15-251

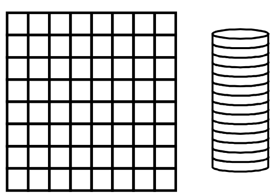
Great Theoretical Ideas in Computer Science

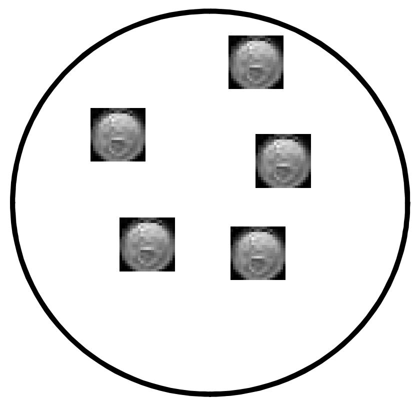
15-251

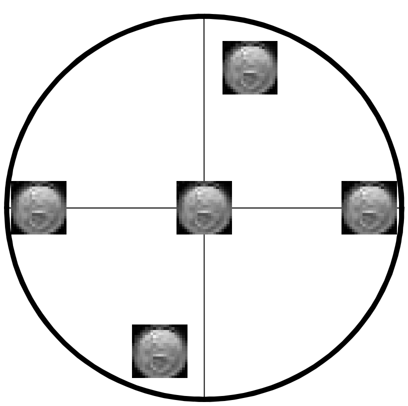
Game Playing for Computer Scientists

Combinatorial Games

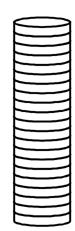
Lecture 3 (September 2, 2008)







A Take-Away Game



21 chips

Two Players: I and II

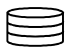
A move consists of removing one, two, or three chips from the pile

Players alternate moves, with Player I starting

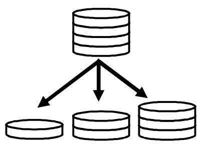
Player that removes the last chip wins

Which player would you rather be?

Try Small Examples!



If there are 1, 2, or 3 only, player who moves next wins

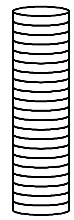


If there are 4 chips left, player who moves next must leave 1, 2 or 3 chips, and his opponent will win

1 2 3 1st player wins 0 2nd player win

With 5, 6 or 7 chips left, the player ~~who moves~~ next can win by leaving 4 chips

4 5



21 chips

multiples of 4 ⇒ 2nd player wins

0, 4, 8, 12, 16, ... are target positions; if a player moves to that position, they can win the game

Therefore, with 21 chips, Player I can win!

non multiples of 4 ⇒ 1st player wins!

What if the last player to move loses?

If there is 1 chip, the player who moves next loses

If there are 2,3, or 4 chips left, the player who moves next can win by leaving only 1

In this case, 1, 5, 9, 13, ... are a win for the second player

Combinatorial Games

There are two players

There is a finite set of possible positions

The rules of the game specify for both players and each position which moves to other positions are legal moves

The players alternate moving

The game ends in a finite number of moves (no draws!)

Normal Versus Misère

Normal Play Rule: The last player to move wins

Misère Play Rule: The last player to move loses

A Terminal Position is one where neither player can move anymore

positions 0, 4, 8 2nd player wins

position 1, 5, 9 . . . 2nd player wins

What is Omitted

No random moves

(This rules out games like poker)

No hidden moves

(This rules out games like battleship)

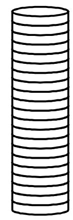
No draws in a finite number of moves

(This rules out tic-tac-toe)

P-Positions and N-Positions

P-Position: Positions that are winning for the Previous player (the player who just moved)

N-Position: Positions that are winning for the Next player (the player who is about to move)



21 chips

0, 4, 8, 12, 16, … are P-positions; if a player moves to that position, they can win the game

21 chips is an N-position

What's a P-Position?

