0	0	0
16	1	16	1	16
32	0	0	0	0
64	0	0	0	0
128	1	128	1	128
256	0	0	0	0
512	0	0	0	0
1024	1	1024	1	1024
2048	0	0	0	0
4096	0	0	0	0
8192	0	0	0	0
16384	1	16384	1	16384
32768	1	*-32768*	1	*32768*
Sum		**-15213**		**50323**

- *uy* = *y* + 2 * 32768 = *y* + 65536

Signed vs. Unsigned in C

Constants

- By default are considered to be signed integers
- Unsigned if have "U" as suffix

```
0U, 4294967259U
```

Casting

- Explicit casting between signed & unsigned same as U2T and T2U

```
int tx, ty;
unsigned ux, uy;
tx = (int) ux;
uy = (unsigned) ty;
```

- Implicit casting also occurs via assignments and procedure calls

```
tx = ux;
uy = ty;
```


Casting Surprises

Expression Evaluation

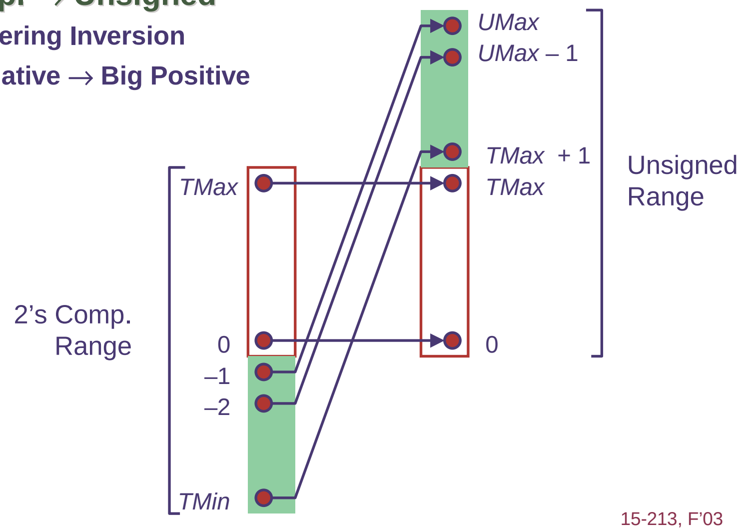
- If mix unsigned and signed in single expression, signed values implicitly cast to unsigned
- Including comparison operations <, >, ==, <=, >=
- Examples for *W* = 32

Constant$_1$	Constant$_2$	Relation	Evaluation
`0`	`0U`	`==`	unsigned
`-1`	`0`	`<`	signed
`-1`	`0U`	`>`	unsigned
`2147483647`	`-2147483648`	`>`	signed
`2147483647U`	`-2147483648`	`<`	unsigned
`-1`	`-2`	`>`	signed
`(unsigned) -1`	`-2`	`>`	unsigned
`2147483647`	`2147483648U`	`<`	unsigned
`2147483647`	`(int) 2147483648U`	`>`	signed

Explanation of Casting Surprises

2's Comp. → Unsigned

- Ordering Inversion
- Negative → Big Positive



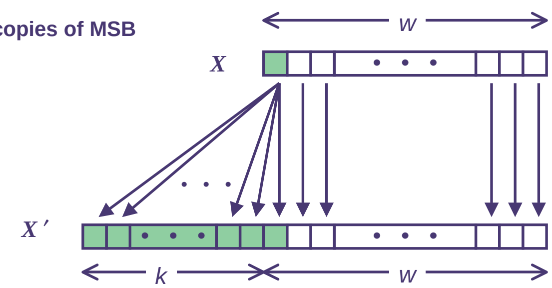
Sign Extension

Task:

- Given *w*-bit signed integer *x*
- Convert it to *w*+*k*-bit integer with same value

Rule:

- Make *k* copies of sign bit:
- $X' = \underbrace{x_{w-1}, \ldots, x_{w-1}}_{k \text{ copies of MSB}}, x_{w-1}, x_{w-2}, \ldots, x_0$



Sign Extension Example

```
short int x =  15213;
int      ix = (int) x;
short int y = -15213;
int      iy = (int) y;
```

