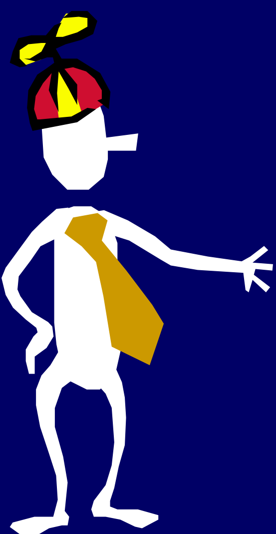
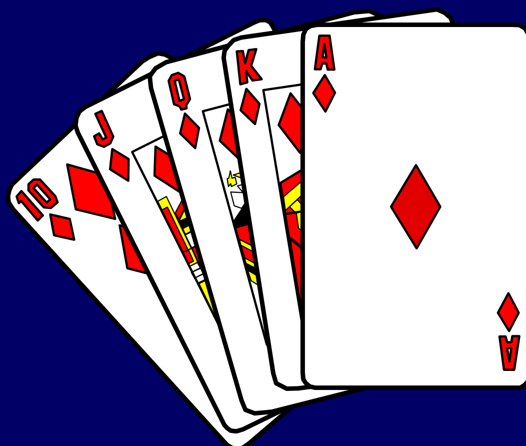
Great Theoretical Ideas In Computer Science

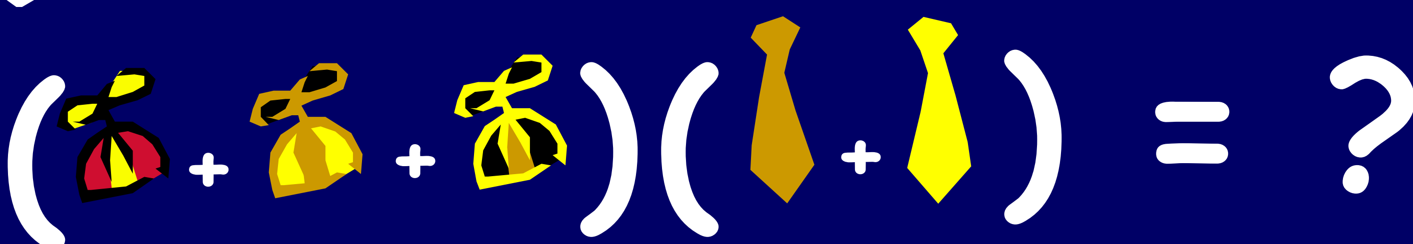
Anupam Gupta CS 15-251 Fall 2006

Lecture 7 Sept 19, 2006 Carnegie Mellon University

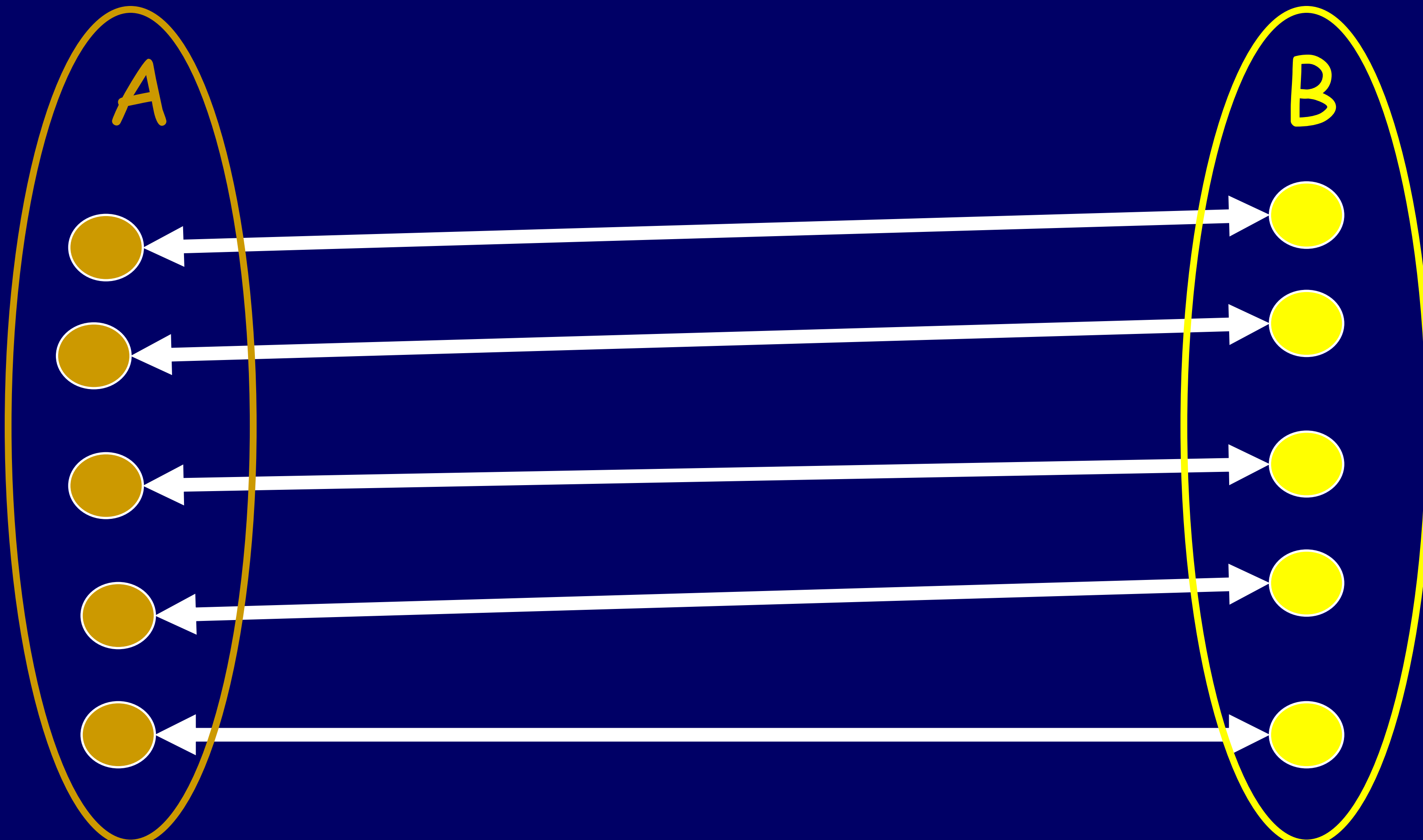
# Counting II: Recurring Problems and Correspondences







# 1-1 onto Correspondence
# (just "correspondence" for short)

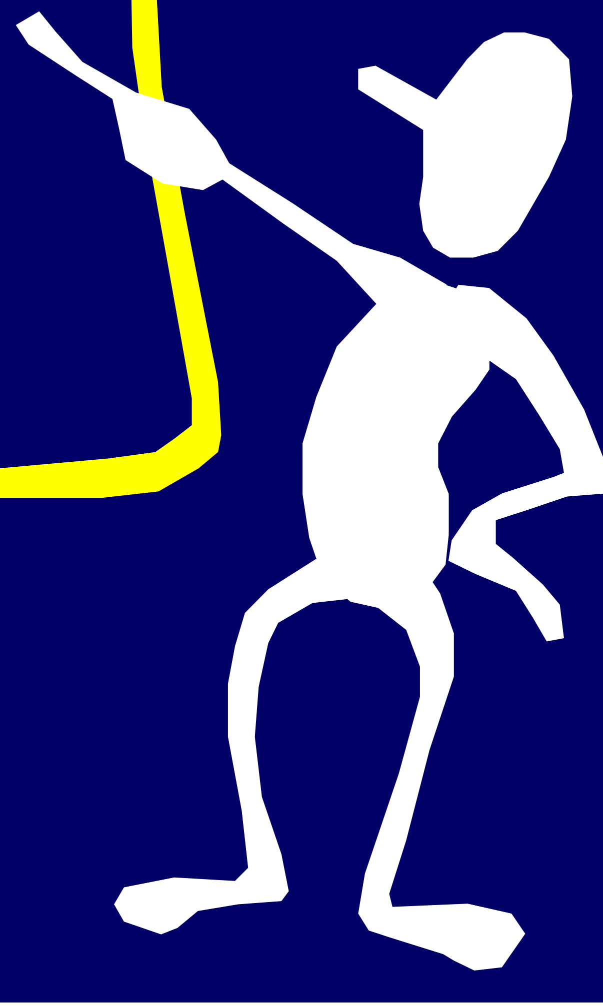


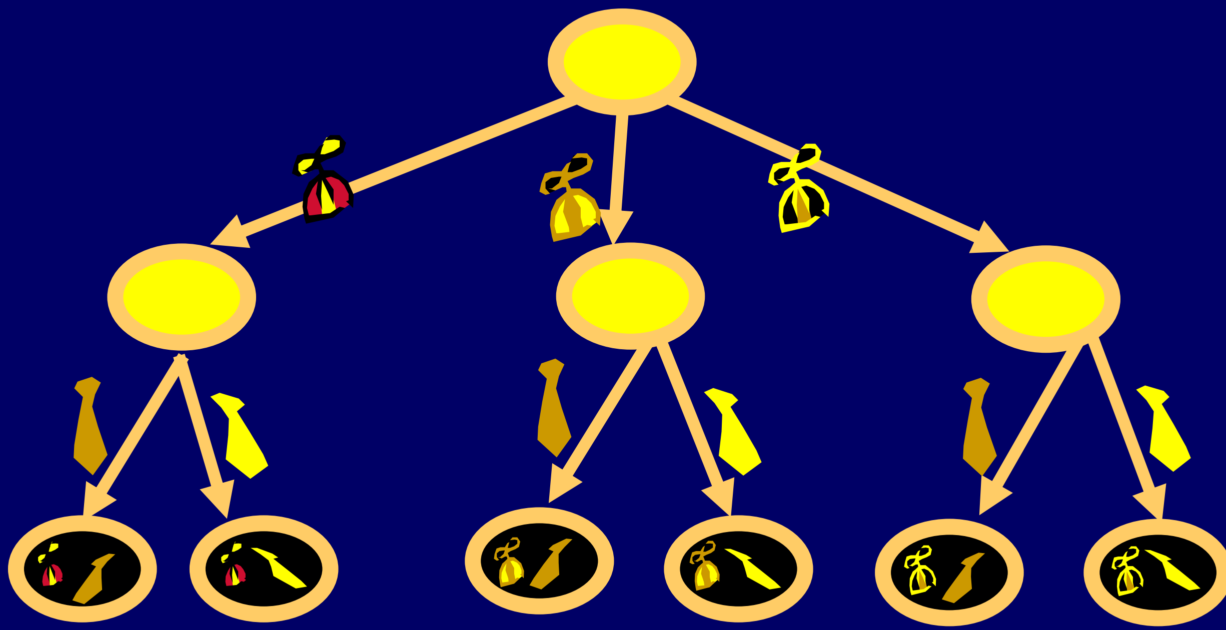
# Correspondence Principle

If two finite sets can be placed into 1-1 onto correspondence, then they have the same size.

If a finite set A has a k-to-1 correspondence to finite set B, then $|B| = |A|/k$

The number of subsets of an n-element set is $2^n$.



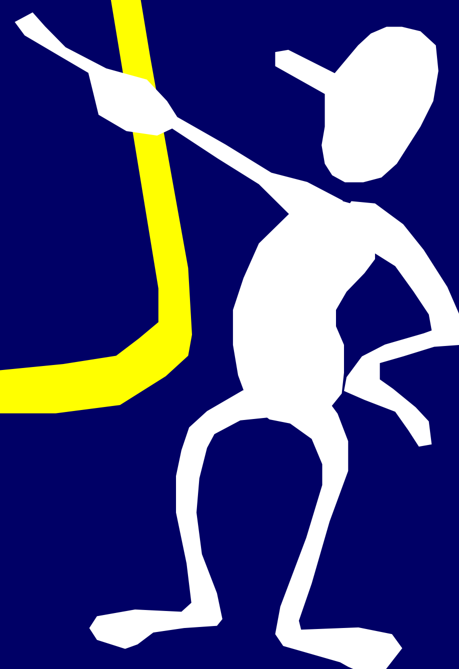


A choice tree provides a "choice tree representation" of a set S, if

1) Each leaf label is in S, and each element of S is some leaf label

2) No two leaf labels are the same

Sometimes it is easiest to count the number of objects with property Q, by counting the number of objects that do not have property Q.



