Great Theoretical Ideas In Computer Science

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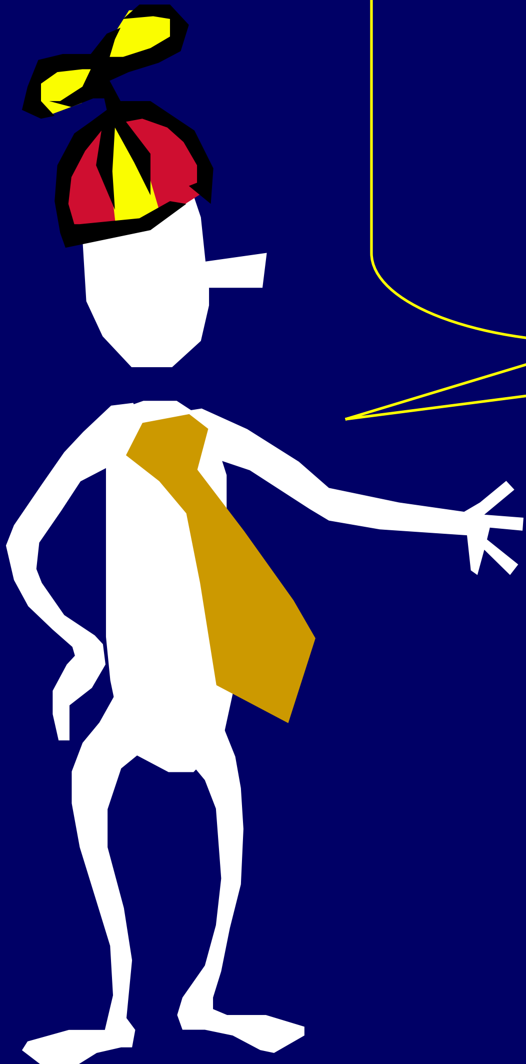
Carnegie Mellon University

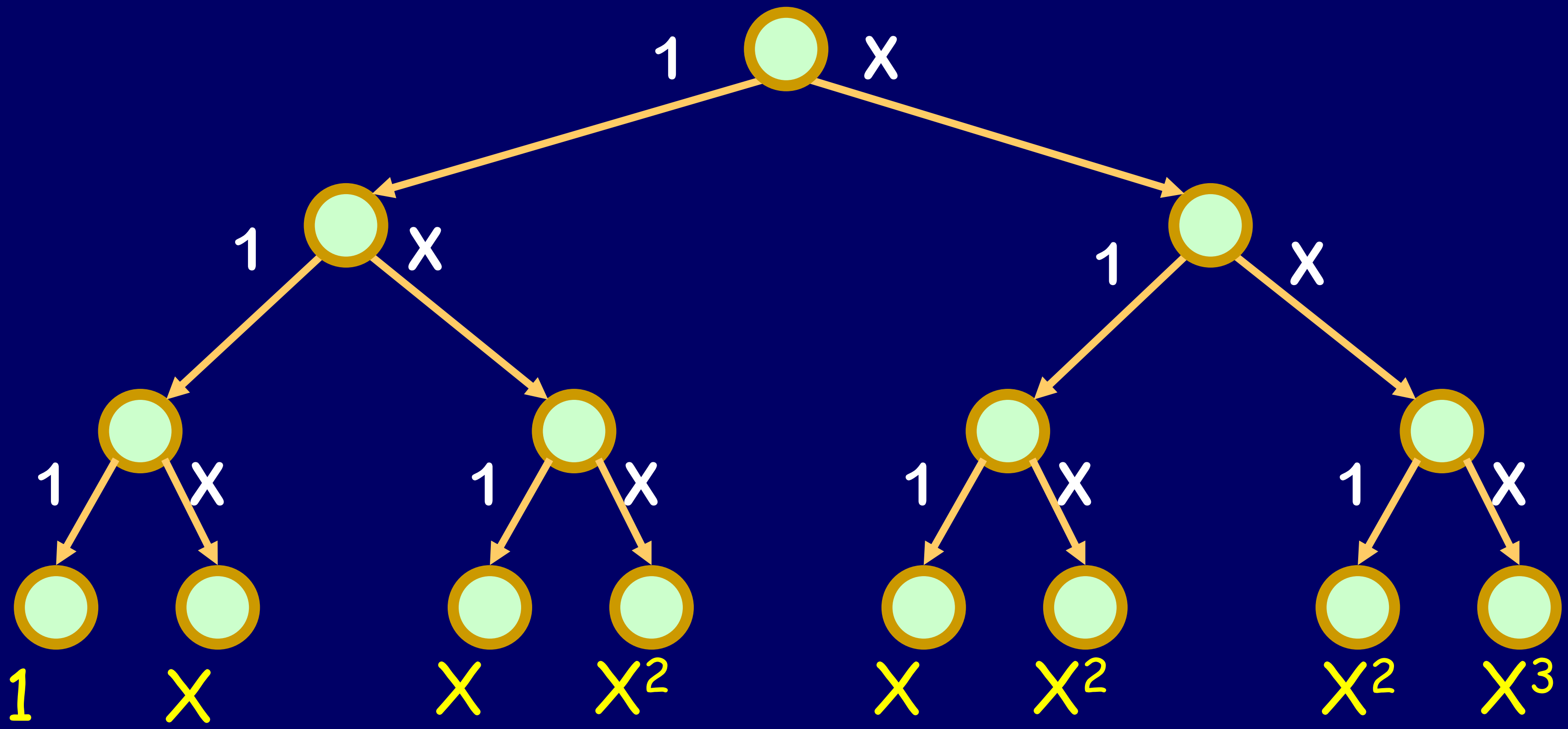
Counting III: Pascal's Triangle, Polynomials, and Vector Programs

Last time, we saw that

Polynomials Count!



Choice tree for terms of $(1+X)^3$



Combine like terms to get $1 + 3X + 3X^2 + X^3$

The Binomial Formula

$$(1+X)^n = \binom{n}{0} + \binom{n}{1}X + \binom{n}{2}X^2 + \ldots + \binom{n}{k}X^k + \ldots + \binom{n}{n}X^n$$

Binomial Coefficients

binomial expression

The Binomial Formula

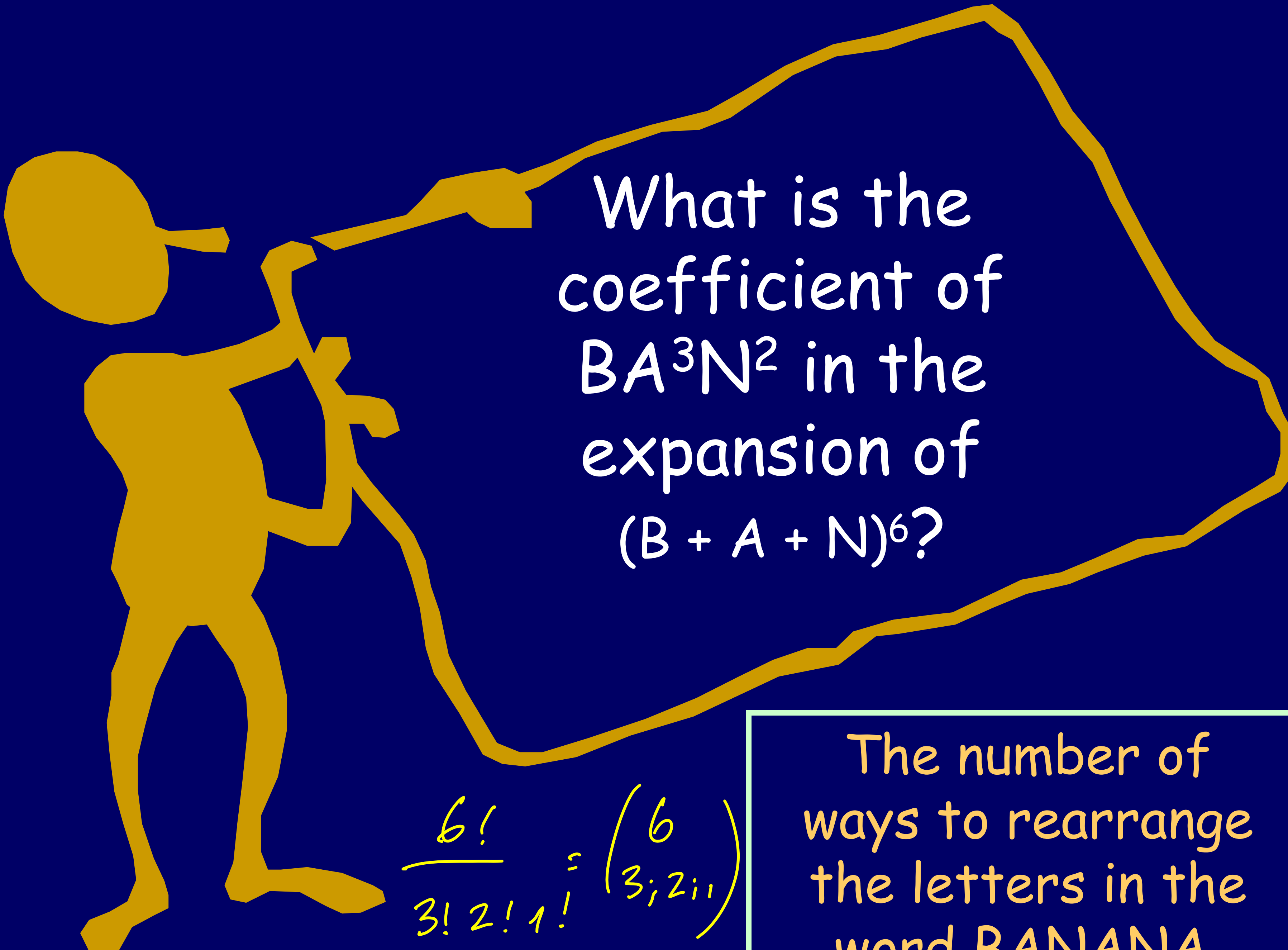
$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

One polynomial, two representations

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

"Product form" or "Generating form"

"Additive form" or "Expanded form"



$$\frac{6!}{3!\,2!\,1!} = \binom{6}{3;2;1}$$

The number of ways to rearrange the letters in the word BANANA.