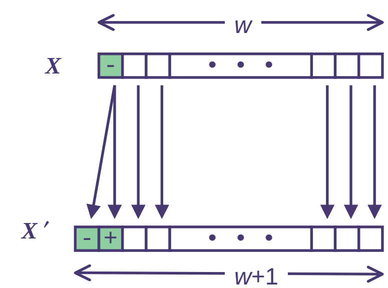
	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
ix	15213	00 00 3B 6D	00000000 00000000 00111011 01101101
y	-15213	C4 93	11000100 10010011
iy	-15213	FF FF C4 93	11111111 11111111 11000100 10010011

- Converting from smaller to larger integer data type
- C automatically performs sign extension

Justification For Sign Extension

Prove Correctness by Induction on *k*

- Induction Step: extending by single bit maintains value



- Key observation: $-2^{w-1} = -2^{w} + 2^{w-1}$
- Look at weight of upper bits:

$X \quad -2^{w-1} x_{w-1}$

$X' \quad -2^{w} x_{w-1} + 2^{w-1} x_{w-1} \quad = \quad -2^{w-1} x_{w-1}$

Why Should I Use Unsigned?

Don't Use Just Because Number Nonzero

- C compilers on some machines generate less efficient code

```
unsigned i;
for (i = 1; i < cnt; i++)
  a[i] += a[i-1];
```

- Easy to make mistakes

```
for (i = cnt-2; i >= 0; i--)
  a[i] += a[i+1];
```

Do Use When Performing Modular Arithmetic

- Multiprecision arithmetic
- Other esoteric stuff

Do Use When Need Extra Bit's Worth of Range

- Working right up to limit of word size

Negating with Complement & Increment

Claim: Following Holds for 2's Complement

`~x + 1 == -x`

Complement

- Observation: `~x + x == 1111…11`$_2$ `== -1`

`x`	1 0 0 1 1 1 0 1
`+ ~x`	0 1 1 0 0 0 1 0
`-1`	1 1 1 1 1 1 1 1

Increment

- ~x + ~~x~~ + (-~~x~~ + 1) == -~~1~~ + (-x + ~~1~~)
- ~x + 1 == -x

Warning: Be cautious treating `int`'s as integers

- OK here

Comp. & Incr. Examples

x = 15213

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
~x	-15214	C4 92	11000100 10010010
~x+1	-15213	C4 93	11000100 10010011
y	-15213	C4 93	11000100 10010011

0

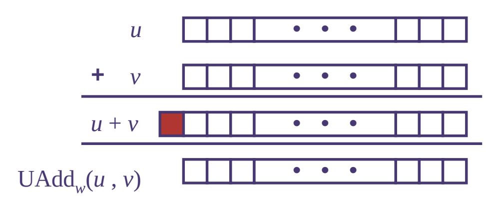
	Decimal	Hex	Binary
0	0	00 00	00000000 00000000
~0	-1	FF FF	11111111 11111111
~0+1	0	00 00	00000000 00000000

Unsigned Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



Standard Addition Function

- Ignores carry output

Implements Modular Arithmetic

$s = \mathrm{UAdd}_w(u, v) = u + v \mod 2^w$

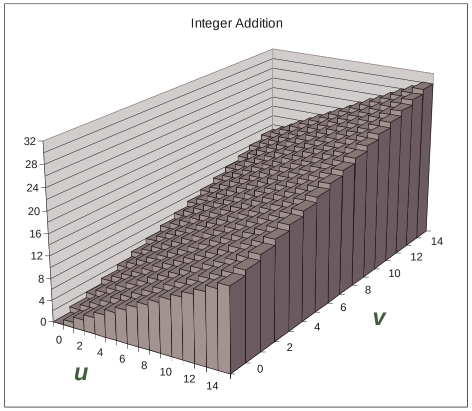
$$UAdd_w(u,v) = \begin{cases} u+v & u+v < 2^w \\ u+v-2^w & u+v \geq 2^w \end{cases}$$

Visualizing Integer Addition

Integer Addition

- 4-bit integers *u*, *v*
- Compute true sum $\mathrm{Add}_4(u\ ,\ v)$
- Values increase linearly with *u* and *v*
- Forms planar surface