The number of subsets of size r that can be formed from an n-element set is:

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

# Product Rule (rephrased)

Suppose every object of a set S can be constructed by a sequence of choices with $P_1$ possibilities for the first choice, $P_2$ for the second, and so on.

IF 1) Each sequence of choices constructs an object of type S, AND

2) No two different sequences create the same object

THEN

there are $P_1 P_2 P_3 \ldots P_n$ objects of type S.

# How many different orderings of deck with 52 cards?

What type of object are we making?

Ordering of a deck

Construct an ordering of a deck by a sequence of 52 choices:

52 possible choices for the first card;

51 possible choices for the second card;

50 possible choices for the third card;

…

1 possible choice for the $52^{nd}$ card.

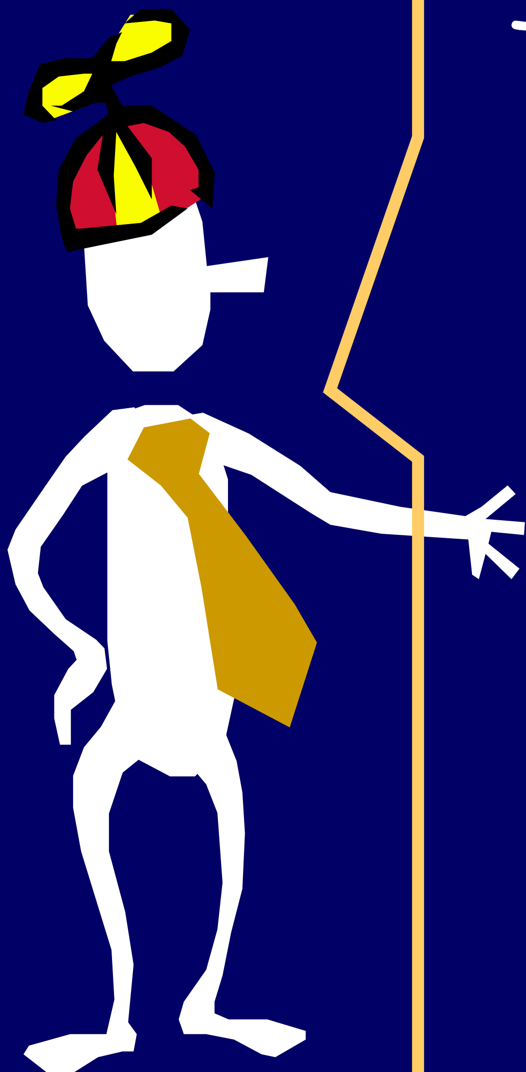
By the product rule: $52 \times 51 \times 50 \times \ldots \times 3 \times 2 \times 1 = 52!$

# The Sleuth's Criterion

There should be a unique way to create every object in S.

in other words:

For any object in S, it should be possible to reconstruct the (unique) sequence of choices which lead to it.

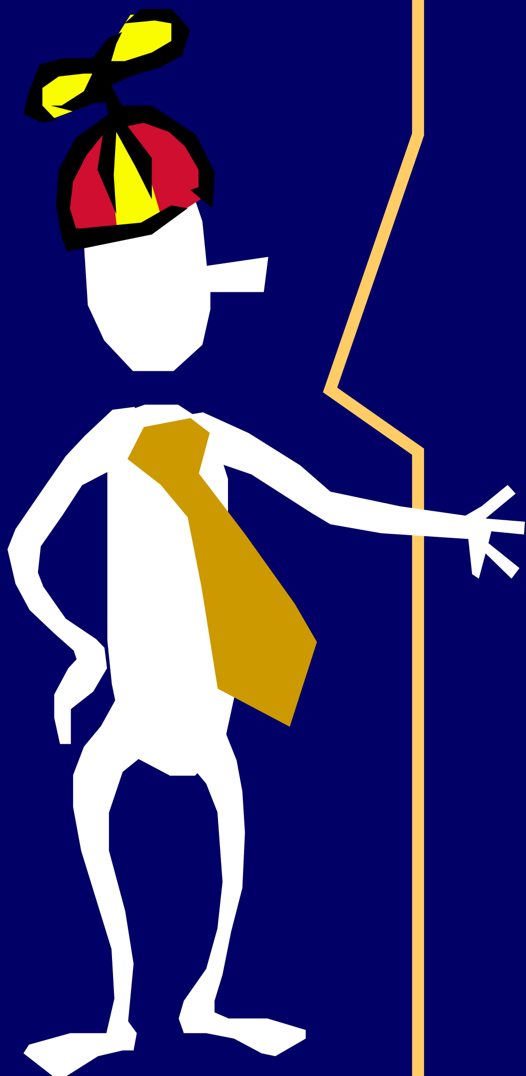


The three big mistakes people make in associating a choice tree with a set S are:

1) Creating objects not in S

2) Missing out some objects from the set S

3) Creating the same object two different ways

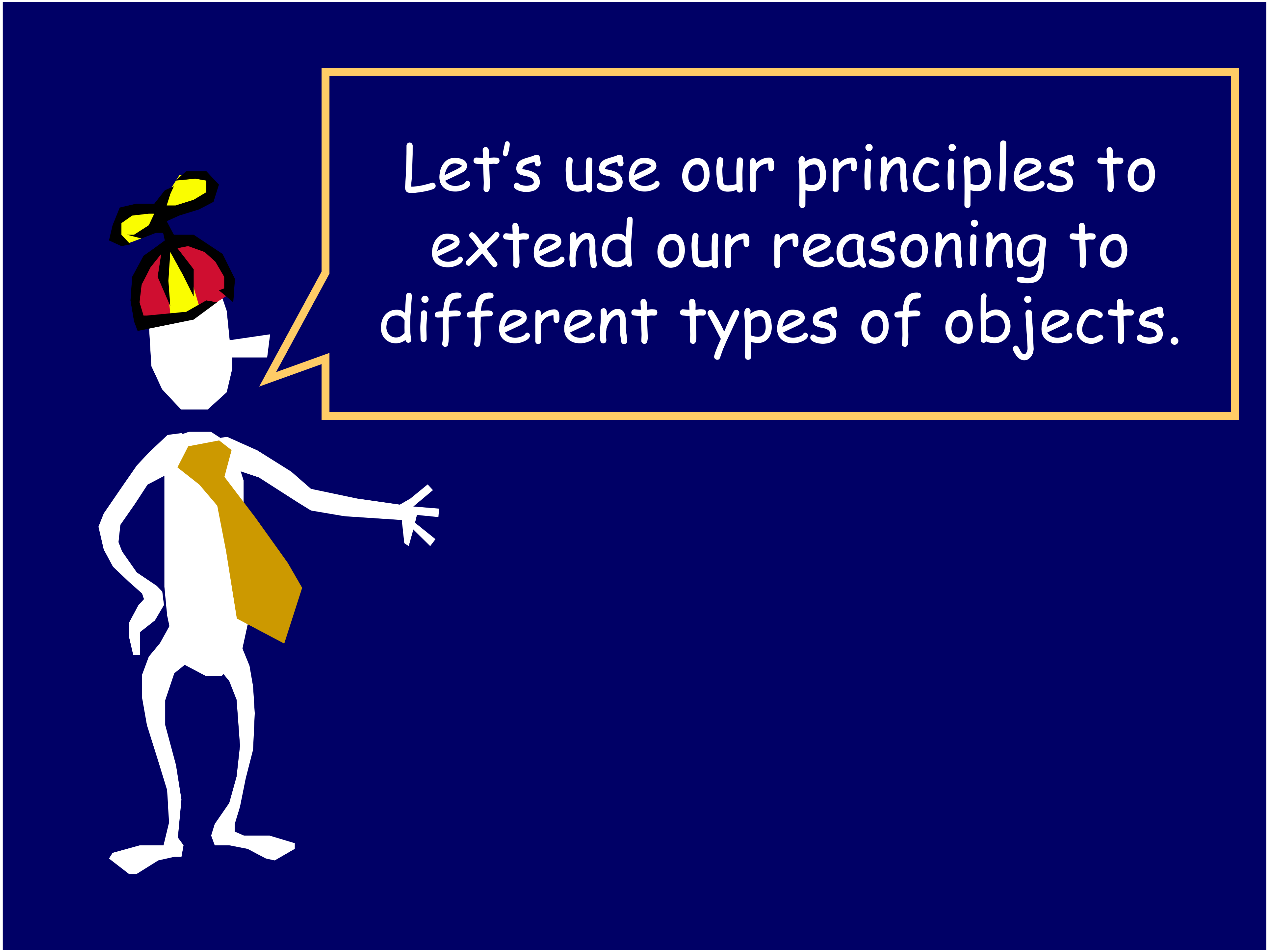


# DEFENSIVE THINKING

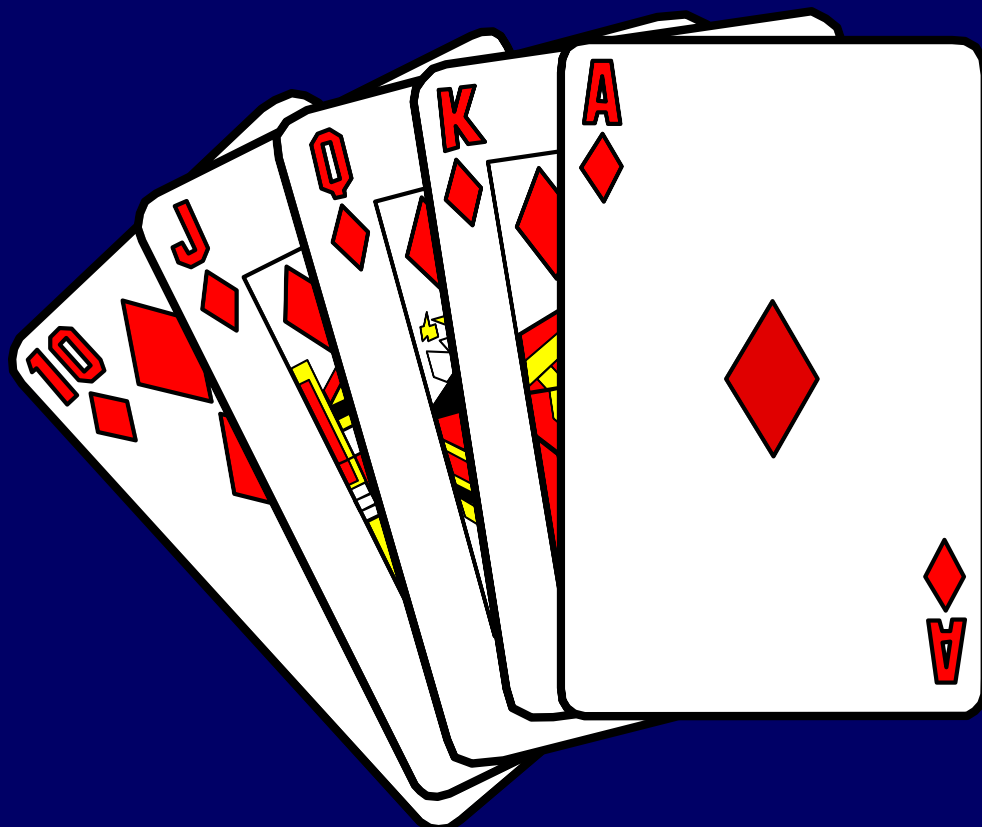
ask yourself:

Am I creating all objects of the right type?

