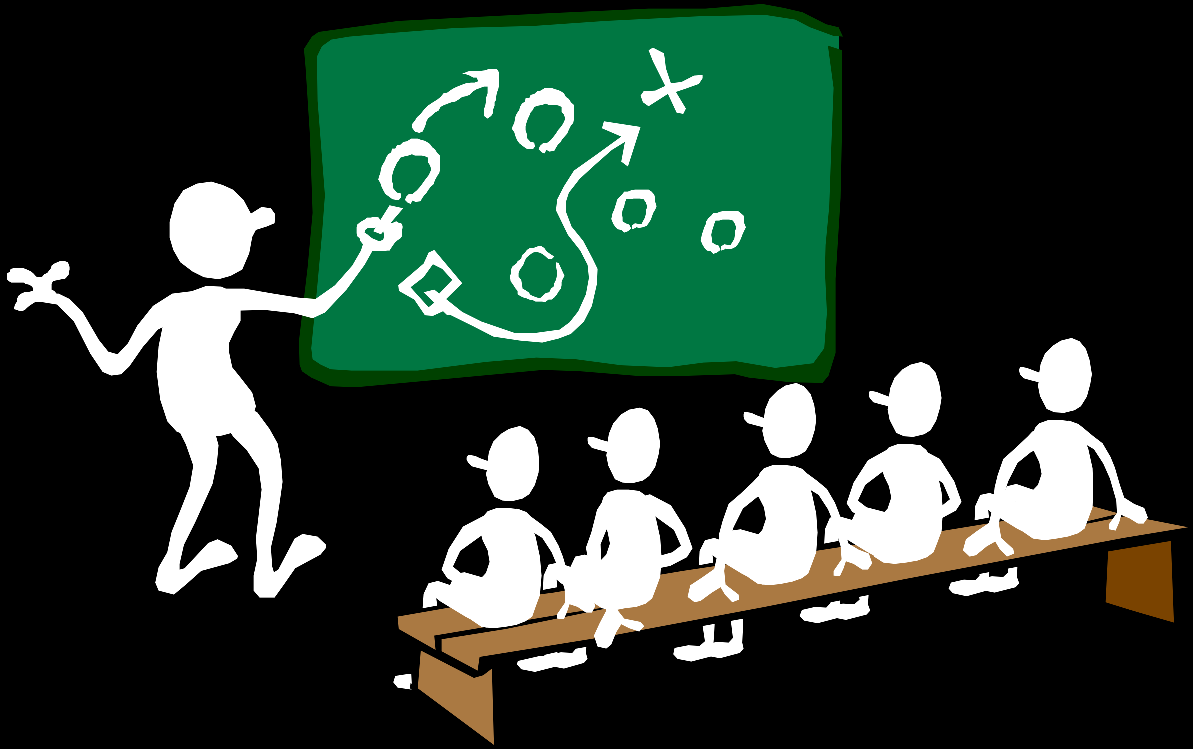
15-251

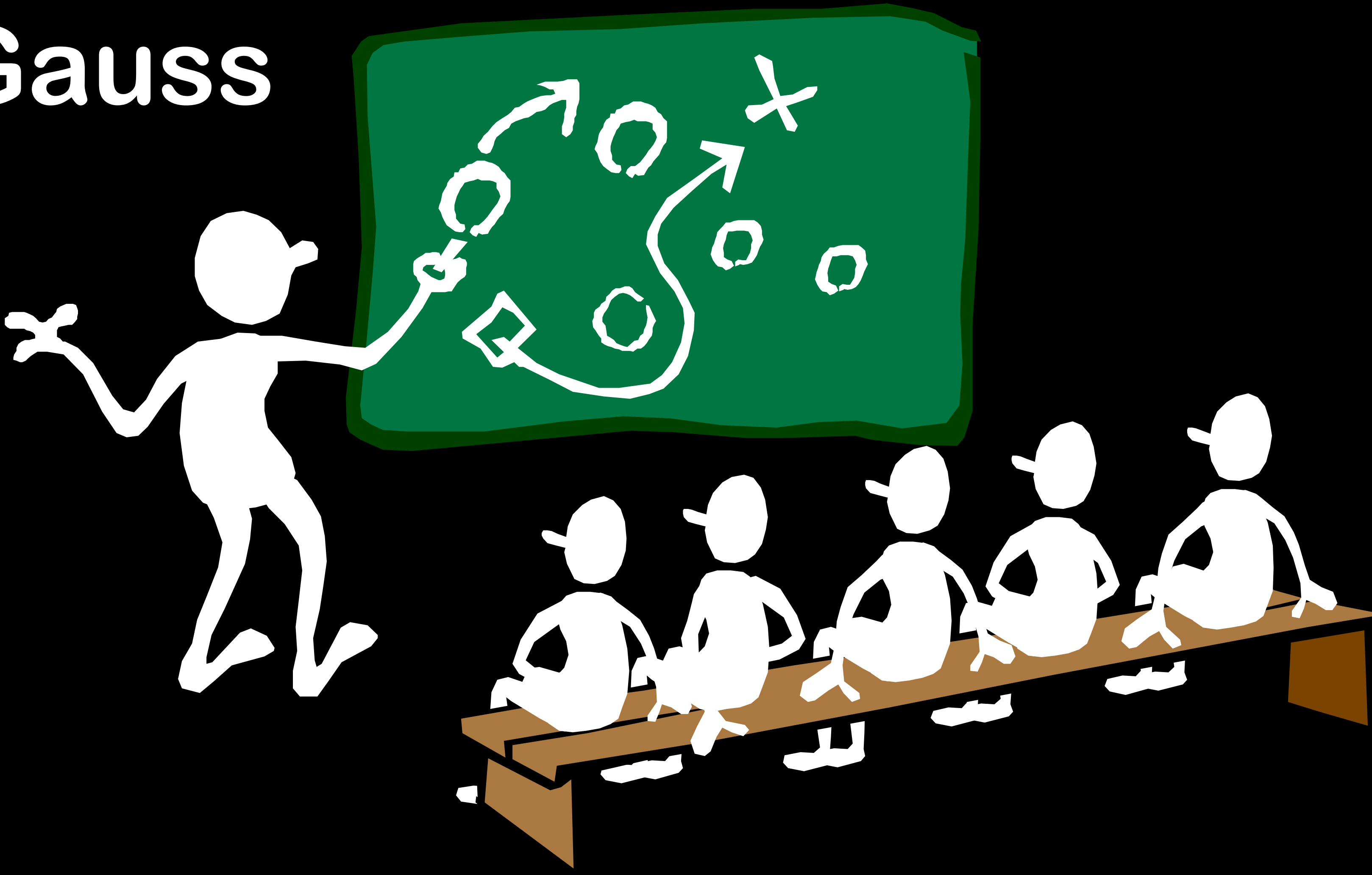
Great Theoretical Ideas in Computer Science

Grade School Revisited: How