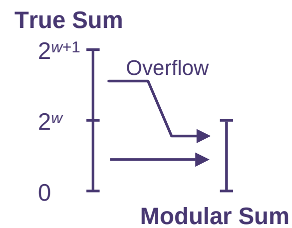
$\mathrm{Add}_4(u\ ,\ v)$

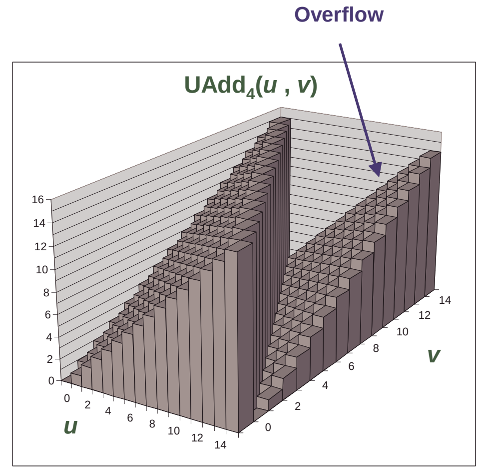


Visualizing Unsigned Addition

Wraps Around

- If true sum ≥ 2^w
- At most once



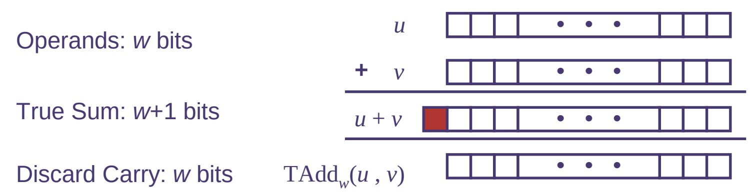


Mathematical Properties

Modular Addition Forms an *Abelian Group*

- Closed under addition
  $0 \le \mathrm{UAdd}_w(u\ ,\ v) \le 2^w - 1$
- Commutative
  $\mathrm{UAdd}_w(u\ ,\ v) = \mathrm{UAdd}_w(v\ ,\ u)$
- Associative
  $\mathrm{UAdd}_w(t, \mathrm{UAdd}_w(u\ ,\ v)) = \mathrm{UAdd}_w(\mathrm{UAdd}_w(t, u\ ), v)$
- 0 is additive identity
  $\mathrm{UAdd}_w(u\ ,\ 0) = u$
- Every element has additive inverse
  - Let $\mathrm{UComp}_w(u\ ) = 2^w - u$
  $\mathrm{UAdd}_w(u\ ,\ \mathrm{UComp}_w(u\ )) = 0$

Two's Complement Addition



TAdd and UAdd have Identical Bit-Level Behavior

- Signed vs. unsigned addition in C:

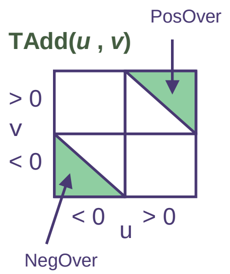
```
int s, t, u, v;
s = (int) ((unsigned) u + (unsigned) v);
t = u + v
```

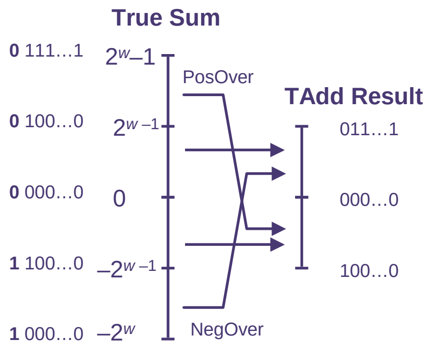
- Will give `s == t`

Characterizing TAdd

Functionality

- True sum requires w+1 bits
- Drop off MSB
- Treat remaining bits as 2's comp. integer





$$TAdd_w(u,v) = \begin{cases} u+v+2^{w-1} & u+v < TMin_w \quad \textbf{(NegOver)} \\ u+v & TMin_w \le u+v \le TMax_w \\ u+v-2^{w-1} & TMax_w < u+v \quad \textbf{(PosOver)} \end{cases}$$