Can I reverse engineer my choice sequence from any given object?


Let's use our principles to extend our reasoning to different types of objects.

# Counting Poker Hands…



# 52 Card Deck, 5 card hands

4 possible suits:

♥ ♦ ♣ ♠

13 possible ranks:

2,3,4,5,6,7,8,9,10,J,Q,K,A

Pair: set of two cards of the same rank

Straight: 5 cards of consecutive rank

Flush: set of 5 cards with the same suit

# Ranked Poker Hands

**Straight Flush**

A straight and a flush

**4 of a kind**

4 cards of the same rank

**Full House**

3 of one kind and 2 of another

**Flush**

A flush, *but not a straight*

**Straight**

A straight, *but not a flush*

**3 of a kind**

3 of the same rank, *but not a full house or 4 of a kind*

**2 Pair**

2 pairs, *but not 4 of a kind or a full house*

**A Pair**

# Straight Flush

9 choices for rank of lowest card at the start of the straight.

4 possible suits for the flush.

$9 \times 4 = 36$

$$\frac{36}{\binom{52}{5}} = \frac{36}{2598960}$$ = 1 in 72,193.33...

# 4 Of A Kind

13 choices of rank

48 choices for remaining card

$13 \times 48 = 624$

$$\frac{624}{\binom{52}{5}} = \frac{624}{2598960} = \frac{1}{4165}$$

# Flush

4 choices of suit

$\binom{13}{5}$ choices of cards

$4 \times 1287$
$= 5148$

"but not a straight flush..."

- 36 straight flushes

5112 flushes

$$\frac{5112}{\binom{52}{5}} = \frac{1}{508.4\cdots}$$

# Straight

9 choices of lowest card in straight

$4^5$ choices of suits for 5 cards

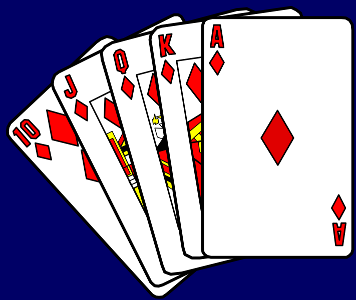
$9 \times 2148$
$= 9216$

"but not a straight flush…"

- 36 straight flushes

9180 flushes

$$\frac{9180}{\binom{52}{5}} = \frac{1}{283.11\cdots}$$



# Storing Poker Hands
# How many bits per hand?

I want to store a 5 card poker hand using the smallest number of bits (space efficient).

each card: 2 bits suit
4 bit rank

6 bits per card

5 cards ⇒ 30 bits (saved 2 bits)


How can we store a poker hand without storing its order?

Order the 2,598,560 Poker hands lexicographically [or in any fixed manner]

To store a hand all I need is to store its index of size $\lceil \log_2(2{,}598{,}560) \rceil = 22$ bits.

Hand 0000000000000000000000
Hand 0000000000000000000001
Hand 0000000000000000000010

.

.

.

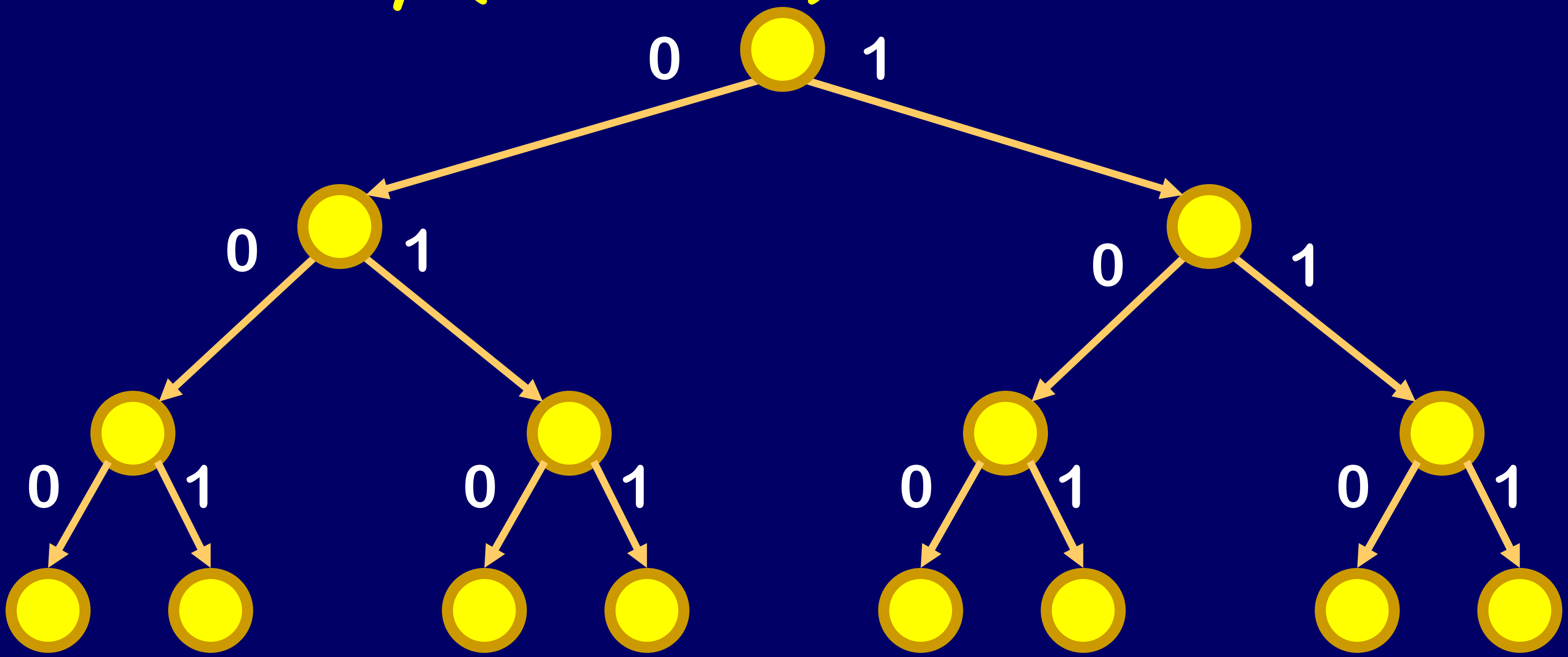
# 22 Bits Is OPTIMAL

$2^{21} = 2{,}097{,}152 < 2{,}598{,}560$

Thus there are more poker hands than there are 21-bit strings.

Hence, you can't have a 21-bit string for each hand.

# Binary (Boolean) Choice Tree




A binary (Boolean) choice tree is a choice tree where each internal node has degree 2.

Usually the choices are labeled 0 and 1.

# 22 Bits Is OPTIMAL

$2^{21} = 2097152 < 2{,}598{,}560$

A binary choice tree of depth 21 can have at most $2^{21}$ leaves.

Hence, there are not enough leaves for all 5-card hands.

An n-element set can be stored so that each element uses $\lceil \log_2(n) \rceil$ bits.

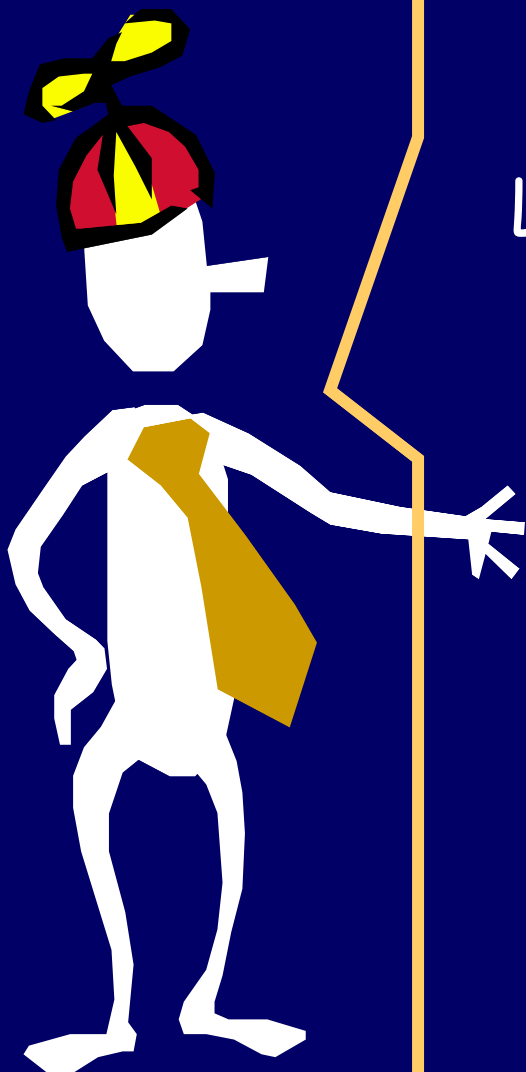
Furthermore, any representation of the set will have some string of at least that length.

## Information Counting Principle:

If each element of a set can be represented using $k$ bits, the size of the set is bounded by $2^k$

## Information Counting Principle:

Let S be a set represented by a depth-k binary choice tree, the size of the set is bounded by $2^k$



ONGOING MEDITATION:

Let S be any set and T be a binary choice tree representation of S.

We can think of each element of S being encoded by the binary sequences of choices that lead to its leaf.

We can also start with a binary encoding of a set and make a corresponding binary choice tree.

Now, for something completely different…

How many ways to rearrange the letters in the word **"SYSTEMS"**?

# SYSTEMS

1) 7 places to put the Y,
6 places to put the T,
5 places to put the E,
4 places to put the M,
and the S's are forced.

$$7 \times 6 \times 5 \times 4 = 840$$

$\binom{7}{3}$ ways of placing the S's

4 ways of placing Y

3 ______ T

2 ______ E

$\binom{7}{3} \times 4 \times 3 \times 2 \times 1$

$= 840.$

# SYSTEMS

2) Let's pretend that the S's are distinct:

$S_1YS_2TEMS_3$

There are 7! permutations of $S_1YS_2TEMS_3$

But when we stop pretending we see that we have counted each arrangement of SYSTEMS 3! times, once for each of 3! rearrangements of $S_1S_2S_3$.

$S_1 Y S_2 TEM S_3$

$S_3 Y S_2 TEM S_1$

$$\frac{7!}{3!} = 840$$

# Arrange n symbols
# $r_1$ of type 1, $r_2$ of type 2, …, $r_k$ of type k

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\binom{n-r_1-r_2}{r_2}\cdots\binom{r_k}{r_k}$$

$$= \frac{n!}{(n-r_1)!\,r_1!} \cdot \frac{(n-r_1)!}{(n-r_1-r_2)!\,r_2!}$$

$$= \frac{n!}{r_1!r_2!\cdots r_k!}$$

# CARNEGIEMELLON

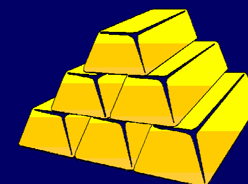
$$\frac{14!}{2!3!2!} = 3,632,428,800$$

Remember:

The number of ways to arrange n symbols with $r_1$ of type 1, $r_2$ of type 2, …, $r_k$ of type k is:

$$\frac{n!}{r_1!\,r_2!\cdots r_k!}$$

5 distinct pirates want to divide 20 identical, indivisible bars of gold. How many different ways can they divide up the loot?



# Sequences with 20 G's and 4 /'s

GG/G//GGGGGGGGGGGGGGGGG/

represents the following division among the pirates: 2, 1, 0, 17, 0

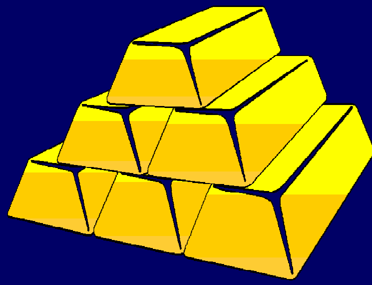
In general, the $i^{th}$ pirate gets the number of G's after the $i$-$1^{st}$ / and before the $i^{th}$ /.