Multinomial Coefficients

$$\binom{n}{r_1; r_2; \ldots; r_k} \equiv \begin{cases} 0 \text{ if } r_1 + r_2 + \ldots + r_k \neq n \\ \dfrac{n!}{r_1! r_2! \ldots r_k!} \end{cases}$$

$$\binom{n}{k;\ n-k} = \binom{n}{k}$$

The Multinomial Formula

$$\left(X_1+X_2+\ldots+X_k\right)^n = \sum_{\substack{r_1,r_2,\ldots,r_k \\ \sum r_i=n}} \binom{n}{r_1;r_2;\ldots;r_k} X_1^{r_1} X_2^{r_2} X_3^{r_3} \ldots X_k^{r_k}$$

Power Series Representation

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

$$= \sum_{k=0}^{\infty} \binom{n}{k} \cdot x^k$$

Since $\binom{n}{k} = 0$ if $k > n$

"Closed form" or "Generating form"

"Power series" ("Taylor series") expansion

By playing these two representations against each other we obtain a new representation of a previous insight:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

Let $x=1$.

$$2^n = \underbrace{\sum_{k=0}^{n} \binom{n}{k}}$$

The number of subsets of an *n*-element set

By varying x, we can discover new identities

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

Let *x= -1.*

$$0 = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k$$

Equivalently,

$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$

The number of even-sized subsets of an *n* element set is the same as the number of odd-sized subsets.

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

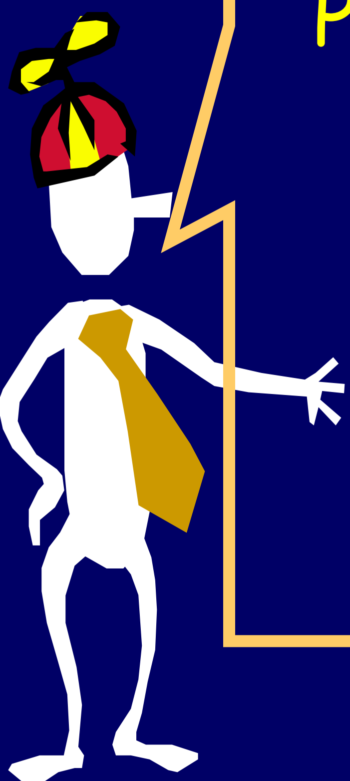
Let *x*= *-1*.

$$0 = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k$$

Equivalently,

$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$



Proofs that work by manipulating algebraic forms are called "algebraic" arguments. Proofs that build a 1-1 onto correspondence are called "combinatorial" arguments.

$$\sum_{k \text{ even}}^{n} \binom{n}{k} = \sum_{k \text{ odd}}^{n} \binom{n}{k} = 2^{n-1}$$

Let O_n be the set of binary strings of length n with an odd number of ones.

Let E_n be the set of binary strings of length n with an even number of ones.

We gave an algebraic proof that

$$|O_n| = |E_n|$$

A Combinatorial Proof

Let O_n be the set of binary strings of length n with an odd number of ones.

Let E_n be the set of binary strings of length n with an even number of ones.

A combinatorial proof must construct a one-to-one correspondence between O_n and E_n

An attempt at a correspondence

Let f_n be the function that takes an n-bit string and flips all its bits.

f_n is clearly a one-to-one and onto function

for odd n. *E.g.* in f_7 we have

0010011 → 1101100

1001101 → 0110010

...but do even n work? In f_6 we have

110011 → 001100

101010 → 010101

Uh oh. Complementing maps evens to evens!

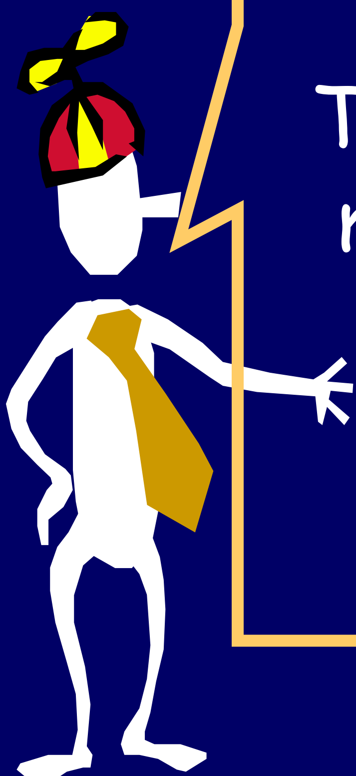
A correspondence that works for all *n*

Let f_n be the function that takes an n-bit string and flips only *the first bit*. For example,

0010011 → 1010011
1001101 → 0001101

110011 → 010011
101010 → 001010

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$



The binomial coefficients have so many representations that many fundamental mathematical identities emerge…

The Binomial Formula

$(1+X)^0 = 1$

$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

Pascal's Triangle:
k^{th} row are the coefficients of $(1+X)^k$

$(1+X)^0 =$ 1

$(1+X)^1 =$ $1 + 1X$

$(1+X)^2 =$ $1 + 2X + 1X^2$

$(1+X)^3 =$ $1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 =$ $1 + 4X + 6X^2 + 4X^3 + 1X^4$

k^{th} Row Of Pascal's Triangle:

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{k}, \ldots \binom{n}{n}$$

$(1+X)^0 =$ 1

$(1+X)^1 =$ $1 + 1X$

$(1+X)^2 =$ $1 + 2X + 1X^2$

$(1+X)^3 =$ $1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 =$ $1 + 4X + 6X^2 + 4X^3 + 1X^4$

Inductive definition of kth entry of nth row:
Pascal(n,0) = Pascal (n,n) = 1;
Pascal(n,k) = Pascal(n-1,k-1) + Pascal(n-1,k)

$(1+X)^0 =$ 1

$(1+X)^1 =$ $1 + 1X$

$(1+X)^2 =$ $1 + 2X + 1X^2$

$(1+X)^3 =$ $1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 =$ $1 + 4X + 6X^2 + 4X^3 + 1X^4$

"Pascal's Triangle"

$$\binom{0}{0} = 1$$

$$\binom{1}{0} = 1 \qquad \binom{1}{1} = 1$$

$$\binom{2}{0} = 1 \qquad \binom{2}{1} = 2 \qquad \binom{2}{2} = 1$$

$$\binom{3}{0} = 1 \qquad \binom{3}{1} = 3 \qquad \binom{3}{2} = 3 \qquad \binom{3}{3} = 1$$

Al-Karaji, Baghdad 953-1029

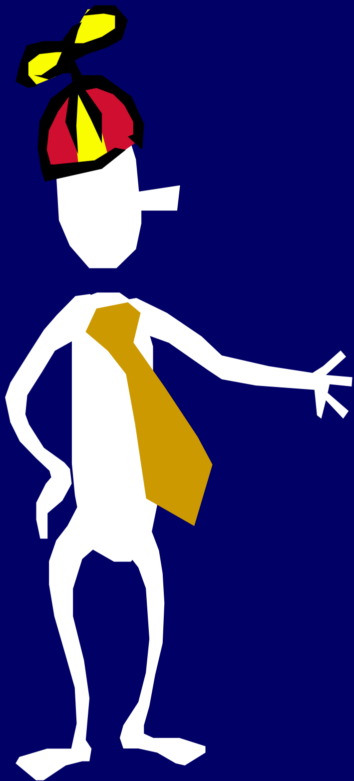
Chu Shin-Chieh 1303

The Precious Mirror of the Four Elements

. . . Known in Europe by 1529

Blaise Pascal 1654

Pascal's Triangle



1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

⋰ ⋮ ⋱

"It is extraordinary how fertile in properties the triangle is. Everyone can try his hand."

Summing The Rows

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

1 =1

1 + 1 =2

1 + 2 + 1 =4

1 + 3 + 3 + 1 =8

1 + 4 + 6 + 4 + 1 =16

1 + 5 + 10 + 10 + 5 + 1 =32

1 + 6 + 15 + 20 + 15 + 6 + 1 =64

⋰ ⋮ ⋱

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

6+20+6 = 1+15+15+1

Summing on 1st Avenue

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

. . .

$$\sum_{i=k}^{n} i = \sum_{i=k}^{n} \binom{i}{1} = \binom{n+1}{2} = \frac{n \cdot (n+1)}{2}$$

Summing on k^{th} Avenue

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

⋮ ⋱

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
=2
=3
=5
=8
=13

1
1 1
1 2 1
2 2 2
1 3 3 1
2 2 2 2
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Al-Karaji Squares

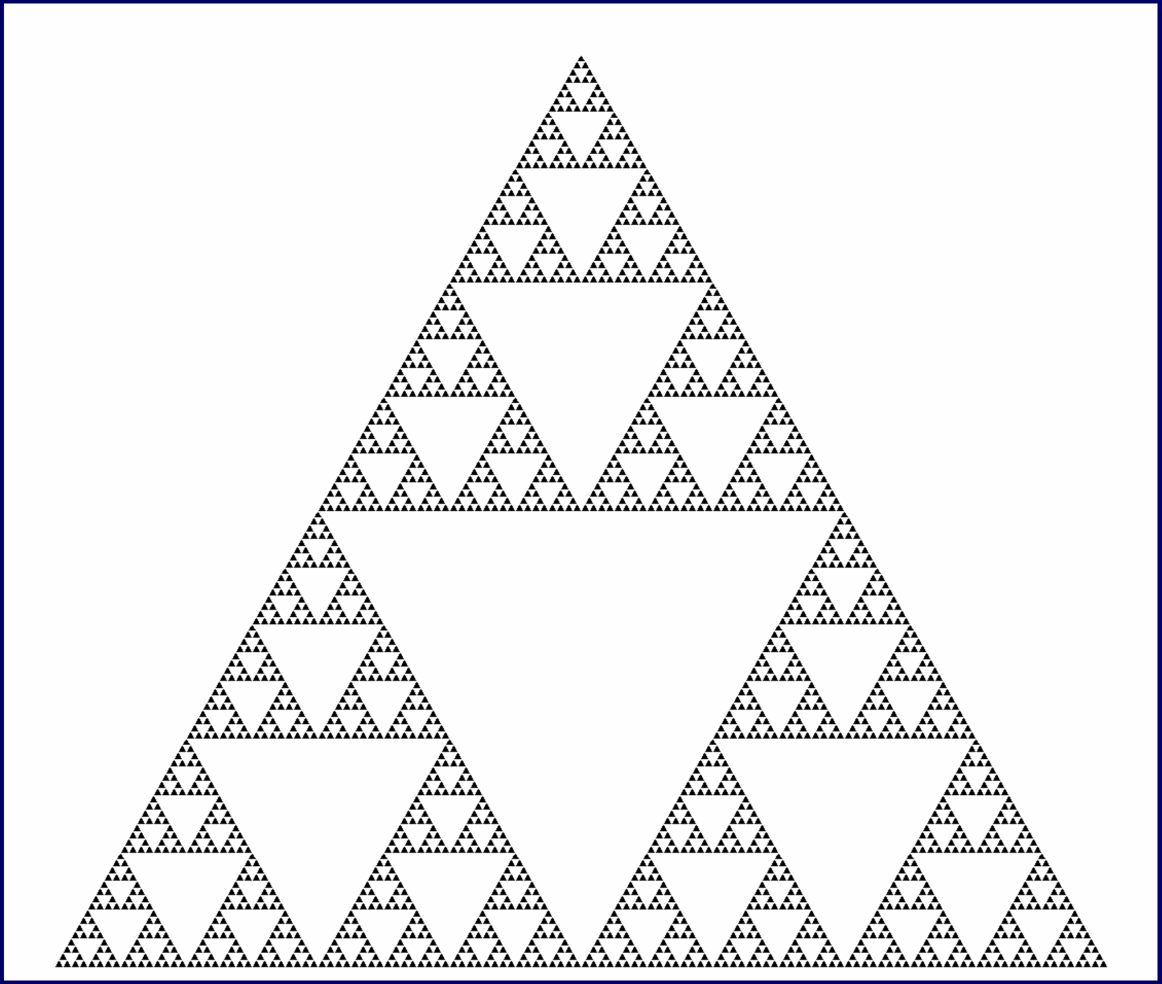
$$n + 2 \times \frac{n \cdot (n-1)}{2}$$