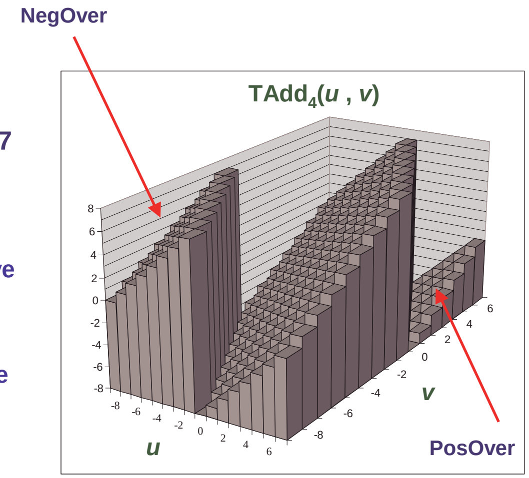
Visualizing 2's Comp. Addition

Values

- 4-bit two's comp.
- Range from -8 to +7

Wraps Around

- If sum $\geq 2^{w-1}$
  - Becomes negative
  - At most once
- If sum $< -2^{w-1}$
  - Becomes positive
  - At most once

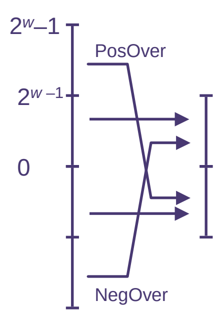


Detecting 2's Comp. Overflow

Task

- Given s = TAdd$_w$(u , v)
- Determine if s = Add$_w$(u , v)
- Example

```
int s, u, v;
s = u + v;
```



Claim

- Overflow iff either:
  - $u, v < 0, s \geq 0$ (NegOver)
  - $u, v \geq 0, s < 0$ (PosOver)

```
ovf = (u<0 == v<0) && (u<0 != s<0);
```


Mathematical Properties of TAdd

Isomorphic Algebra to UAdd

- TAdd$_w$(u , v) = U2T(UAdd$_w$(T2U(u), T2U(v)))
  - Since both have identical bit patterns

Two's Complement Under TAdd Forms a Group

- Closed, Commutative, Associative, 0 is additive identity
- Every element has additive inverse

  Let TComp$_w$(u) = U2T(UComp$_w$(T2U(u))

  TAdd$_w$(u , TComp$_w$(u)) = 0

$$TComp_w(u) = \begin{cases} -u & u \neq TMin_w \\ TMin_w & u = TMin_w \end{cases}$$

Multiplication

Computing Exact Product of w-bit numbers x, y

- **Either signed or unsigned**

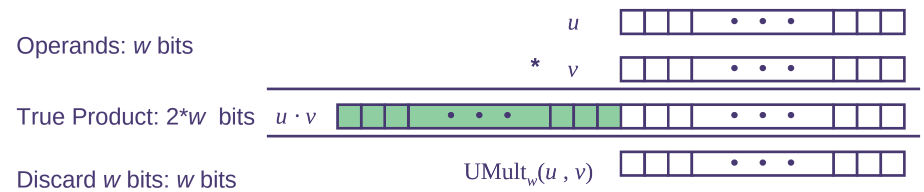
Ranges

- **Unsigned:** $0 \le x * y \le (2^w - 1)^2 = 2^{2w} - 2^{w+1} + 1$
  - **Up to $2w$ bits**
- **Two's complement min:** $x * y \ge (-2^{w-1})*(2^{w-1}-1) = -2^{2w-2} + 2^{w-1}$
  - **Up to $2w-1$ bits**
- **Two's complement max:** $x * y \le (-2^{w-1})^2 = 2^{2w-2}$
  - **Up to $2w$ bits, but only for $(TMin_w)^2$**

Maintaining Exact Results

- **Would need to keep expanding word size with each product computed**
- **Done in software by "arbitrary precision" arithmetic packages**

Unsigned Multiplication in C



Standard Multiplication Function

- **Ignores high order w bits**

Implements Modular Arithmetic

$UMult_w(u, v) = u \cdot v \bmod 2^w$

Unsigned vs. Signed Multiplication

Unsigned Multiplication

```
unsigned ux = (unsigned) x;
unsigned uy = (unsigned) y;
unsigned up = ux * uy
```

- **Truncates product to w-bit number** $up = UMult_w(ux, uy)$
- **Modular arithmetic: $up = ux \cdot uy \bmod 2^w$**

Two's Complement Multiplication

```
int x, y;
int p = x * y;
```

- **Compute exact product of two w-bit numbers x, y**
- **Truncate result to w-bit number** $p = TMult_w(x, y)$

Unsigned vs. Signed Multiplication

Unsigned Multiplication

```
unsigned ux = (unsigned) x;
unsigned uy = (unsigned) y;
unsigned up = ux * uy
```