This gives a correspondence between divisions of the gold and sequences with 20 G's and 4 /'s.

How many different ways to divide up the loot?

Sequences with 20 G's and 4 /'s

$$\binom{24}{4}$$



How many different ways can n distinct pirates divide k identical, indivisible bars of gold?

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

# How many integer solutions to the following equations?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

$$x_1 \;, x_2, x_3, x_4, x_5 \geq 0$$

Think of $X_k$ as being the number of gold bars that are allotted to pirate k.

$$\binom{24}{4}$$

How many integer solutions to the following equations?

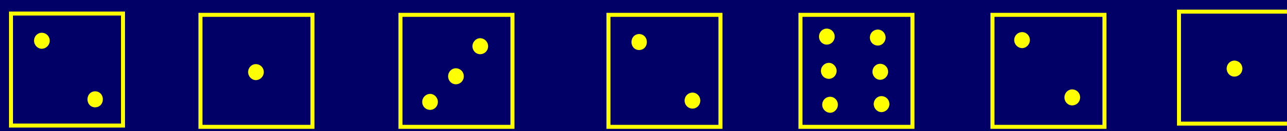
$$x_1 + x_2 + x_3 + \ldots + x_{n-1} + x_n = k$$

$$x_1, x_2, x_3, \ldots, x_{n-1}, x_n \geq 0$$

$$\binom{n + k - 1}{n - 1} = \binom{n + k - 1}{k}$$

# Identical/Distinct Dice

Suppose that we roll seven dice.



How many different outcomes are there, if order matters?

$$6^7$$

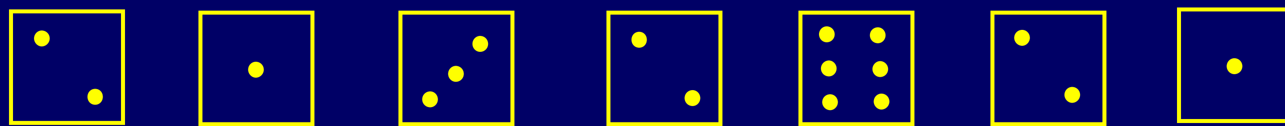
What if order doesn't matter?
(E.g., Yahtzee)

$$\binom{12}{7}$$

# 7 Identical Dice

| | | | | | G G G G G G G

$$\frac{12!}{7!\,5!}$$



How many different outcomes?

$X_1 + X_2 + \cdots + X_6 = 7$

$X_i \geq 0$

Corresponds to 6 pirates
and 7 bars of gold!

$$\binom{6+7-1}{7}$$

Let $X_k$ be the number of dice showing k.
The $k^{th}$ pirate gets $X_k$ gold bars.

$$\binom{6 + 7 - 1}{7}$$

# Multisets

A multiset is a set of elements, each of which has a *multiplicity*.

The size of the multiset is the sum of the multiplicities of all the elements.

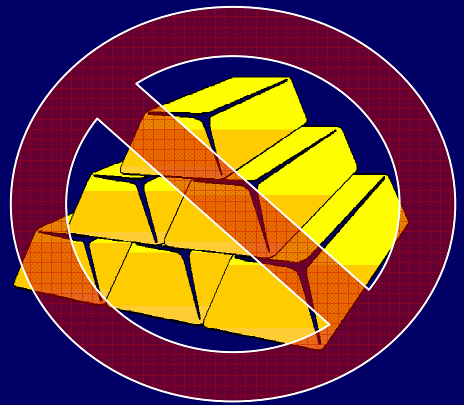
Example:
{X, Y, Z} with m(X)=0 m(Y)=3, m(Z)=2

Unary visualization: {Y, Y, Y, Z, Z}

# Counting Multisets

There are $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$ ways to choose a multiset of size k from n types of elements

# Back to the pirates



How many ways are there of choosing 20 pirates from a set of 5 pirates, with repetitions allowed?

$$\binom{5+20-1}{20} = \binom{24}{20} = \binom{24}{4}$$

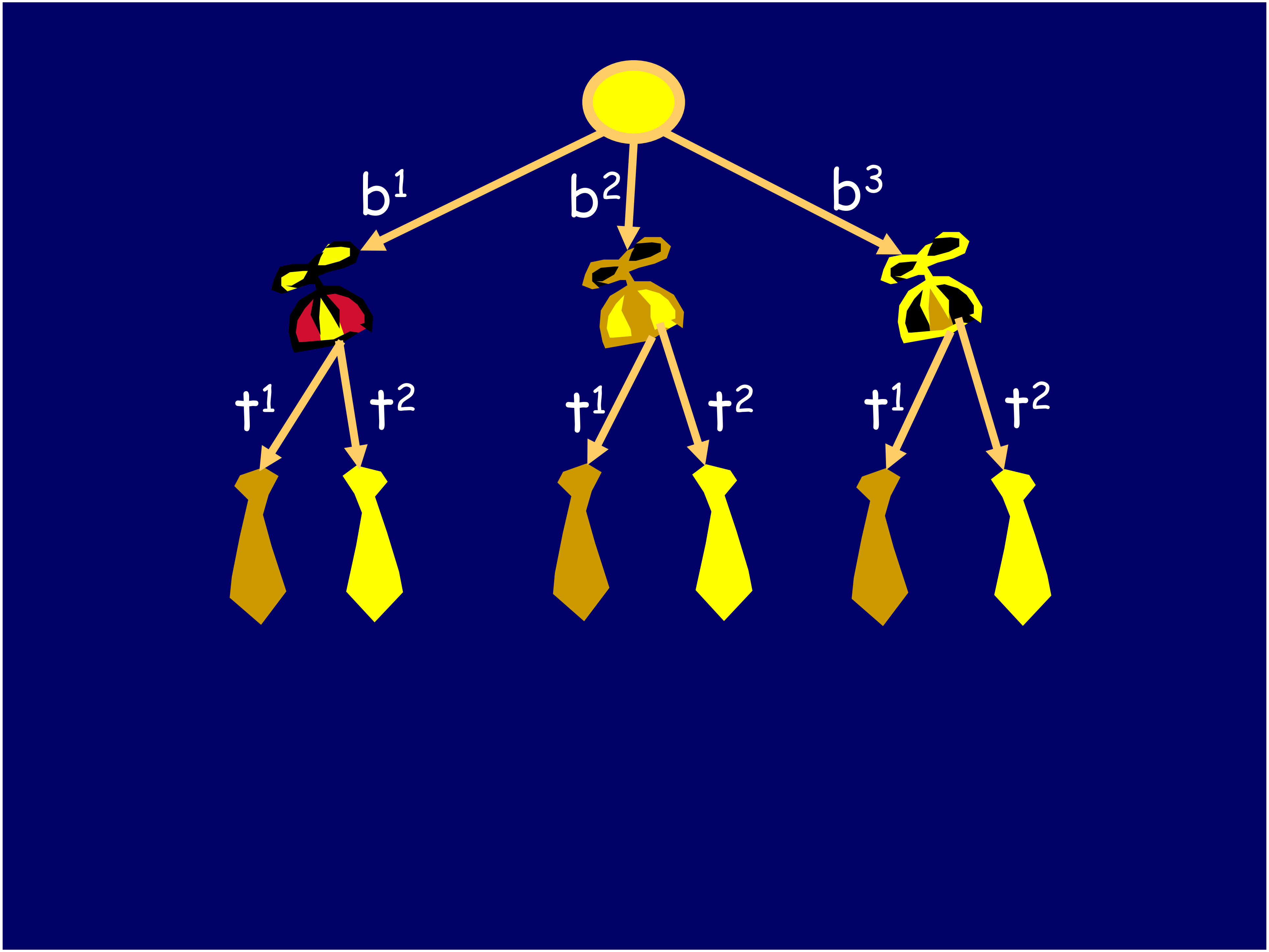
$$x_1 + x_2 + x_3 + \ldots + x_{n-1} + x_n = k$$

$$x_1, x_2, x_3, \ldots, x_{n-1}, x_n \geq 0$$

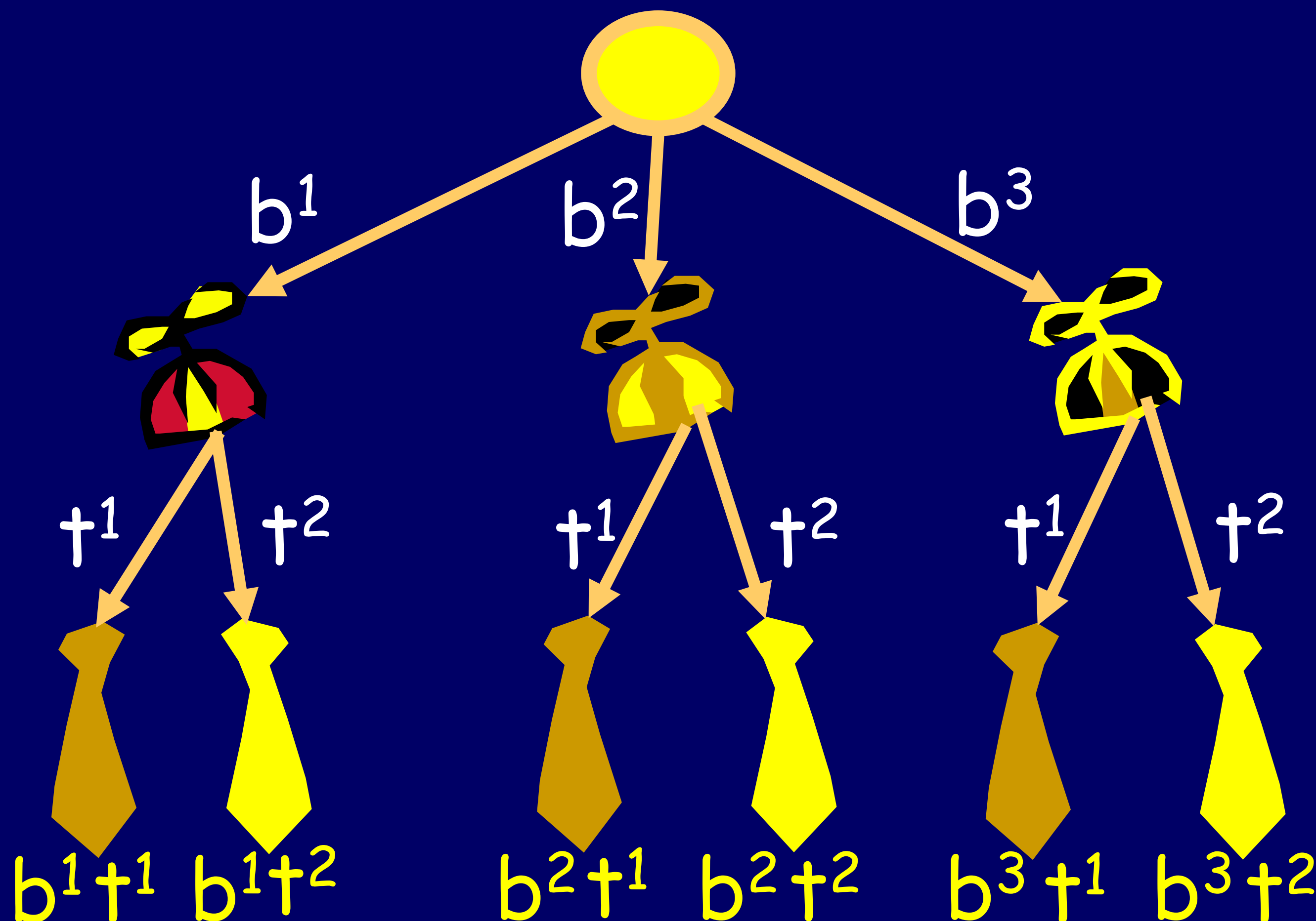
has $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$ integer solutions.