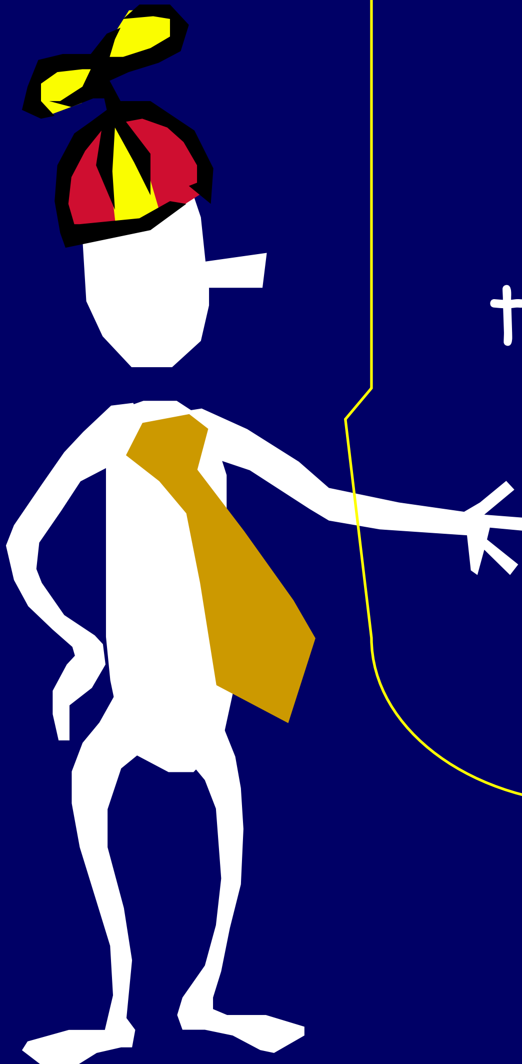
$$= n + n^2 - n$$

$$= n^2$$

1	=0
1 1	=1
1 2 + 2*1	=4
1 3 + 2*3 1	=9
1 4 + 2*6 4 1	=16
1 5 + 2*10 10 5 1	=25
1 6 + 2*15 20 15 6 1	=36

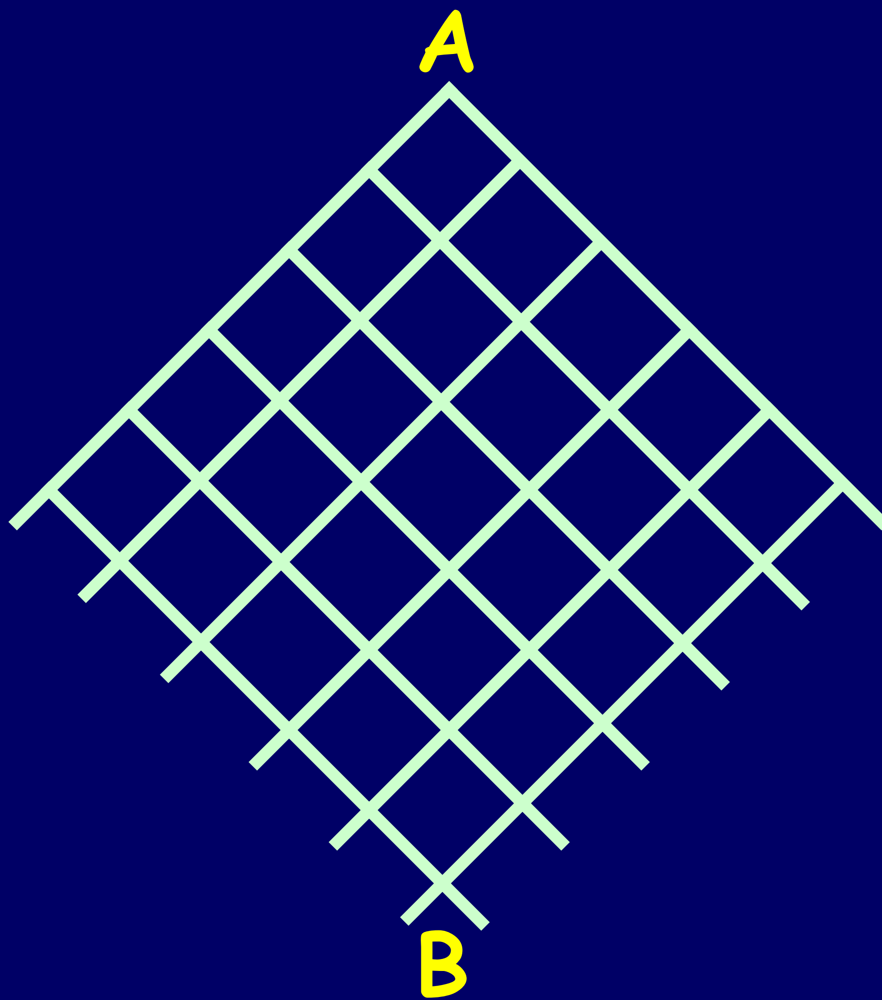
Pascal Mod 2





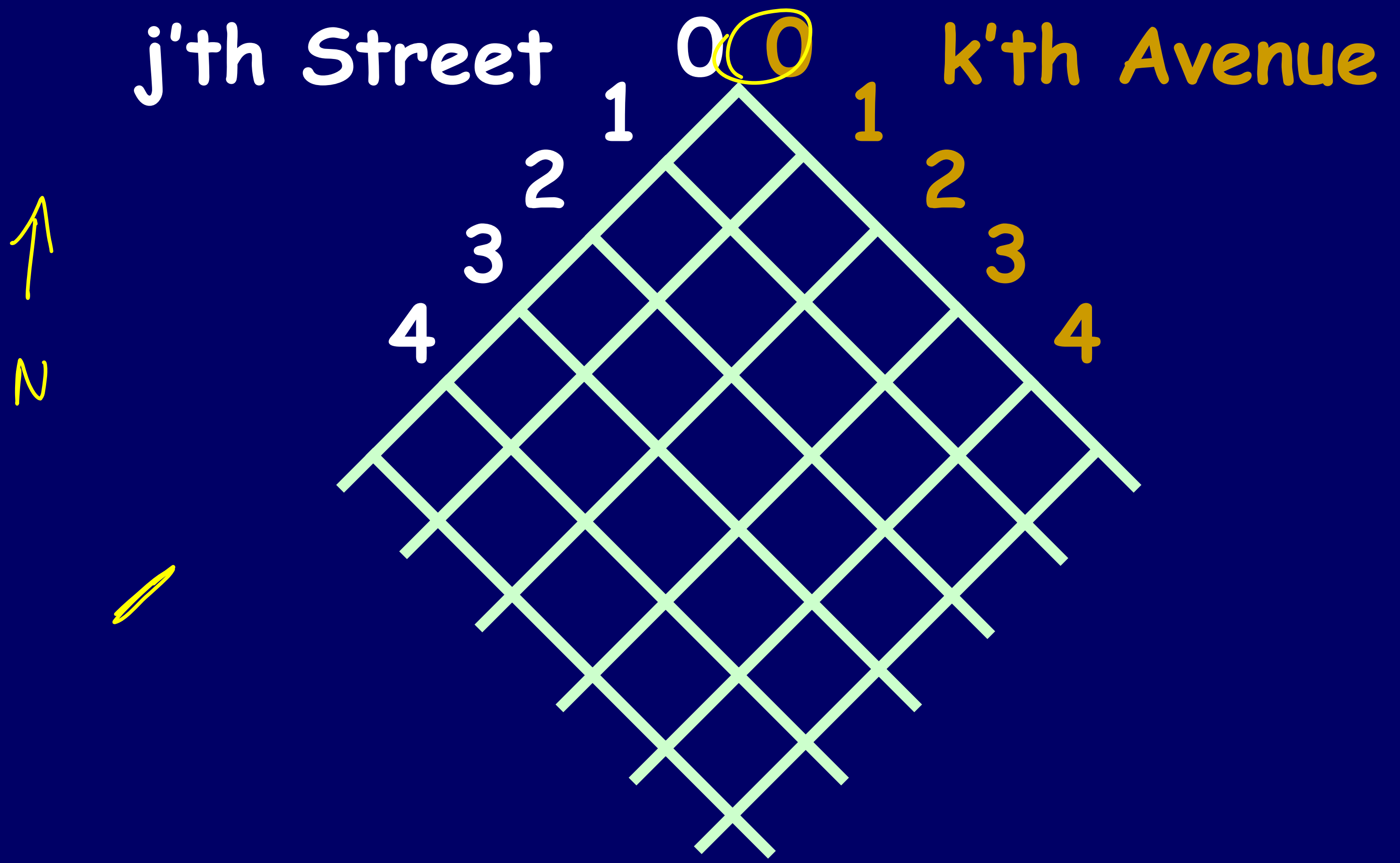
All these properties can be proved inductively and algebraically. We will give *combinatorial* proofs using the Manhattan block walking representation of binomial coefficients.

How many shortest routes from A to B?