Two's Complement Multiplication

```
int x, y;
int p = x * y;
```

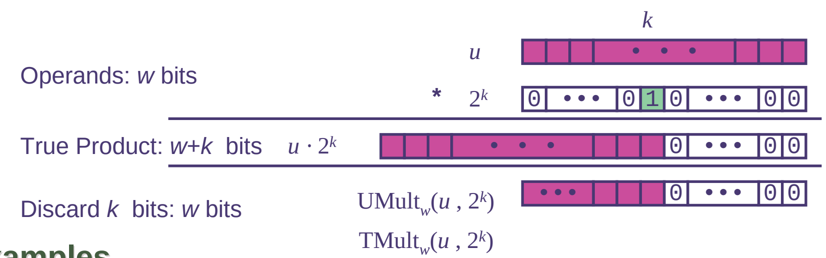
Relation

- **Signed multiplication gives same bit-level result as unsigned**
- `up == (unsigned) p`

Power-of-2 Multiply with Shift

Operation

- `u << k` gives `u * ` 2^k
- Both signed and unsigned



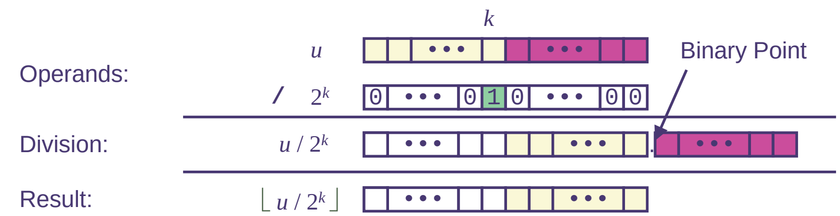
Examples

- `u << 3        ==  u * 8`
- `u << 5 - u << 3   ==   u * 24`
- Most machines shift and add much faster than multiply
  - Compiler generates this code automatically

Unsigned Power-of-2 Divide with Shift

Quotient of Unsigned by Power of 2

- `u >> k` gives ⌊ `u / ` 2^k ⌋
- Uses logical shift

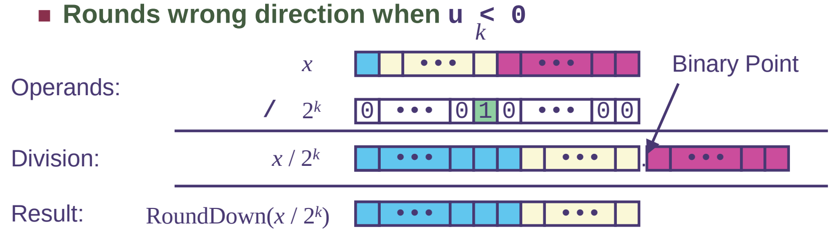


	Division	Computed	Hex	Binary
x	15213	15213	3B 6D	00111011 01101101
x >> 1	7606.5	7606	1D B6	**0**0011101 10110110
x >> 4	950.8125	950	03 B6	**0000**0011 10110110
x >> 8	59.4257813	59	00 3B	**00000000** 00111011

Signed Power-of-2 Divide with Shift

Quotient of Signed by Power of 2

- `x >> k` gives ⌊ `x / ` 2^k ⌋
- Uses arithmetic shift
- Rounds wrong direction when `u < 0`



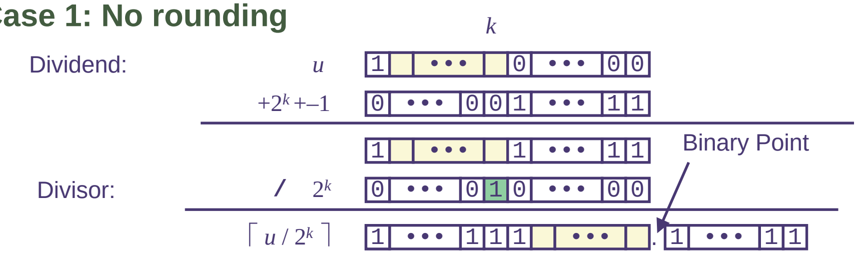
	Division	Computed	Hex	Binary
y	-15213	-15213	C4 93	11000100 10010011
y >> 1	-7606.5	-7607	E2 49	**1**1100010 01001001
y >> 4	-950.8125	-951	FC 49	**1111**1100 01001001
y >> 8	-59.4257813	-60	FF C4	**11111111** 11000100