# POLYNOMIALS EXPRESS CHOICES AND OUTCOMES

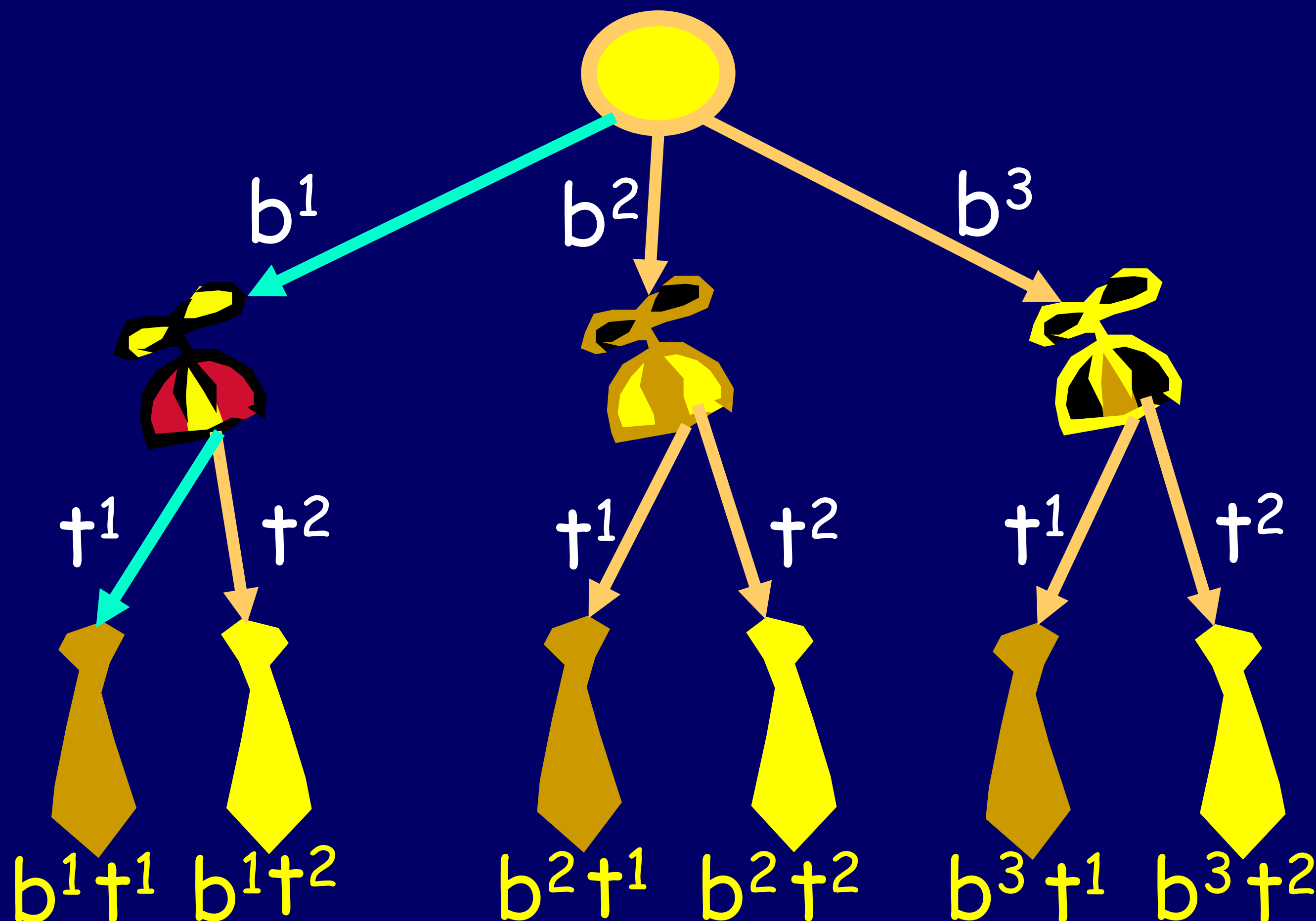
## Products of Sum = Sums of Products



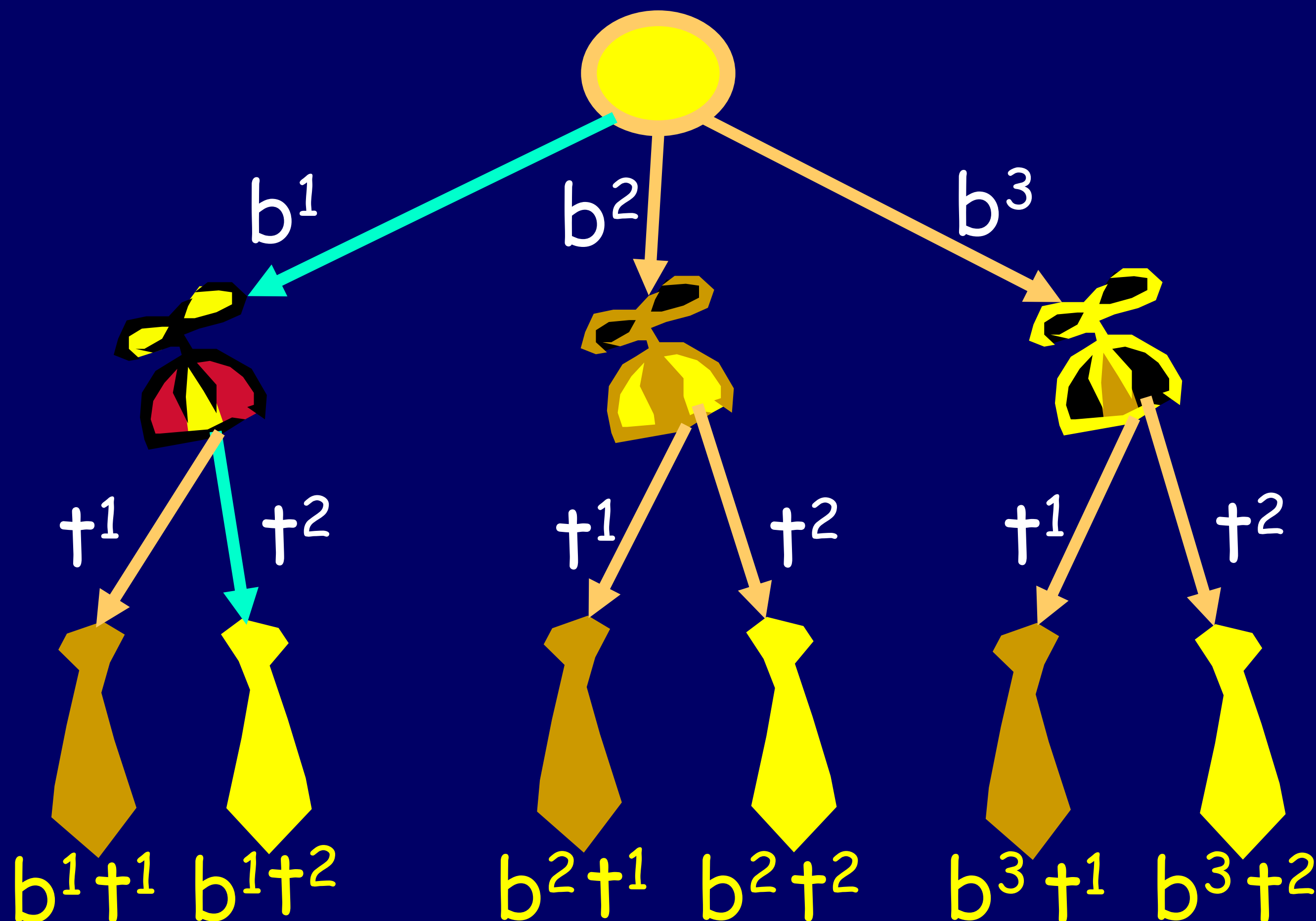
b1
b2
b3
t1
t2
t1
t2
t1
t2



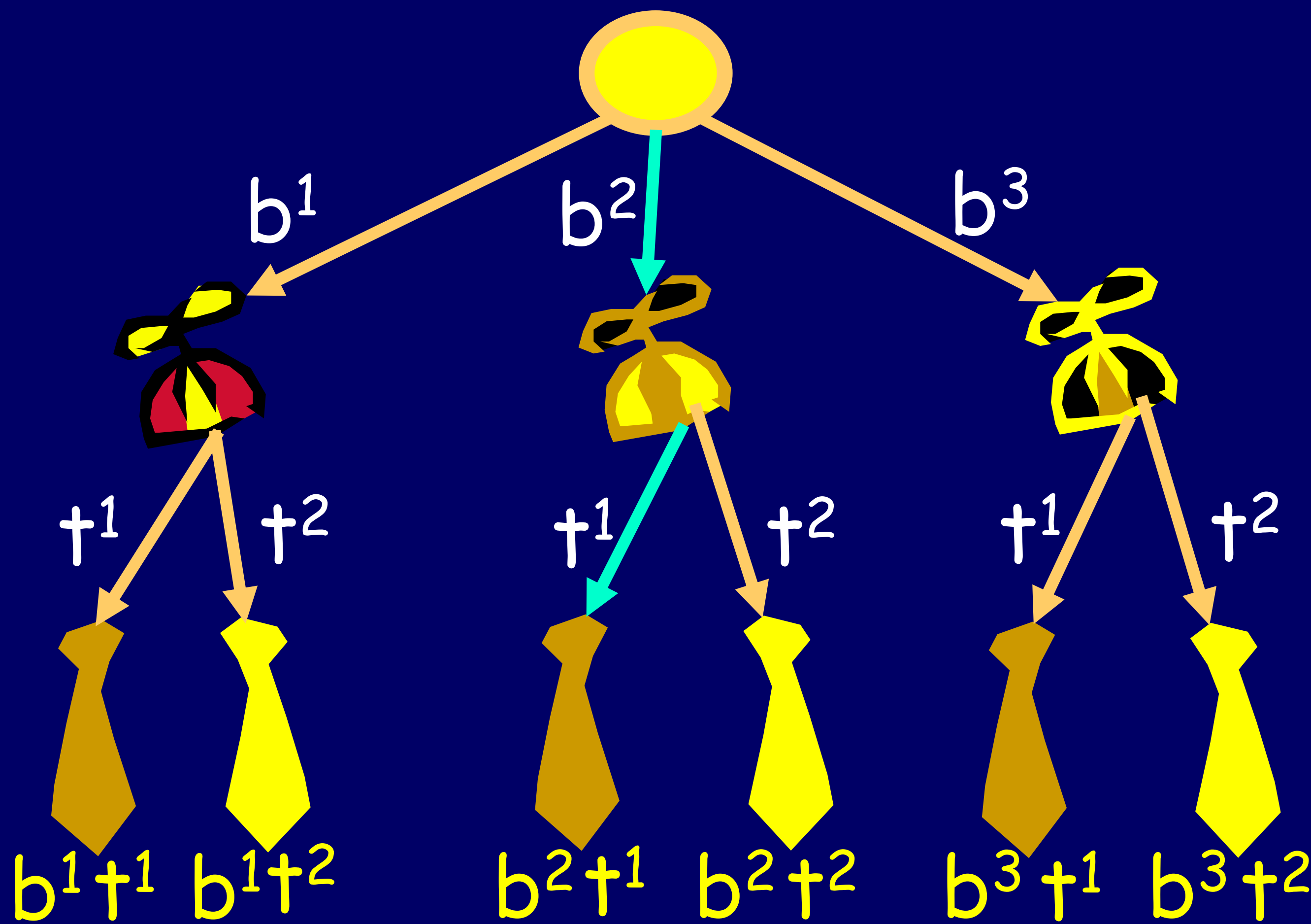

$(b^1 + b^2 + b^3)(t^1 + t^2) =$



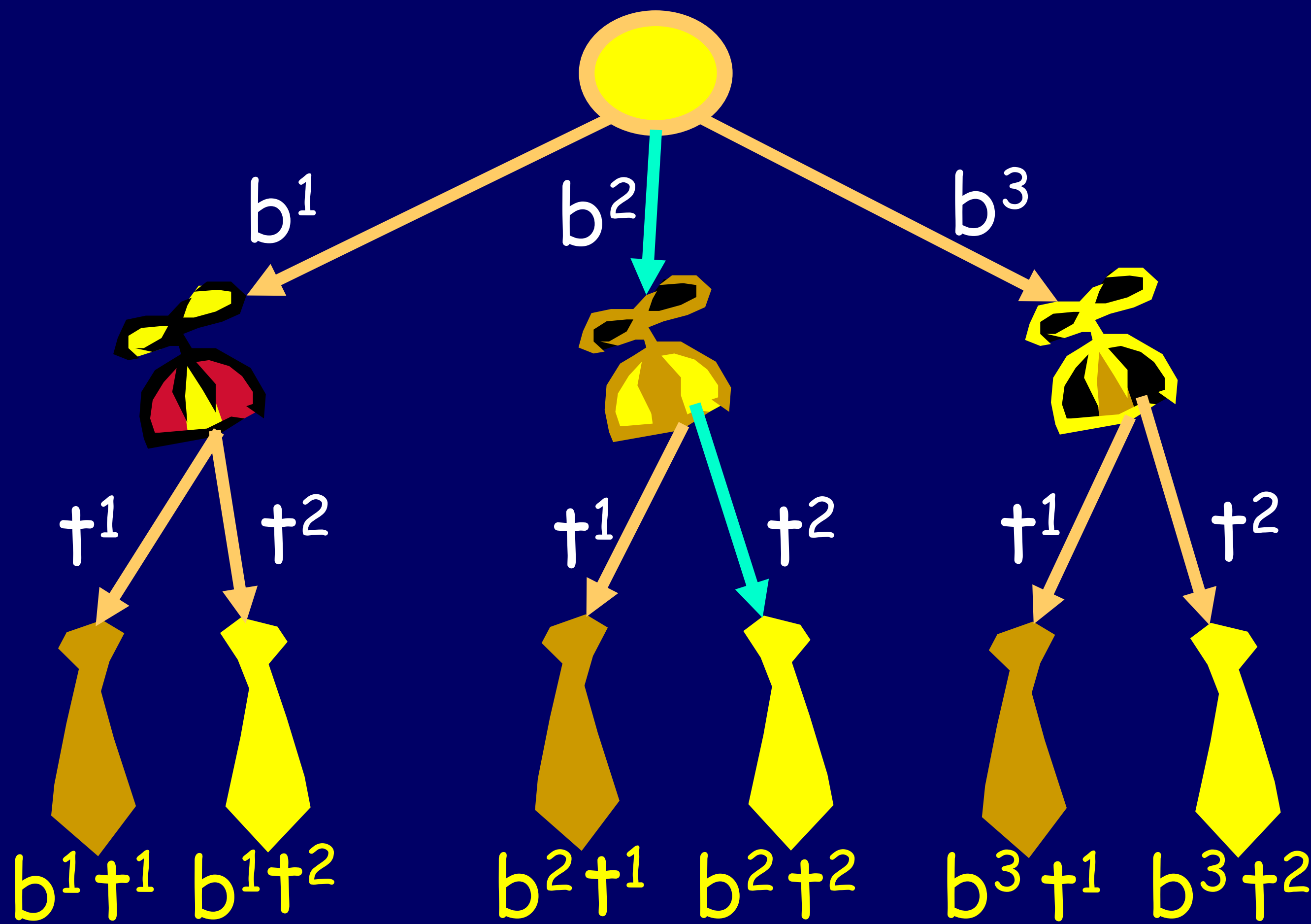

$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 +$




$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 +$

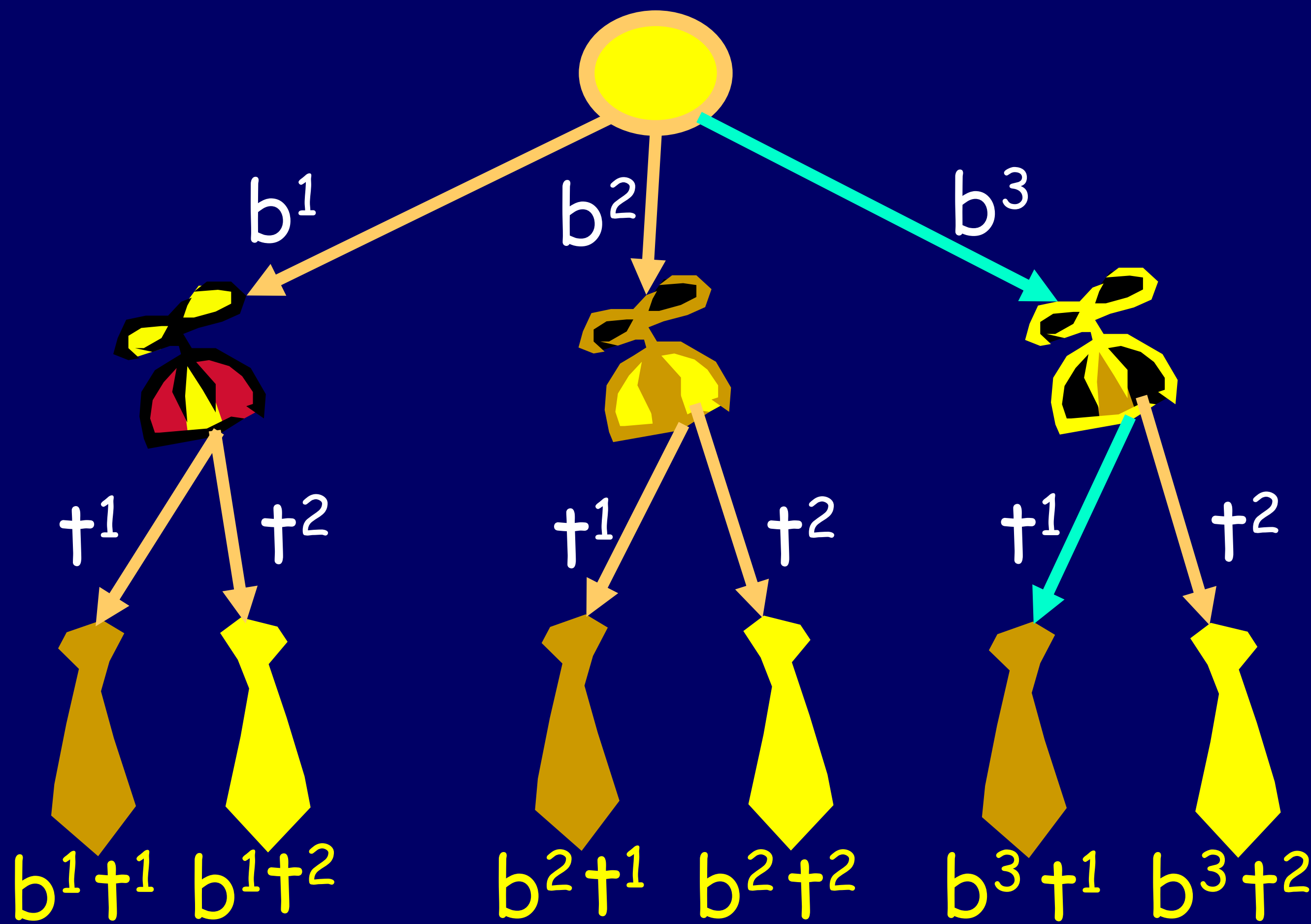



$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 + b^2t^1 +$

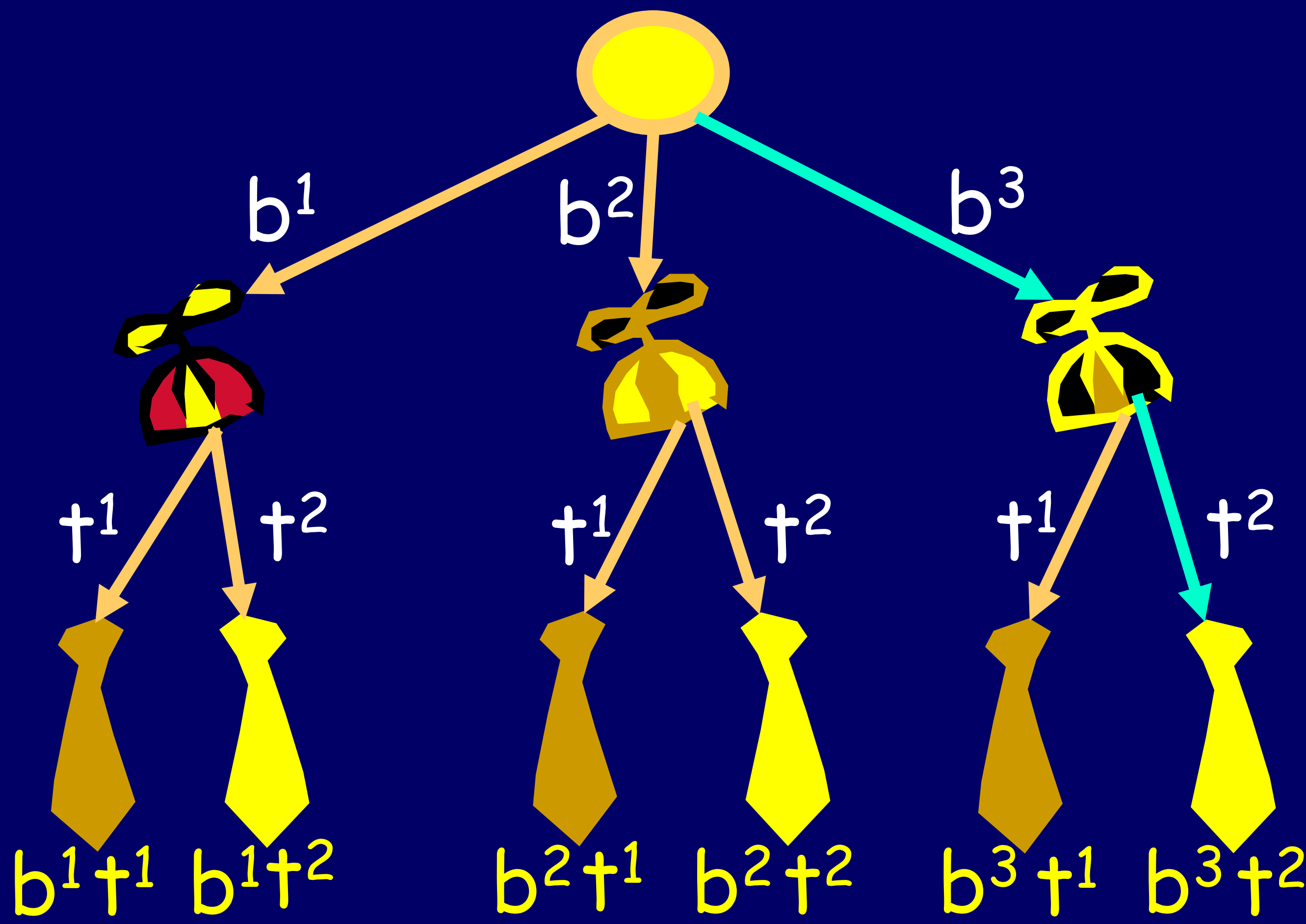



$$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 + b^2t^1 + b^2t^2$$

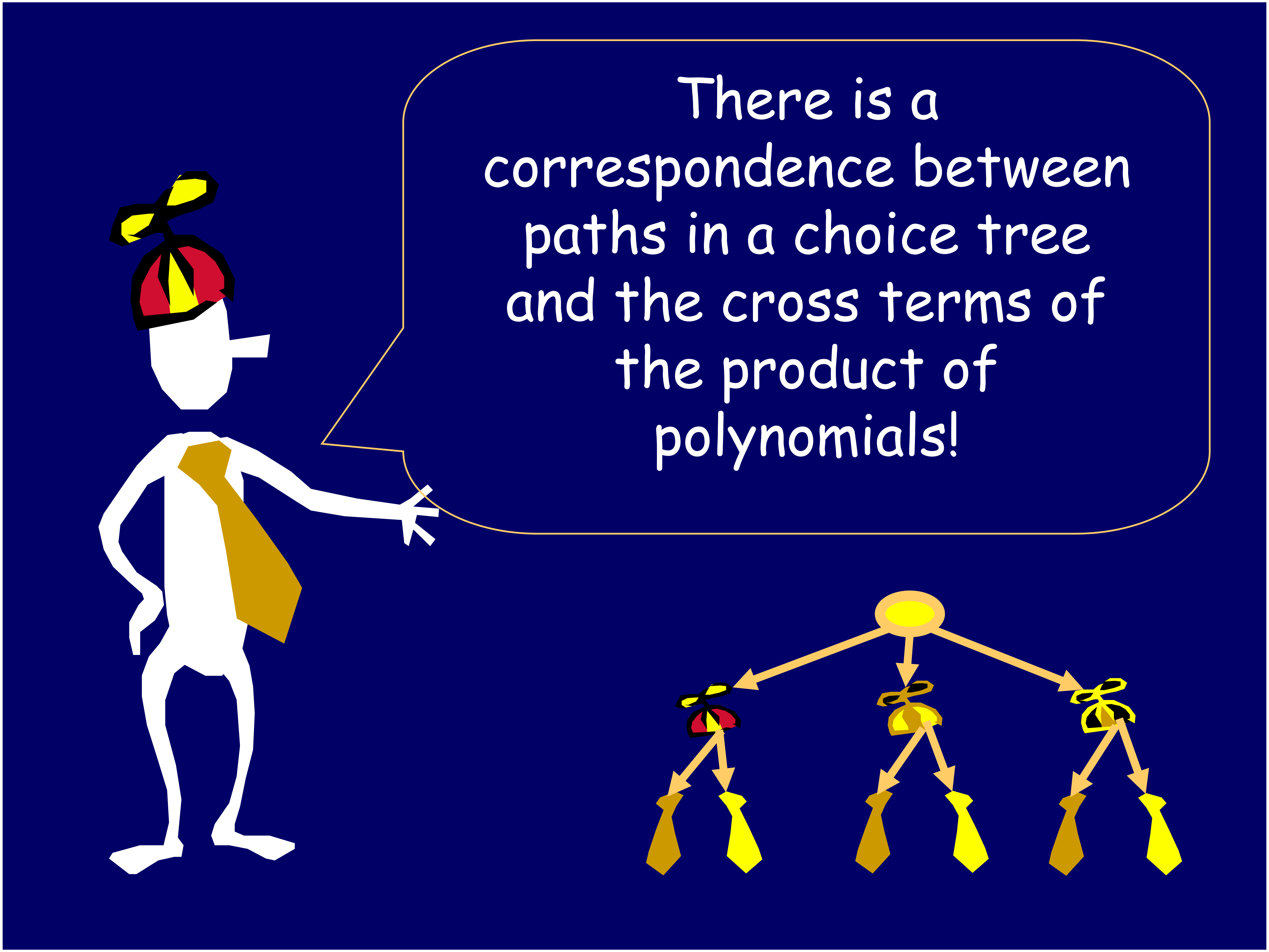
$$+$$




$$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 + b^2t^1 + b^2t^2$$
$$+ b^3t^1 +$$

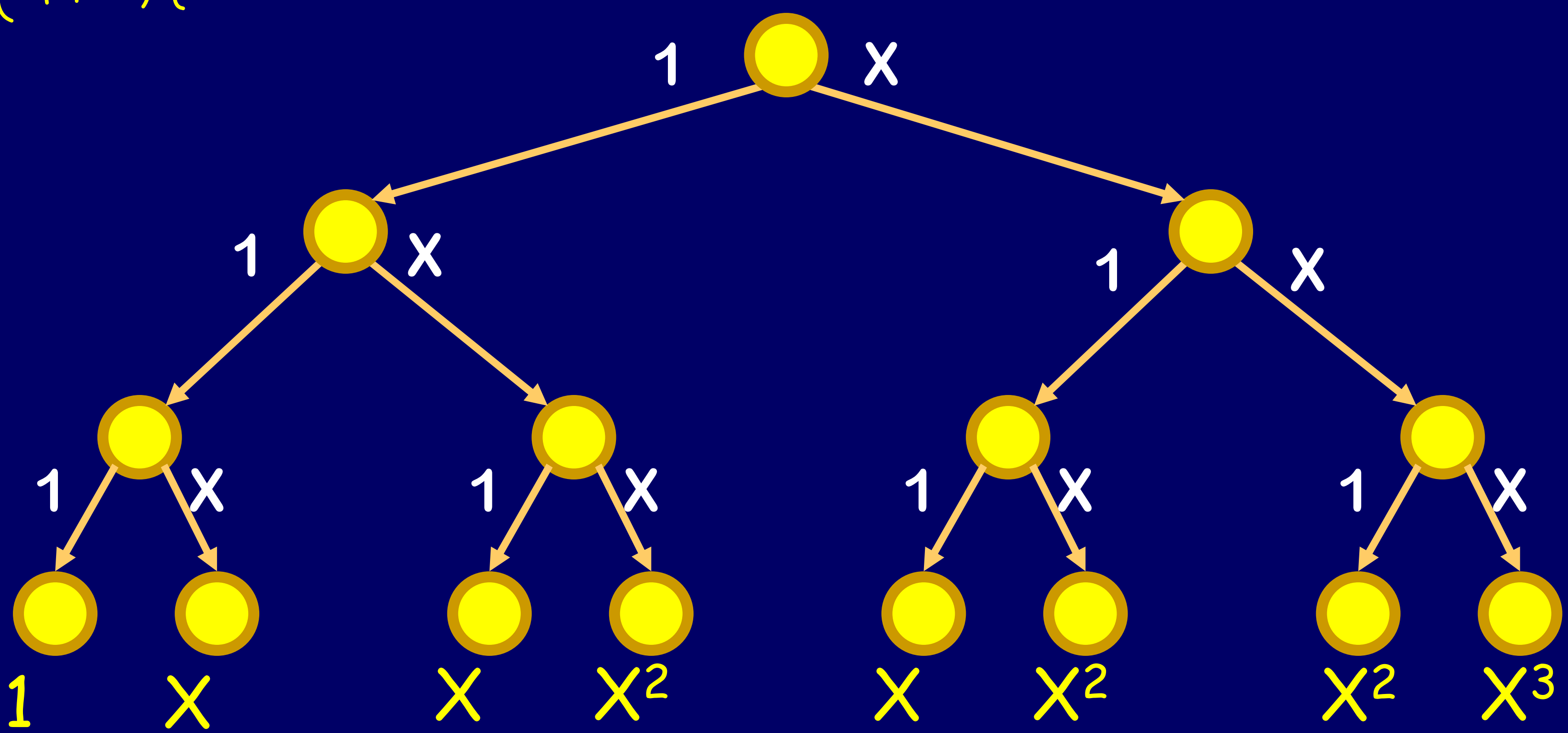



$$(b^1 + b^2 + b^3)(t^1 + t^2) = b^1t^1 + b^1t^2 + b^2t^1 + b^2t^2 + b^3t^1 + b^3t^2$$