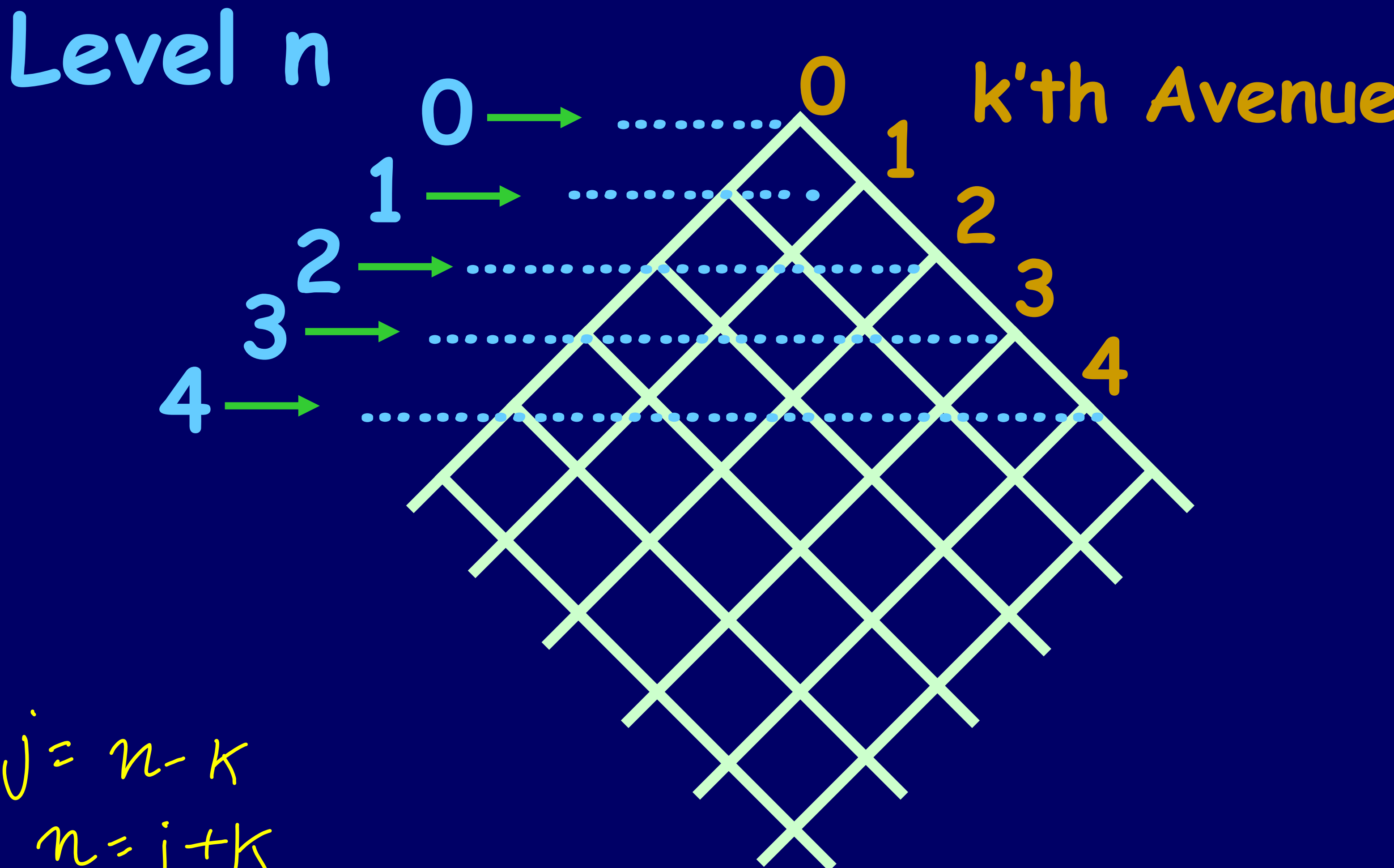
$$\binom{10}{5}$$

Manhattan



There are $\binom{j+k}{k}$ shortest routes from (0,0) to (j,k).

Manhattan

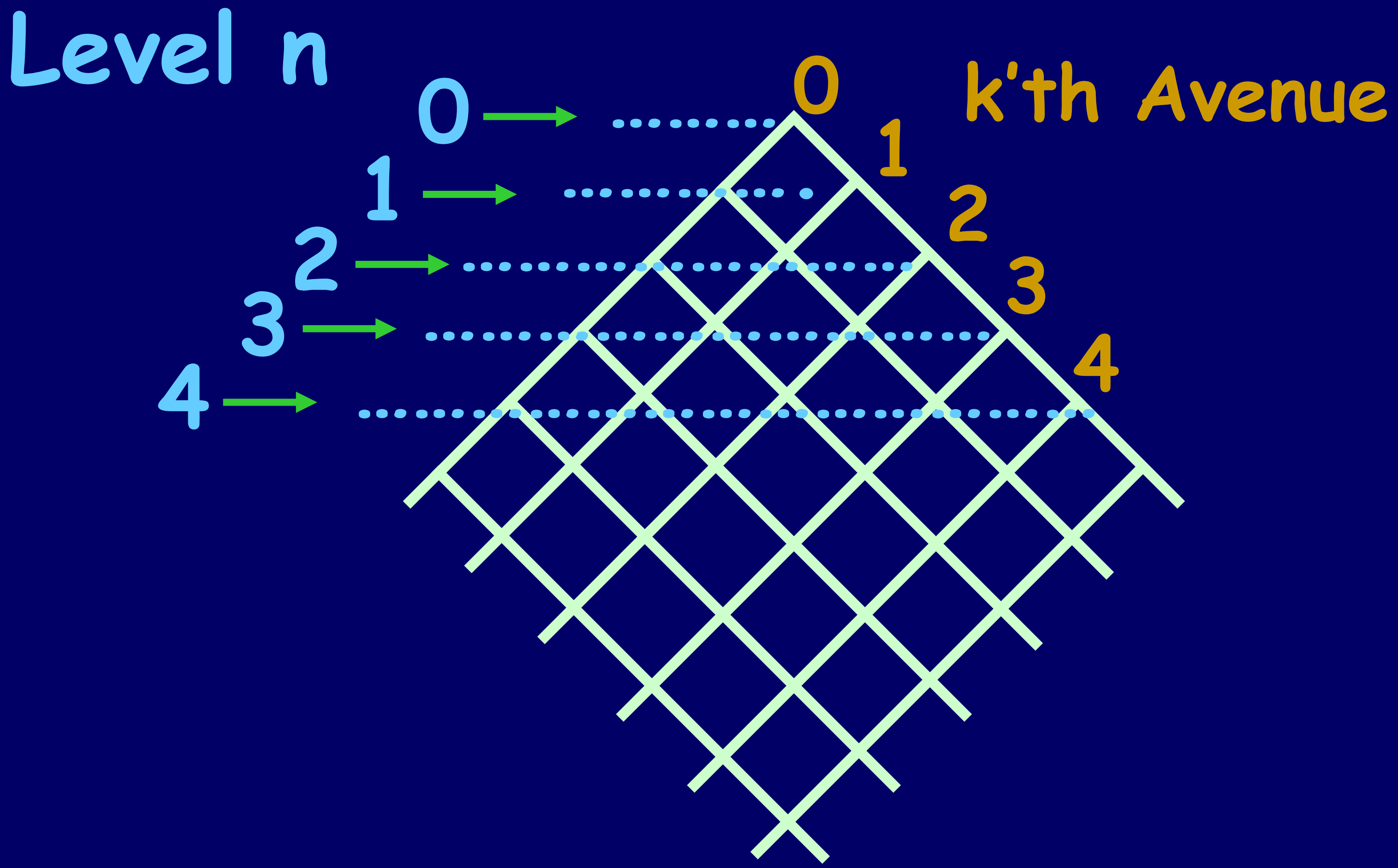


$j = n - k$

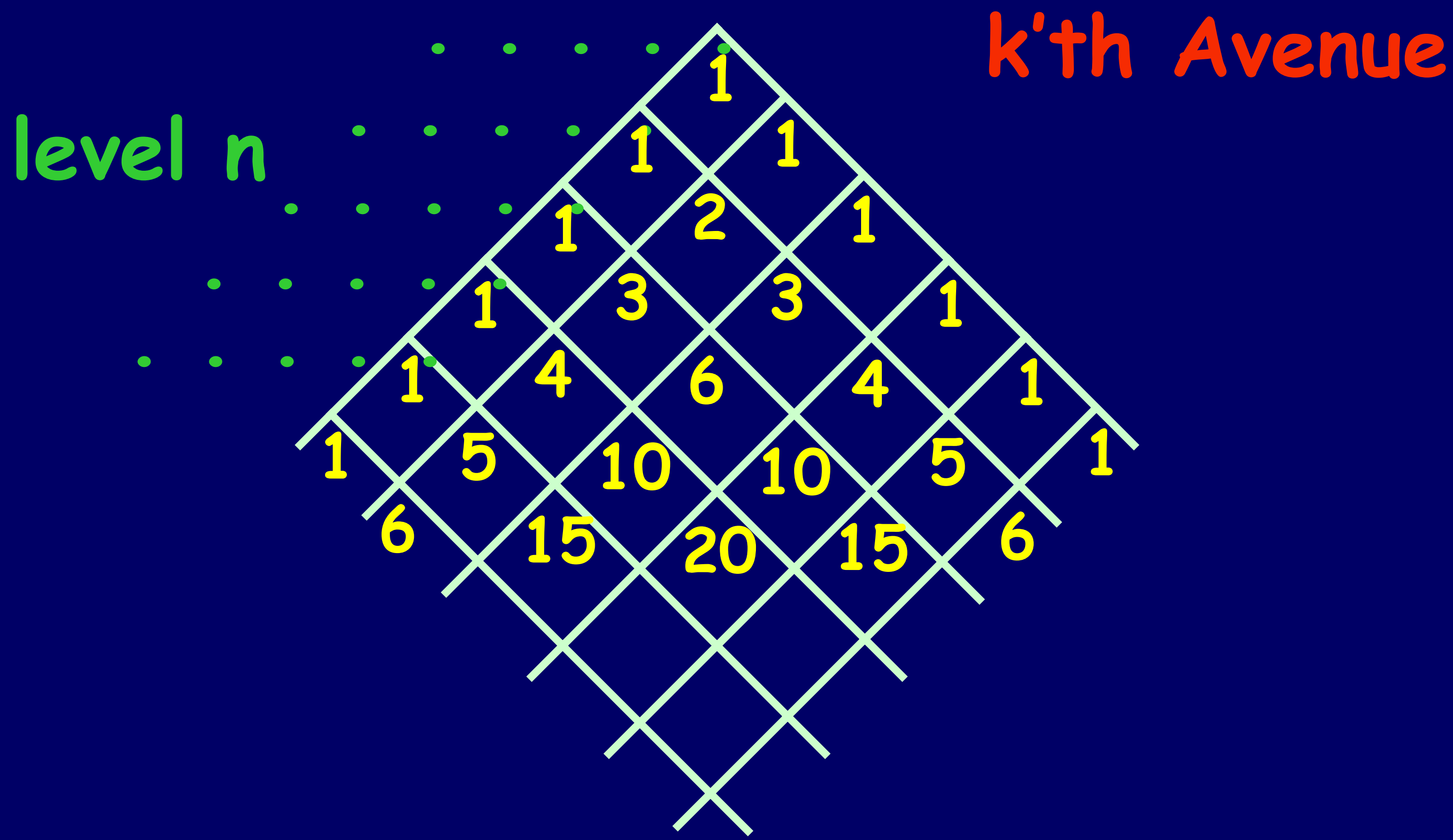
$n = j + k$

There are $\binom{n}{k}$ shortest routes from (0,0) to (n-k,k).