Correct Power-of-2 Divide

Quotient of Negative Number by Power of 2

- Want ⌈ `x / ` 2^k ⌉ (Round Toward 0)
- Compute as ⌊ `(x+`2^k`-1)/ `2^k ⌋
  - In C: `(x + (1<<k)-1) >> k`
  - Biases dividend toward 0

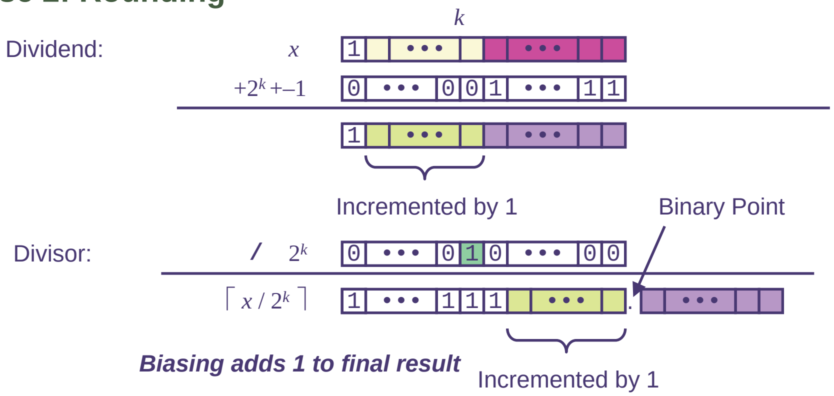
Case 1: No rounding



Biasing has no effect

Correct Power-of-2 Divide (Cont.)

Case 2: Rounding



Properties of Unsigned Arithmetic

Unsigned Multiplication with Addition Forms Commutative Ring

- Addition is commutative group
- Closed under multiplication
  $0 \leq \mathrm{UMult}_w(u\,,\,v) \leq 2^w - 1$
- Multiplication Commutative
  $\mathrm{UMult}_w(u\,,\,v) = \mathrm{UMult}_w(v\,,\,u)$
- Multiplication is Associative
  $\mathrm{UMult}_w(t, \mathrm{UMult}_w(u\,,\,v)) = \mathrm{UMult}_w(\mathrm{UMult}_w(t, u\,), v)$
- 1 is multiplicative identity
  $\mathrm{UMult}_w(u\,,\,1) = u$
- Multiplication distributes over addtion
  $\mathrm{UMult}_w(t, \mathrm{UAdd}_w(u\,,\,v)) = \mathrm{UAdd}_w(\mathrm{UMult}_w(t, u\,), \mathrm{UMult}_w(t, v))$

Properties of Two's Comp. Arithmetic

Isomorphic Algebras

- Unsigned multiplication and addition
  - Truncating to *w* bits
- Two's complement multiplication and addition
  - Truncating to *w* bits

Both Form Rings

- Isomorphic to ring of integers mod 2^w

Comparison to Integer Arithmetic

- Both are rings
- Integers obey ordering properties, e.g.,
  $u > 0 \quad \Rightarrow \quad u + v > v$
  $u > 0, v > 0 \quad \Rightarrow \quad u \cdot v > 0$
- These properties are not obeyed by two's comp. arithmetic
  TMax `+ 1` `==` *TMin*
  `15213 * 30426` `==` `-10030` (16-bit words)

C Puzzle Answers

- Assume machine with 32 bit word size, two's comp. integers
- *TMin* makes a good counterexample in many cases

`x < 0`	⇒	`((x*2) < 0)`	**False:**	*TMin*
`ux >= 0`			**True:**	0 = *UMin*
`x & 7 == 7`	⇒	`(x<<30) < 0`	**True:**	$x_1 = 1$
`ux > -1`			**False:**	0
`x > y`	⇒	`-x < -y`	**False:**	-1, *TMin*
`x * x >= 0`			**False:**	30426
`x > 0 && y > 0`	⇒	`x + y > 0`	**False:**	*TMax*, *TMax*
`x >= 0`	⇒	`-x <= 0`	**True:**	–*TMax* < 0
`x <= 0`	⇒	`-x >= 0`	**False:**	*TMin*