To Multiply Two Numbers

Lecture 22, November 6, 2008



Gauss



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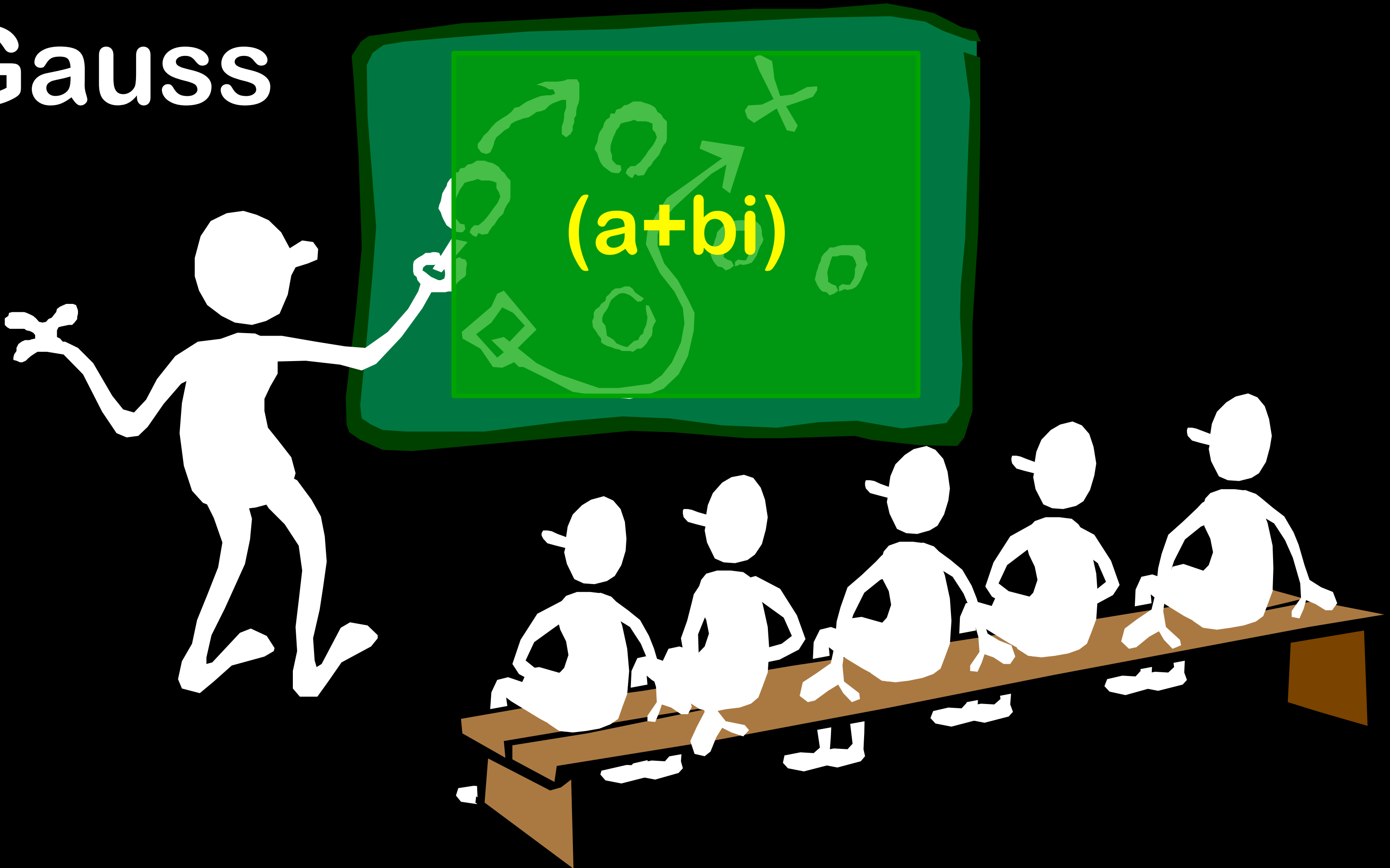