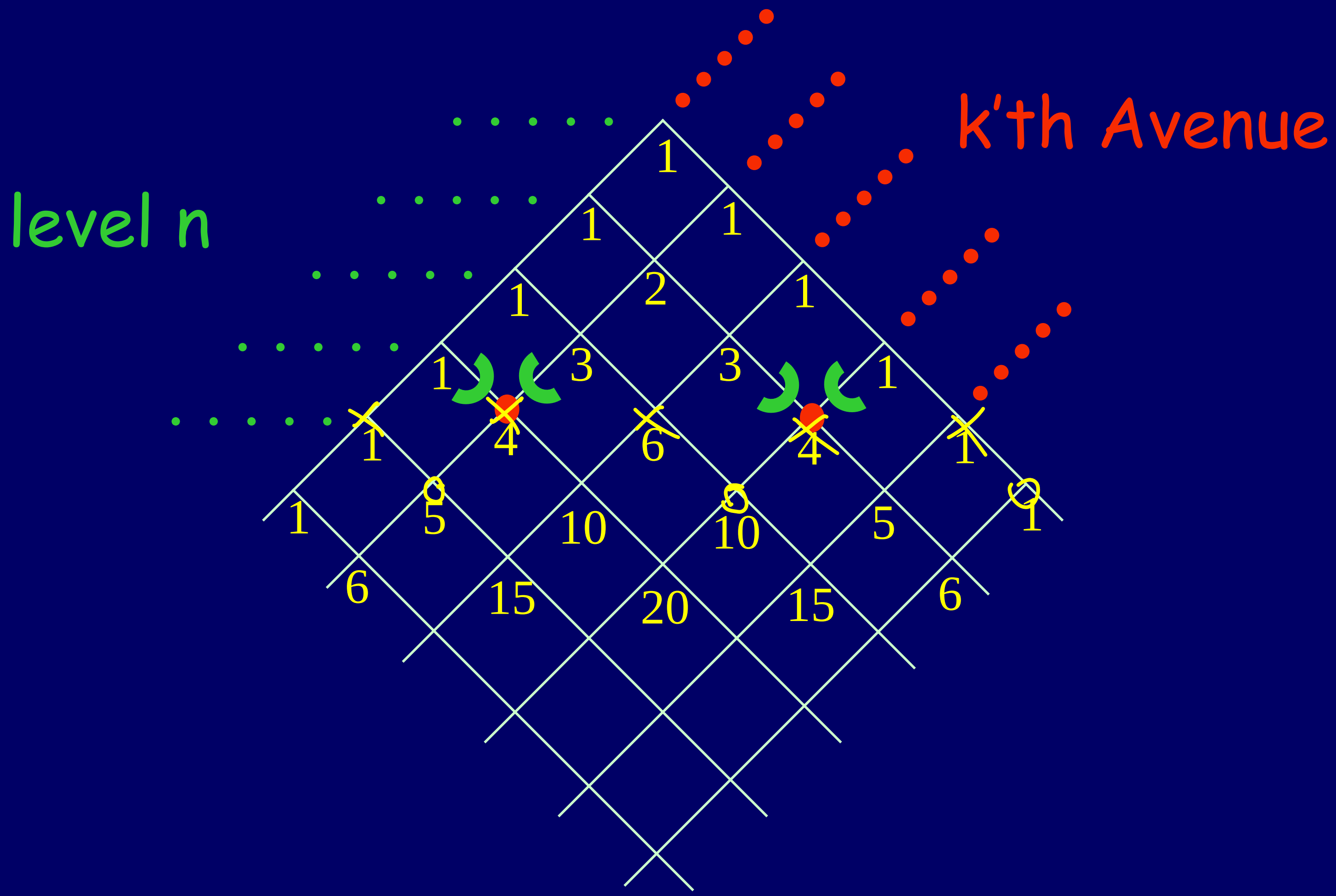
Manhattan



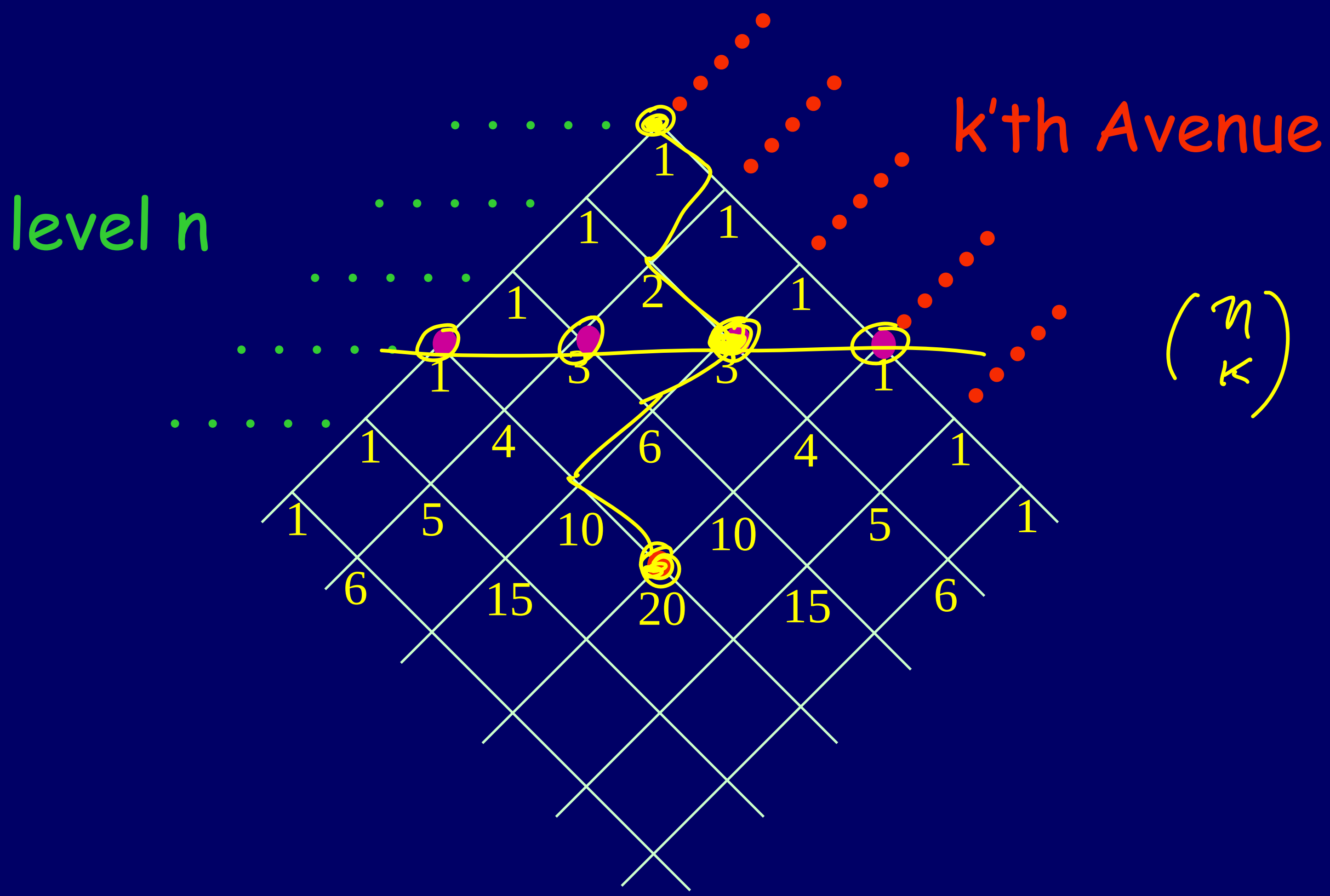
There are $\binom{n}{k}$ shortest routes from (0,0) to Level n and k^{th} Avenue.


k'th Avenue
level n
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
6 15 20 15 6

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$



$$\sum_{k \text{ even}} \binom{n}{k} = 2^{n-1}$$



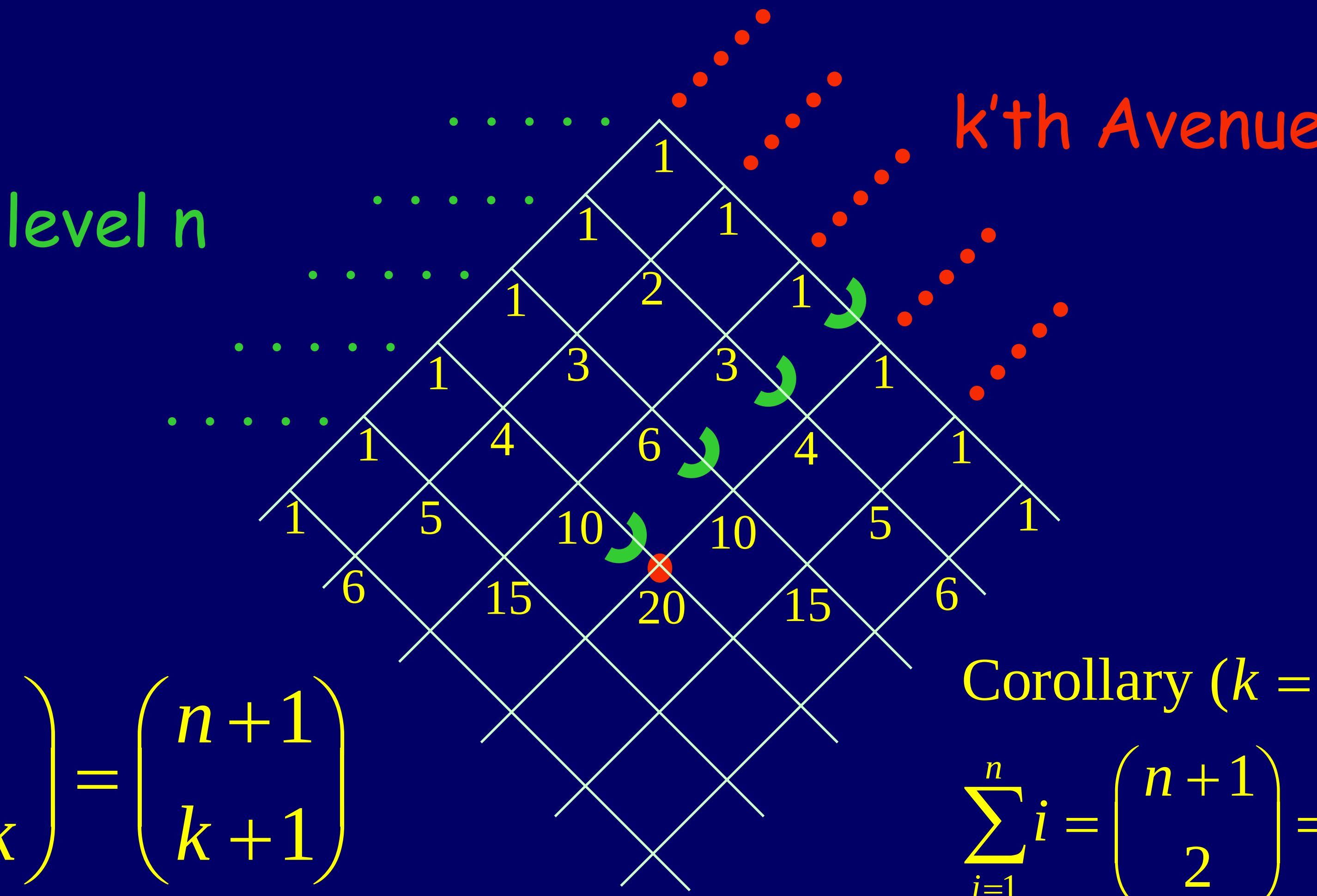
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