There is a correspondence between paths in a choice tree and the cross terms of the product of polynomials!

# Choice tree for terms of $(1+X)^3$

$(1+x)(1+x)(1+x)$




Combine like terms to get $1 + 3X + 3X^2 + X^3$

# What is a closed form expression for $c_k$?

$$(1+X)^n = c_0 + c_1 X + c_2 X^2 + \ldots + c_n X^n$$

# What is a closed form expression for $c_n$?

$(1+X)^n$

$$= \underbrace{(1+X)(1+X)(1+X)(1+X)\ldots(1+X)}_{\text{n times}}$$

"choose" k (1+X) groups from which you pick up an X

After multiplying things out, but *before combining like terms*, we get $2^n$ cross terms, each corresponding to a path in the choice tree.

$c_k$, the coefficient of $X^k$, is the number of paths with *exactly k X's*.

$$c_k = \binom{n}{k}$$

# The Binomial Formula

$$(1+X)^n = \binom{n}{0} + \binom{n}{1}X + \binom{n}{2}X^2 + \ldots + \binom{n}{k}X^k + \ldots + \binom{n}{n}X^n$$

Binomial Coefficients

binomial expression

# The Binomial Formula

$$(1+X)^0 = 1$$

$$(1+X)^1 = 1 + 1X$$

$$(1+X)^2 = 1 + 2X + 1X^2$$

$$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$$

$$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$$

$$(1+3X+3X^2+1X^3)(1+X)$$

# The Binomial Formula

$$(X+Y)^n$$

$$= \binom{n}{0}X^0Y^n + \binom{n}{1}X^1Y^{n-1} + \binom{n}{2}X^2Y^{n-2} + \ldots + \binom{n}{k}X^kY^{n-k} + \ldots + \binom{n}{n}X^nY^0$$

$$(x+y)^n = x^n\left(1+\frac{y}{x}\right)^n$$

$$= x^n\left[\binom{n}{0} + \binom{n}{1}\left(\frac{y}{x}\right)^1 + \quad \ldots \quad + \binom{n}{n}\left(\frac{y}{x}\right)^n\right]$$

$$= \binom{n}{0}x^ny^0 + \binom{n}{1}x^{n-1}y^1 \quad \ldots \quad + \binom{n}{n}y^nx^0$$

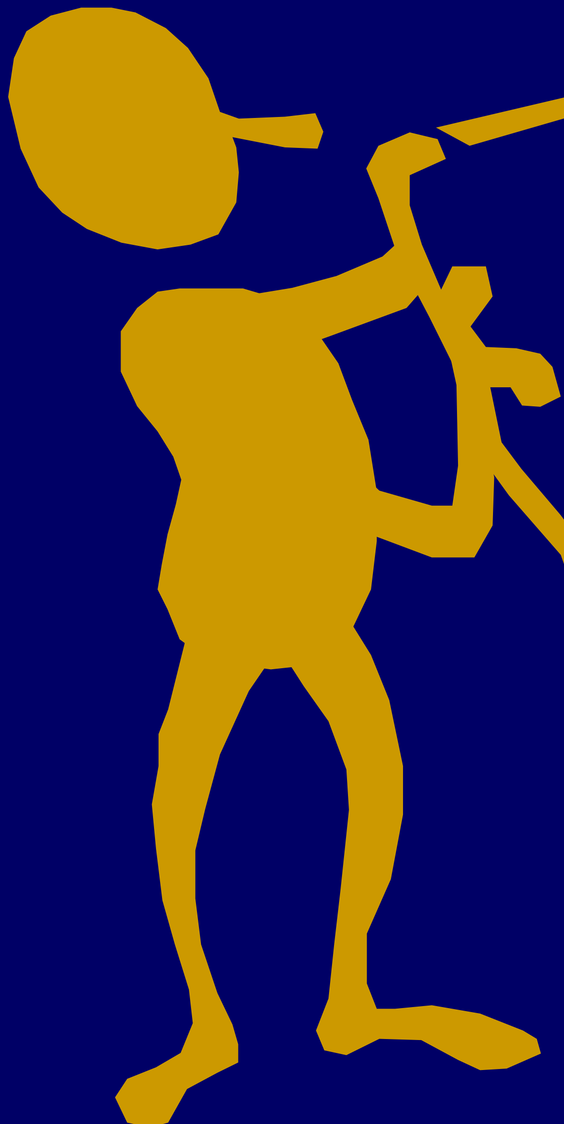
# The Binomial Formula

$$(X+Y)^n = \sum_{k=0}^{k=n} \binom{n}{k} X^k Y^{n-k}$$

What is the coefficient of EMSTY in the expansion of $(E + M + S + T + Y)^5$?