Gauss' Complex Puzzle

Remember how to multiply two complex numbers $a + bi$ and $c + di$?

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Remember how to multiply two complex numbers a + bi and c + di?

$$(a+bi)(c+di) = [ac - bd] + [ad + bc]\, i$$

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Input: a,b,c,d

Output: ac-bd, ad+bc

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$(a+bi)(c+di) = [ac - bd] + [ad + bc]\, i$

Input: a,b,c,d

Output: ac-bd, ad+bc

If multiplying two real numbers costs $1 and adding them costs a penny, what is the cheapest way to obtain the output from the input?

Gauss' Complex Puzzle

Remember how to multiply two complex numbers a + bi and c + di?

$$(a+bi)(c+di) = [ac - bd] + [ad + bc]\, i$$

Input: a,b,c,d

Output: ac-bd, ad+bc

If multiplying two real numbers costs $1 and adding them costs a penny, what is the cheapest way to obtain the output from the input?

Can you do better than $4.02?

Gauss' $3.05 Method

Input: a,b,c,d

Output: ac-bd, ad+bc

Gauss' $3.05 Method

Input: a,b,c,d
Output: ac-bd, ad+bc

$X_1 = a + b$

Gauss' $3.05 Method

Input: a,b,c,d

Output: ac-bd, ad+bc

c $X_1 = a + b$

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c $X_1 = a + b$

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Gauss' $3.05 Method

Input: a,b,c,d

Output: ac-bd, ad+bc

c $X_1 = a + b$

c $X_2 = c + d$

$X_3 = X_1 X_2 = ac + ad + bc + bd$

Gauss' $3.05 Method

Input: a,b,c,d

Output: ac-bd, ad+bc

c $X_1 = a + b$

c $X_2 = c + d$

\$ $X_3 = X_1 X_2$ $= ac + ad + bc + bd$

Gauss' \$3.05 Method

Input: a,b,c,d

Output: ac-bd, ad+bc

c $X_1 = a + b$

c $X_2 = c + d$

\$ $X_3 = X_1 X_2 = ac + ad + bc + bd$

$X_4 = ac$

Gauss' $3.05 Method

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c $X_1 = a + b$

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Gauss' $3.05 Method

Input: a,b,c,d
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c $X_1 = a + b$

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\$ $X_4 = ac$

$X_5 = bd$

Gauss' \$3.05 Method

Input: a,b,c,d

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Gauss' $3.05 Method

Input: a,b,c,d
Output: ac-bd, ad+bc

c	$X_1 = a + b$	
c	$X_2 = c + d$	
$	$X_3 = X_1 X_2$	$= ac + ad + bc + bd$
$	$X_4 = ac$	
$	$X_5 = bd$	
	$X_6 = X_4 - X_5$	$= ac - bd$