By convention:

$$0! = 1$$ (empty product = 1)

$$\binom{n}{k} = 1 \quad \text{if } k = 0$$

$$\binom{n}{k} = 0 \quad \text{if } k < 0 \text{ or } k > n$$



$$\sum_{i=1}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

Corollary ($k=1$)

$$\sum_{i=1}^{n} i = \binom{n+1}{2} = \frac{n(n+1)}{2}$$

Application (Al-Karaji):

$$\sum_{i=0}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$$

$$= (1\cdot 0 + 1) + (2\cdot 1 + 2) + (3\cdot 2 + 3) + \cdots + (n(n-1) + n)$$

$$= 1\cdot 0 + 2\cdot 1 + 3\cdot 2 + \cdots + n(n-1) + \sum_{i=1}^{n} i$$

$$= 2\left[\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \cdots + \binom{n}{2}\right] + \binom{n+1}{2}$$

$$= 2\binom{n+1}{3} + \binom{n+1}{2} = \frac{(2n+1)(n+1)n}{6}$$

Vector Programs

Let's define a (parallel) programming language called VECTOR that operates on possibly infinite vectors of numbers. Each variable $V^{\rightarrow}$ can be thought of as:

< *, *, *, *, *, *, ... >

0 1 2 3 4 5 . . . .

Vector Programs

Let k stand for a scalar constant

$\langle k \rangle$ will stand for the vector $\langle k,0,0,0,\ldots \rangle$

$\langle 0 \rangle = \langle 0,0,0,0,\ldots. \rangle$

$\langle 1 \rangle = \langle 1,0,0,0,\ldots \rangle$

$\vec{V} + \vec{T}$ means to add the vectors position-wise.

$\langle 4,2,3,\ldots \rangle + \langle 5,1,1,\ldots. \rangle = \langle 9,3,4,\ldots \rangle$

Vector Programs

RIGHT($\vec{V}$) means to shift every number in $\vec{V}$ one position to the right and to place a 0 in position 0.

RIGHT(⟨1,2,3, ...⟩) = ⟨0,1,2,3,. ...⟩

Vector Programs

Example:	Store
$V^{\rightarrow} := \langle 6 \rangle;$	$V^{\rightarrow} = \langle 6,0,0,0,.. \rangle$
$V^{\rightarrow} := \mathrm{RIGHT}(V^{\rightarrow}) + \langle 42 \rangle;$	$V^{\rightarrow} = \langle 42,6,0,0,.. \rangle$
$V^{\rightarrow} := \mathrm{RIGHT}(V^{\rightarrow}) + \langle 2 \rangle;$	$V^{\rightarrow} = \langle 2,42,6,0,.. \rangle$
$V^{\rightarrow} := \mathrm{RIGHT}(V^{\rightarrow}) + \langle 13 \rangle;$	$V^{\rightarrow} = \langle 13,2,42,6,. \rangle$

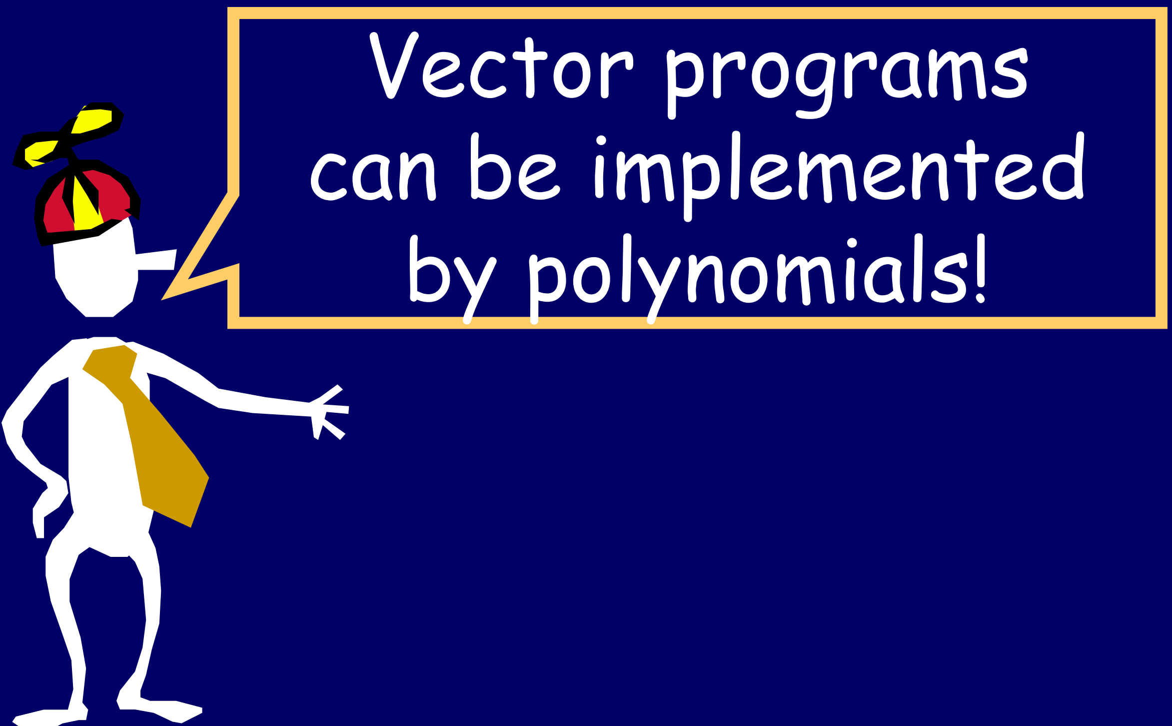
$$V^{\rightarrow} = \langle\ 13,\ 2,\ 42,\ 6,\ 0,\ 0,\ 0,\ .\ .\ .\ \rangle$$

Vector Programs

Example:	Store
$\vec{V} := \langle 1 \rangle;$	$\vec{V} = \langle 1,0,0,0,.. \rangle$
Loop n times:	$\vec{V} = \langle 1,1,0,0,.. \rangle$
$\vec{V} := \vec{V} + \mathrm{RIGHT}(\vec{V});$	$\vec{V} = \langle 1,2,1,0,.. \rangle$
	$\vec{V} = \langle 1,3,3,1,. \rangle$

$\vec{V}$ = n^{th} row of Pascal's triangle.

 $X^1 +$  $X^2 +$  X^3



Programs -----> Polynomials

The vector $\vec{V}$ = < a_0, a_1, a_2, . . . > will be represented by the polynomial:

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

Formal Power Series

The vector $V^{\rightarrow}$ = < a_0, a_1, a_2, . . . > will be represented by the formal power series:

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

$$\vec{V} = \langle a_0, a_1, a_2, \ldots \rangle$$

$$P_V = \sum_{i=0}^{i=\infty} a_i X^i$$

$\langle 0 \rangle$ is represented by 0

$\langle k \rangle$ is represented by k

$\vec{V} + \vec{T}$ is represented by $(P_V + P_T)$

RIGHT($\vec{V}$) is represented by $(P_V\ X)$

Vector Programs

Example:

$\vec{V} := \langle 1 \rangle;$ $P_V := 1;$

Loop n times:

$\vec{V} := \vec{V} + \mathrm{RIGHT}(\vec{V});$ $P_V := P_V + P_V X;$

$\vec{V}$ = n^{th} row of Pascal's triangle.

Vector Programs

Example:

$\vec{V} := \langle 1 \rangle;$ $\qquad P_V := 1;$

Loop n times:

$\vec{V} := \vec{V} + \text{RIGHT}(\vec{V});$ $\qquad P_V := P_V (1+X);$

$\vec{V}$ = n^{th} row of Pascal's triangle.