"Positions that are winning for the Previous player (the player who just moved)"

That means:

For any move that N makes

  There exists a move for P such that

    For any move that N makes

      There exists a move for P such that

⋮

        There exists a move for P such that

          There are no possible moves for N

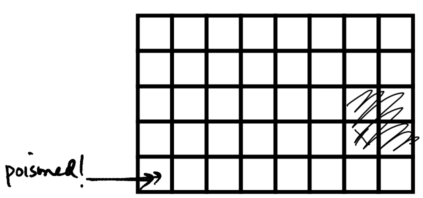
P-positions and N-positions can be defined recursively by the following: normal form game

(1) All terminal positions are P-positions

(2) From every N-position, there is at least one move to a P-position

(3) From every P-position, every move is to an N-position

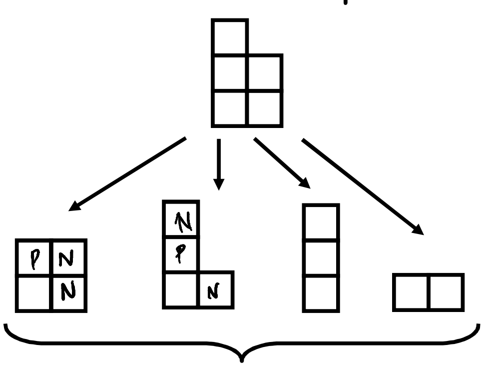
Chomp!



Two-player game, where each move consists of taking a square and removing it and all squares to the right and above.

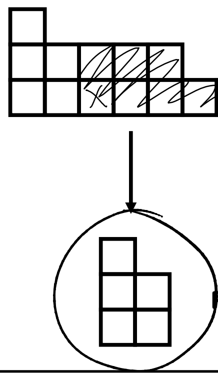
Player who takes position (0,0) loses

Show That This is a P-position



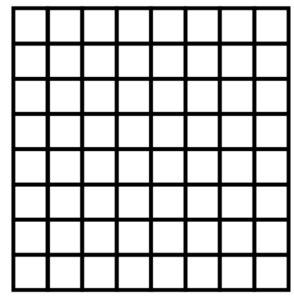
N-Positions!

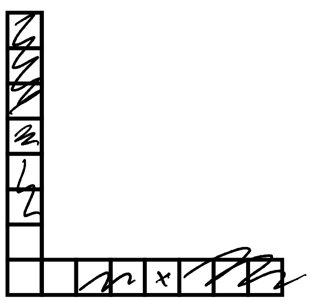
Show That This is an N-position



P-position!

Let's Play! I'm player I





No matter what you do, I can mirror it!

Mirroring is an extremely important strategy in combinatorial games!

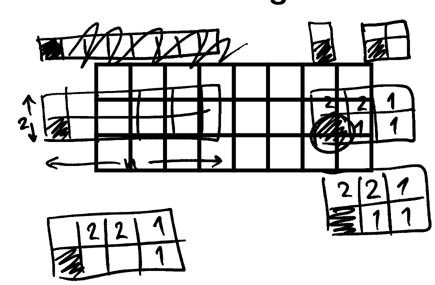
Theorem: Player I can win in any square starting position of Chomp

Proof:

The winning strategy for player I is to chomp on (1,1), leaving only an "L" shaped position

Then, for any move that Player II takes, Player I can simply mirror it on the flip side of the "L"

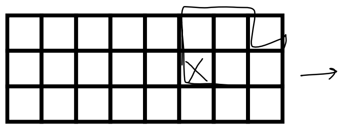
What about rectangular boards?



Theorem: Player I can win in any rectangular starting position

Proof:

Look at this first move:

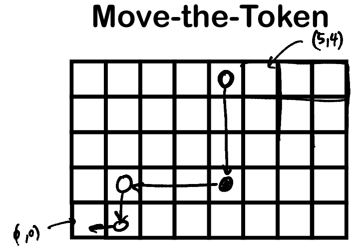


If this is a P-position, then player 1 wins

Otherwise, there exists a P-position that can be obtained from this position

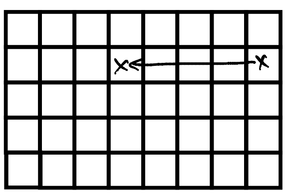
And player I could have just taken that move originally

Move-the-Token



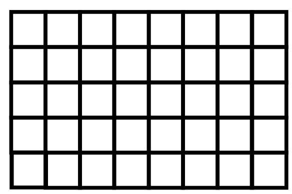
Two-player game, where each move consists of taking the token and moving it either downwards or to the left (but not both).