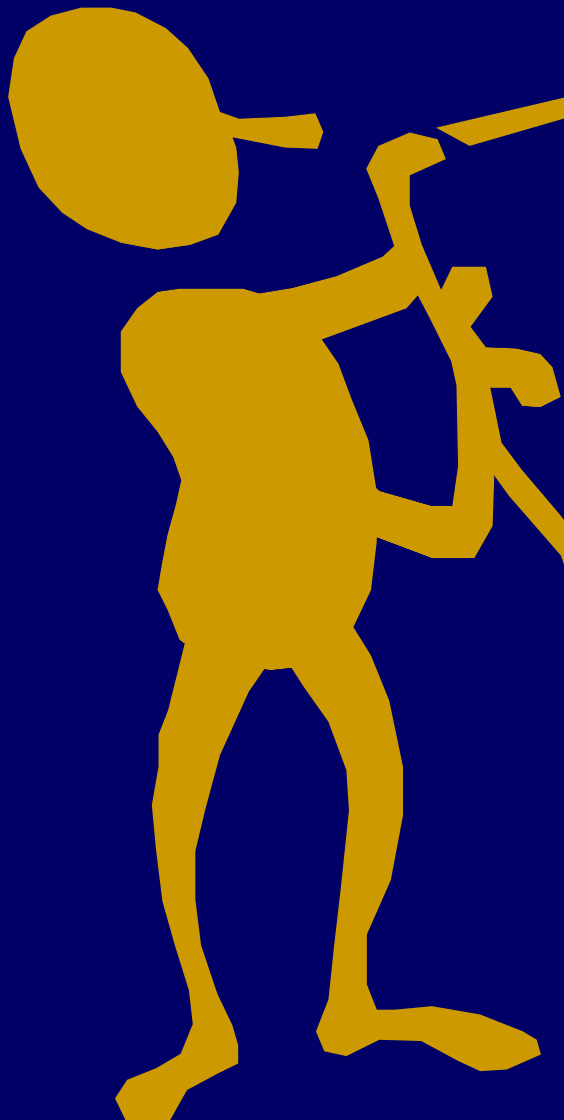
$(E+M+S+T+Y)(E+M+S+T+Y) \cdots (E+M+S+T+Y)$

$= \_\_\_\_\_\_ + 5!\, EMSTY$



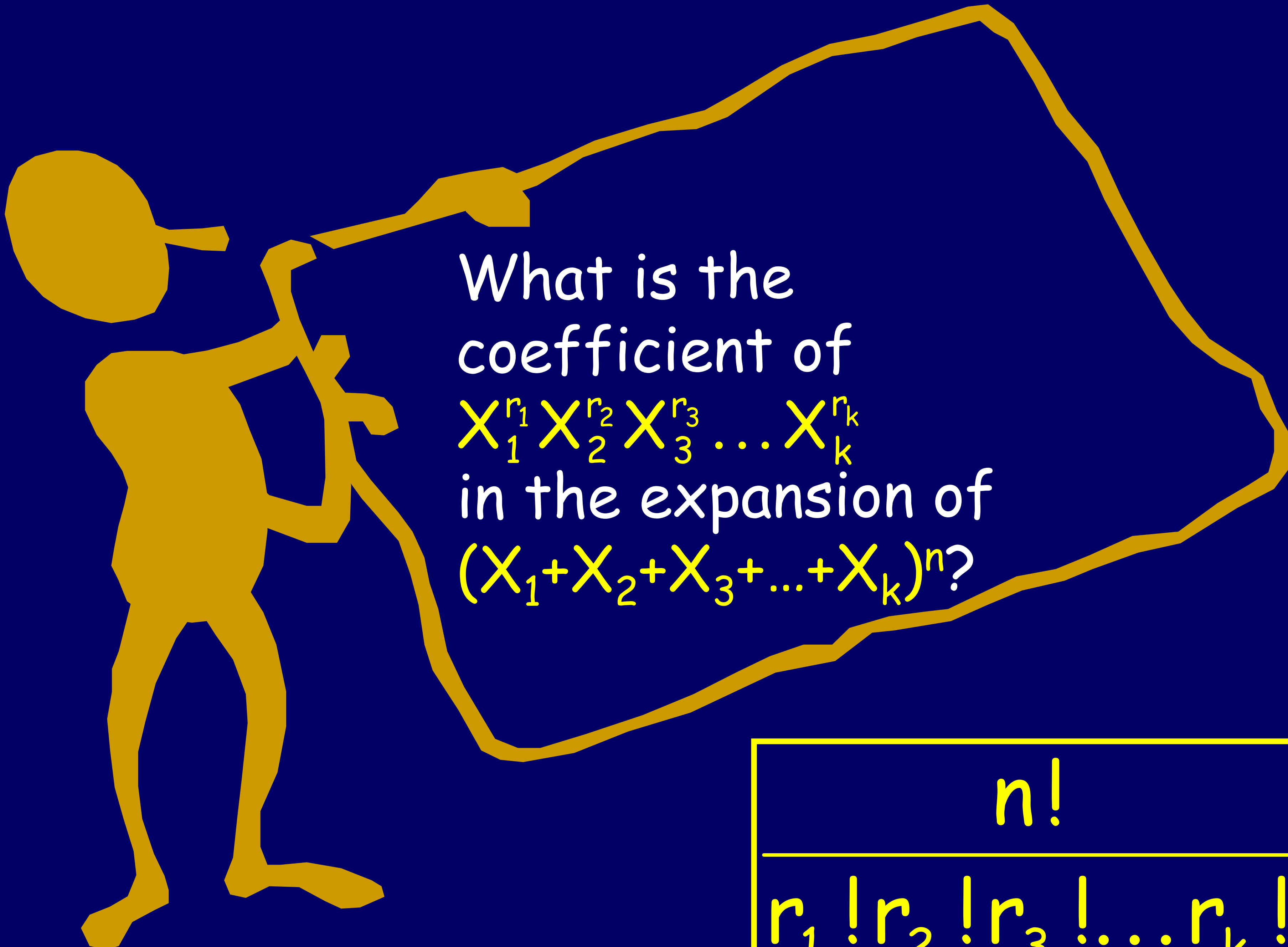
What is the coefficient of $EMS^3TY$ in the expansion of $(E + M + S + T + Y)^7$?

The number of ways to rearrange the letters in the word SYSTEMS.



What is the coefficient of $BA^3N^2$ in the expansion of $(B + A + N)^6$?

The number of ways to rearrange the letters in the word BANANA.



What is the coefficient of $X_1^{r_1} X_2^{r_2} X_3^{r_3} \ldots X_k^{r_k}$ in the expansion of $(X_1+X_2+X_3+\ldots+X_k)^n$?

$$\frac{n!}{r_1!\,r_2!\,r_3!\ldots r_k!}$$

# Multinomial Coefficients

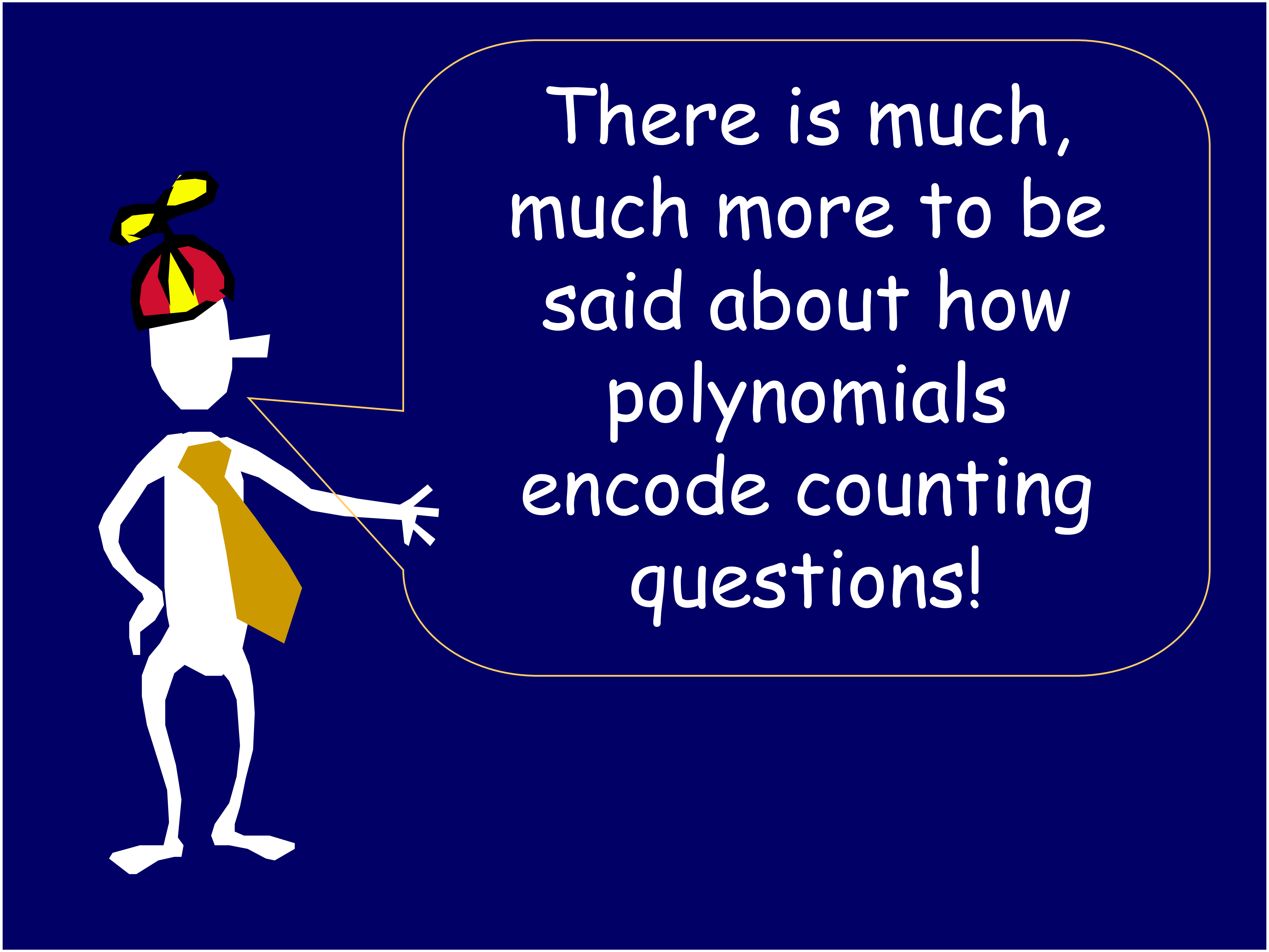
$$\binom{n}{r_1; r_2; \ldots; r_k} \equiv \begin{cases} 0 \text{ if } r_1 + r_2 + \ldots + r_k \neq n \\ \dfrac{n!}{r_1! r_2! \ldots r_k!} \end{cases}$$

$$\binom{n}{k;\ n-k} = \binom{n}{k}$$

# The Multinomial Formula

$$\left(X_1 + X_2 + \ldots + X_k\right)^n$$

$$= \sum_{\substack{r_1, r_2, \ldots, r_k \\ \sum r_i = n}} \underbrace{\binom{n}{r_1; r_2; \ldots; r_k}}_{\text{multinomial coefficient.}} X_1^{r_1} X_2^{r_2} X_3^{r_3} \ldots X_k^{r_k}$$


There is much, much more to be said about how polynomials encode counting questions!

# References

*Applied Combinatorics*, by Alan Tucker