Vector Programs

Example:

$\vec{V} := \langle 1 \rangle;$

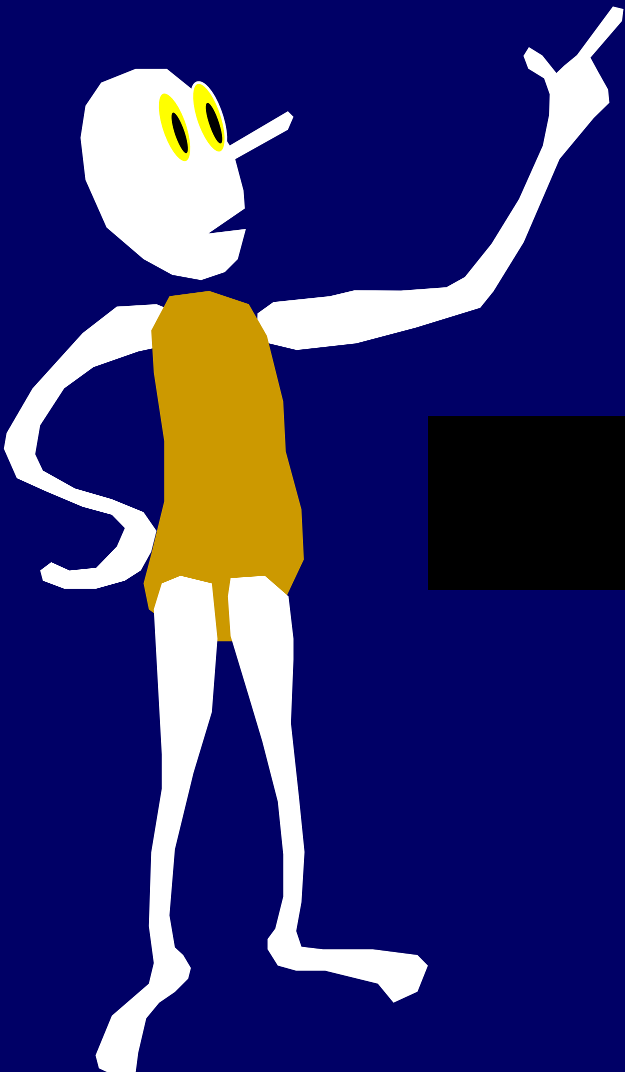
Loop n times:

$\vec{V} := \vec{V} + \text{RIGHT}(\vec{V});$

$P_V = (1+X)^n$

$\vec{V}$ = n^{th} row of Pascal's triangle.

$$1 + X^1 + X^2 + X^3 + \dots + X^{n-1} + X^n = \frac{X^{n+1} - 1}{X - 1}$$

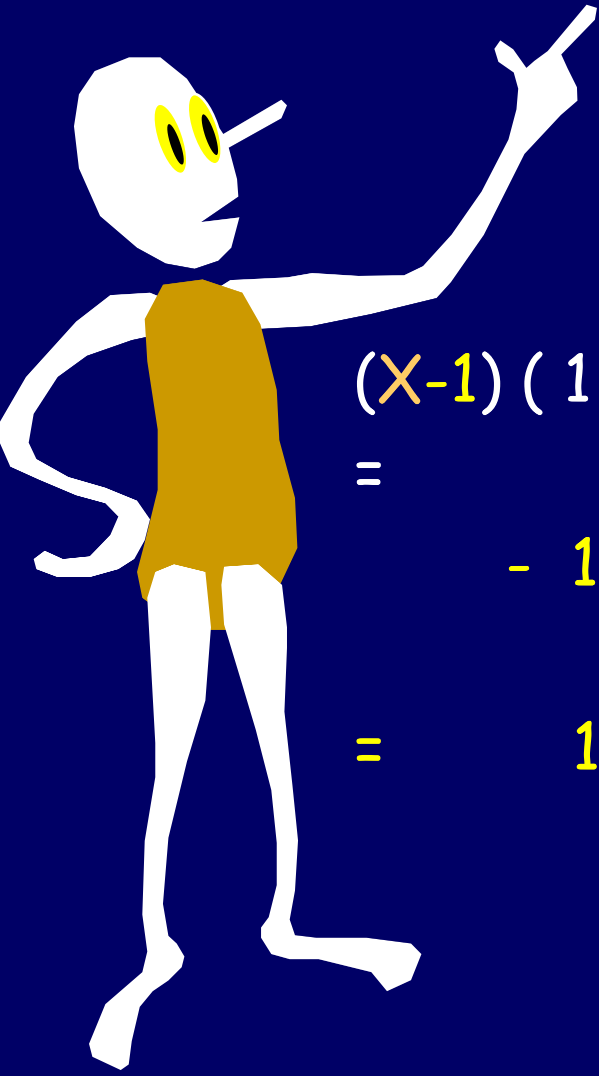


The Geometric Series

$$1 + X^1 + X^2 + X^3 + \ldots + X^n + \ldots = \frac{1}{1 - X}$$

The Infinite Geometric Series

$$1 + X^1 + X^2 + X^3 + \dots + X^n + \dots\dots = \frac{1}{1 - X}$$



$$(X-1)\,(1 + X^1 + X^2 + X^3 + \dots + X^n + \dots)$$

$$= X^1 + X^2 + X^3 + \dots + X^n + X^{n+1} + \dots$$

$$- 1 - X^1 - X^2 - X^3 - \dots - X^{n-1} - X^n - X^{n+1} - \dots$$

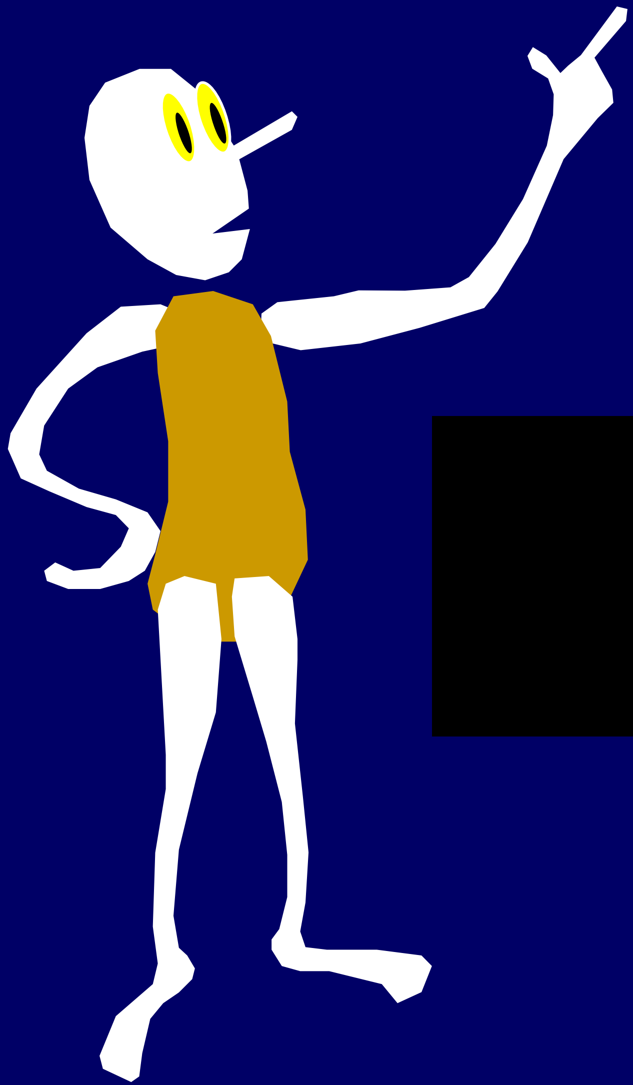
$$= 1$$

$$1 + aX^1 + a^2X^2 + a^3X^3 + \dots + a^nX^n + \dots\dots = \frac{1}{1 - aX}$$

Geometric Series (Linear Form)

$$\left(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots\ldots\right)\left(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots\ldots\right) =$$

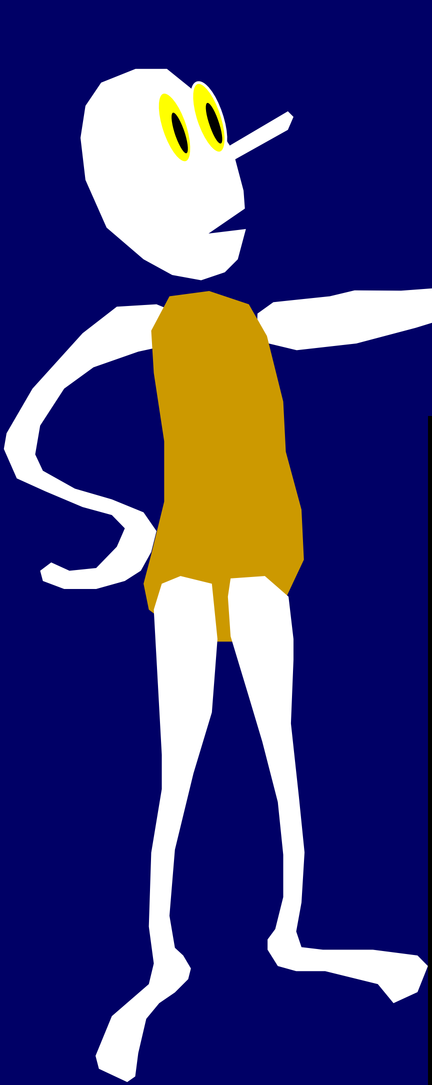
$$\frac{1}{(1 - aX)(1-bX)}$$



Geometric Series (Quadratic Form)

$$(1 + aX^1 + a^2X^2 + \ldots + a^nX^n + \ldots..)(1 + bX^1 + b^2X^2 + \ldots + b^nX^n + \ldots..) =$$

$$1 + c_1X^1 + .. + c_k X^k + \ldots$$

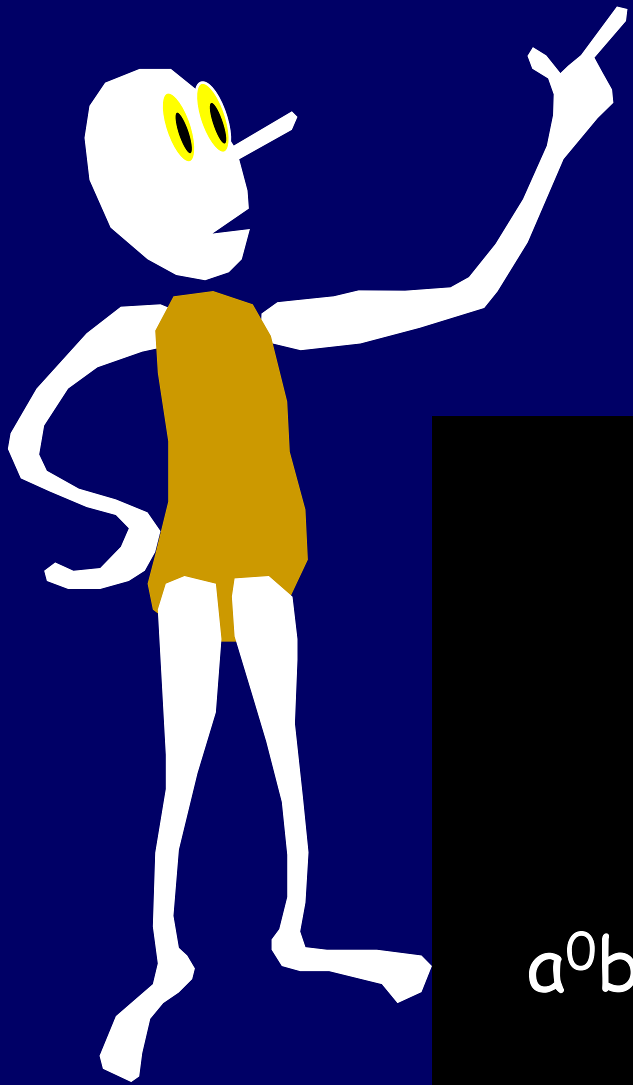


Suppose we multiply this out to get a single, infinite polynomial.