Player who makes the last move (to (0,0)) wins

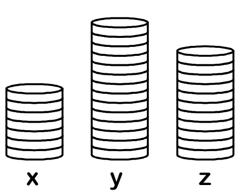


$(x,x) \leftarrow$ 2nd player win (or P position)

(x,y) st $y \neq x \leftarrow$ N position



The Game of Nim



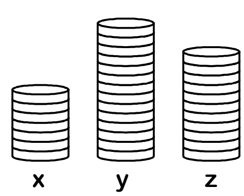
Two players take turns moving

Winner is the last player to remove chips

A move consists of selecting a pile and removing chips from it (you can take as many as you want, but you have to at least take one)

In one move, you cannot remove chips from more than one pile

Analyzing Simple Positions



We use (x,y,z) to denote this position

(0,0,0) is a: P-position

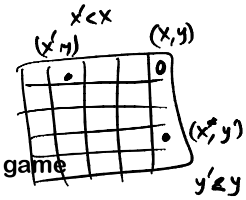
One-Pile Nim

What happens in positions of the form (x,0,0)?

The first player can just take the entire pile, so (x,0,0) is an N-position

Two-Pile Nim

Seen this before?



It's the "Move-the-Token" game

Two-Pile Nim

P-positions are those for which the two piles have an equal number of chips

If it is the opponent's turn to move from such a position, he must change to a position in which the two piles have different number of chips

From a position with an unequal number of chips, you can easily go to one with an equal number of chips (perhaps the terminal position)

3-Pile Nim

x y z

Two players take turns moving

Winner is the last player to remove chips

Nim-Sum

The nim-sum of two non-negative integers is their addition (<u>without carry</u>) in base 2

We will use $\oplus$ to denote the nim-sum

$2 \oplus 3 = (10)_2 \oplus (11)_2 = (01)_2 = 1$

$5 \oplus 3 = (101)_2 \oplus (011)_2 = (110)_2 = 6$

$7 \oplus 4 = (111)_2 \oplus (100)_2 = (011)_2 = 3$

$\oplus$ is associative: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$

$\oplus$ is commutative: $a \oplus b = b \oplus a$

For any non-negative integer x,

$$x \oplus x = 0$$

Cancellation Property Holds

If $x \oplus y = x \oplus z$

Then $x \oplus x \oplus y = x \oplus x \oplus z$

So $y = z$

Bouton's Theorem: A position (x,y,z) in Nim is a P-position if and only if $x \oplus y \oplus z = 0$

Proof:

Let Z denote the set of Nim positions with nim-sum zero

Want: P position

Let NZ denote the set of Nim positions with non-zero nim-sum

Want: N positions

We prove the theorem by proving that Z and NZ satisfy the three conditions of P-positions and N-positions

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(1) All terminal positions are in Z

The only terminal position is (0,0,0)

(2) From each position in NZ, there is a move to a position in Z

$$
\begin{array}{rrl}
 & 001010001 & x \\
 & 100010111 & y \\
\oplus & 111010000 & z \\
\hline
 & 010010110 & p \neq 0
\end{array}
\quad \Rightarrow \quad
\begin{array}{rr}
 & 001010001 \\
 & 100010111 \\
\oplus & 101000110 \\
\hline
 & 000000000
\end{array}
$$

Look at leftmost column with an odd # of 1s

Rig any of the numbers with a 1 in that column so that everything adds up to zero

---

(3) Every move from a position in Z is to a position in NZ

If (x,y,z) is in Z, and x is changed to $x' < x$, then we cannot have

$\in Z$

$$x \oplus \cancel{y} \oplus \cancel{z} = 0 = x' \oplus \cancel{y} \oplus \cancel{z}$$

(x',y,z)

$\notin Z$

Because then $x = x'$

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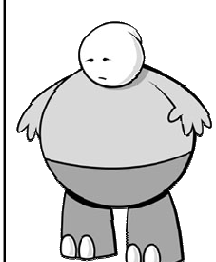
k-Pile Nim



Here's What You Need to Know…

Combinatorial Games

- P-positions versus N-positions
- When there are no draws, every position is either P or N

Nim

- Definition of the game
- Nim-sum
- Bouton's Theorem