What is an expression for C_n?

$$(1 + aX^1 + a^2X^2 + \dots + a^nX^n + \dots\dots)(1 + bX^1 + b^2X^2 + \dots + b^nX^n + \dots\dots) =$$

$$1 + c_1X^1 + \dots + c_k X^k + \dots$$

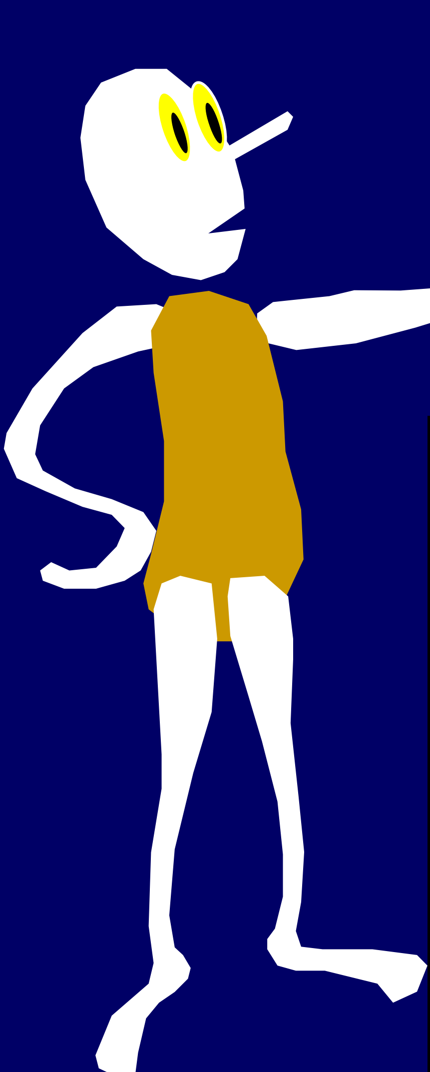


$$c_n =$$

$$a^0b^n + a^1b^{n-1} + \dots a^ib^{n-i} \dots + a^{n-1}b^1 + a^nb^0$$

$$(1 + aX^1 + a^2X^2 + \dots + a^nX^n + \dots\dots)\,(1 + bX^1 + b^2X^2 + \dots + b^nX^n + \dots\dots) =$$

$$1 + c_1X^1 + \dots + c_k X^k + \dots$$

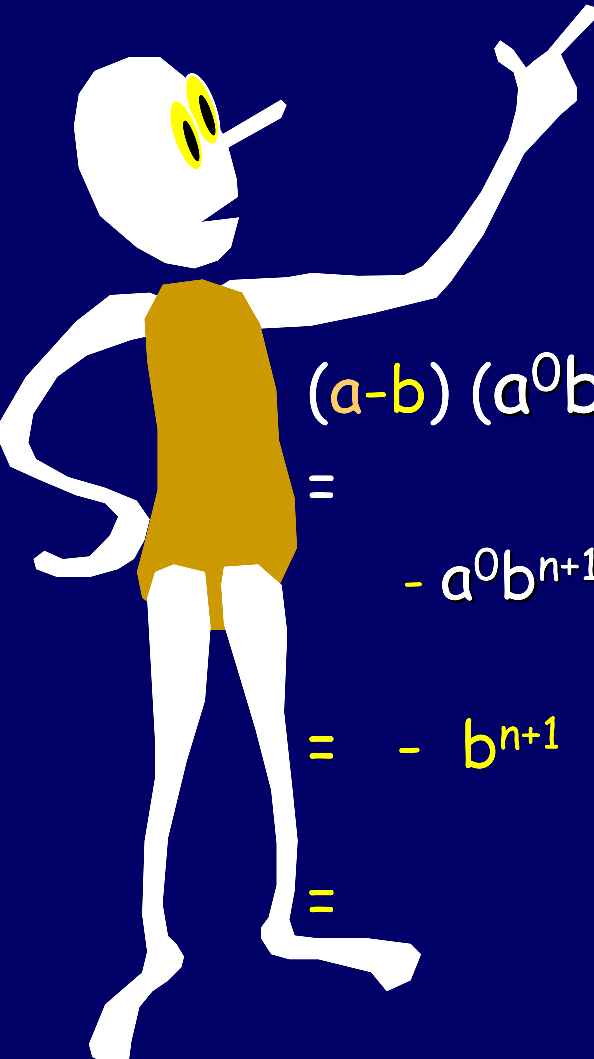


If a = b then

$$c_n = (n+1)(a^n)$$

$$a^0b^n + a^1b^{n-1} + \dots a^ib^{n-i} \dots + a^{n-1}b^1 + a^nb^0$$

$$a^0b^n + a^1b^{n-1} + \dots a^ib^{n-i} \dots + a^{n-1}b^1 + a^nb^0 = \frac{a^{n+1} - b^{n+1}}{a - b}$$



$$(a-b)\,(a^0b^n + a^1b^{n-1} + \dots a^ib^{n-i} \dots + a^{n-1}b^1 + a^nb^0)$$

$$= \quad a^1b^n + \dots a^{i+1}b^{n-i} \dots \quad + a^nb^1 + a^{n+1}b^0$$

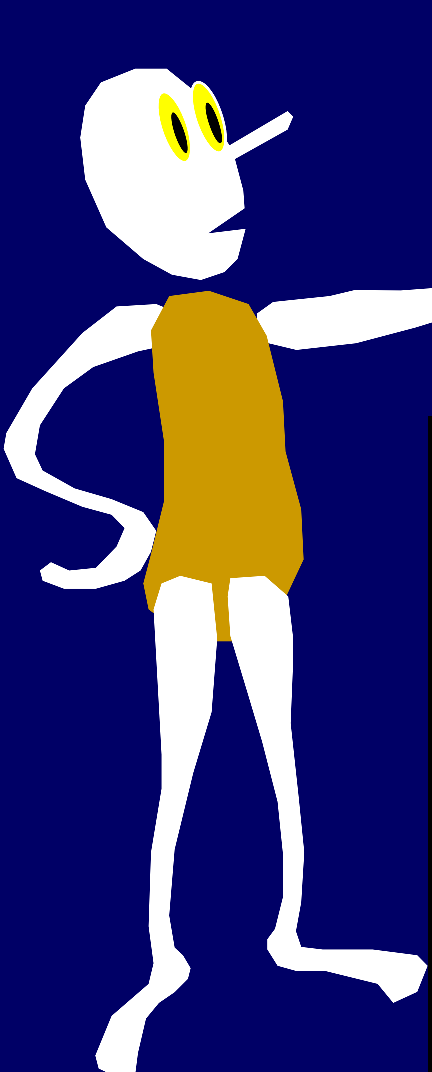
$$- a^0b^{n+1} \quad - a^1b^n \dots \quad a^{i+1}b^{n-i} \dots \quad - a^{n-1}b^2 - a^nb^1$$

$$= \; - \; b^{n+1} \quad + \quad a^{n+1}$$

$$= \quad a^{n+1} - b^{n+1}$$

$$(1 + aX^1 + a^2X^2 + \dots + a^nX^n + \dots\dots)(1 + bX^1 + b^2X^2 + \dots + b^nX^n + \dots\dots) =$$

$$1 + c_1X^1 + \dots + c_k X^k + \dots$$

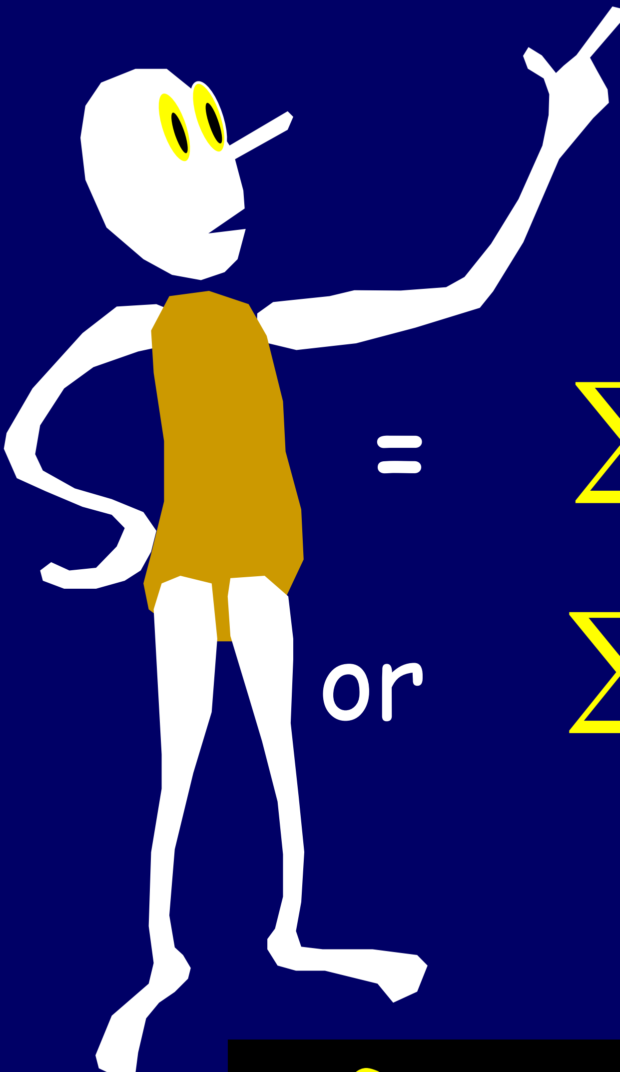


if $a \neq b$ then

$$c_n = \frac{a^{n+1} - b^{n+1}}{a - b}$$

$$a^0b^n + a^1b^{n-1} + \dots a^ib^{n-i} \dots + a^{n-1}b^1 + a^nb^0$$

$$\left(1 + aX^1 + a^2X^2 + \dots + a^nX^n + \dots\right)\left(1 + bX^1 + b^2X^2 + \dots + b^nX^n + \dots\right) =$$



$$= \frac{1}{(1 - aX)(1-bX)}$$

$$= \sum_{n=0..\infty} \frac{a^{n+1} - b^{n+1}}{a - b} X^n$$

or

$$\sum_{n=0..\infty} (n+1)a^n \, X^n$$

when $a=b$

Geometric Series (Quadratic Form)

Study Bee

- Polynomials count
- Binomial formula
- Multinomial coefficients
- Combinatorial proofs of binomial identities
- Vector programs
- Geometric series