Gauss' $3.05 Method

Input: a,b,c,d
Output: ac-bd, ad+bc

c $X_1 = a + b$

c $X_2 = c + d$

\$ $X_3 = X_1 X_2$ $= ac + ad + bc + bd$

\$ $X_4 = ac$

\$ $X_5 = bd$

c $X_6 = X_4 – X_5$ $= ac - bd$

Gauss' $3.05 Method

Input: a,b,c,d
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c	$X_1 = a + b$	
c	$X_2 = c + d$	
$	$X_3 = X_1 X_2$	$= ac + ad + bc + bd$
$	$X_4 = ac$	
$	$X_5 = bd$	
c	$X_6 = X_4 – X_5$	$= ac - bd$
	$X_7 = X_3 – X_4 – X_5$	$= bc + ad$

Gauss' $3.05 Method

Input: a,b,c,d
Output: ac-bd, ad+bc

c $X_1 = a + b$

c $X_2 = c + d$

\$ $X_3 = X_1 X_2 = ac + ad + bc + bd$

\$ $X_4 = ac$

\$ $X_5 = bd$

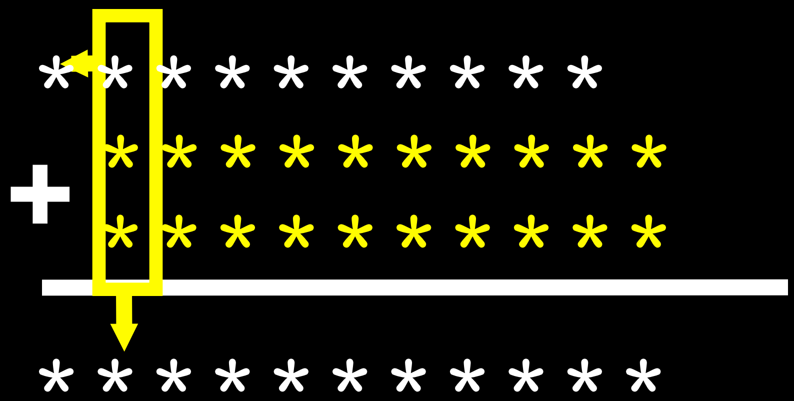
c $X_6 = X_4 – X_5 = ac - bd$

cc $X_7 = X_3 – X_4 – X_5 = bc + ad$

The Gauss optimization saves one multiplication out of four.

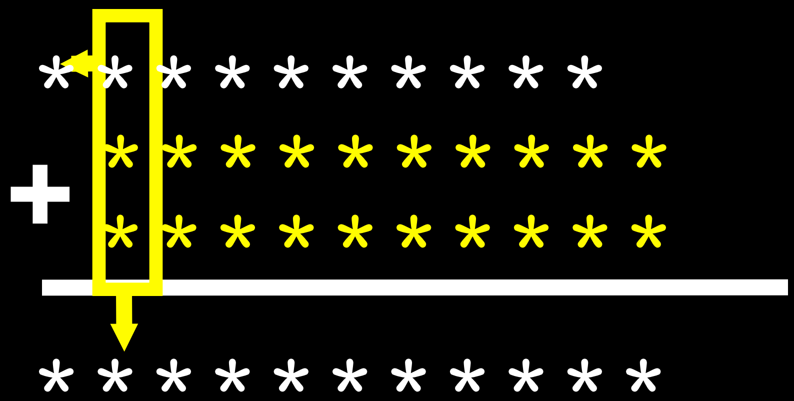
It requires 25% less work.

Time complexity of grade school addition



T(n) = amount of time grade school addition uses to add two n-bit numbers

Time complexity of grade school addition

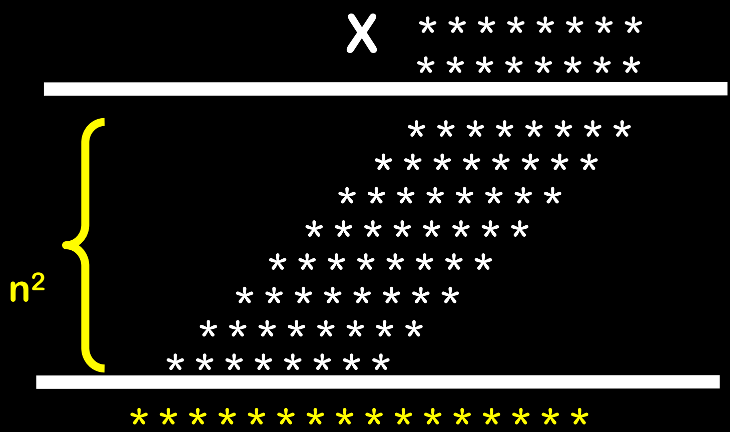


T(n) = amount of time grade school addition uses to add two n-bit numbers

We saw that T(n) was linear

$T(n) = \Theta(n)$

Time complexity of grade school multiplication



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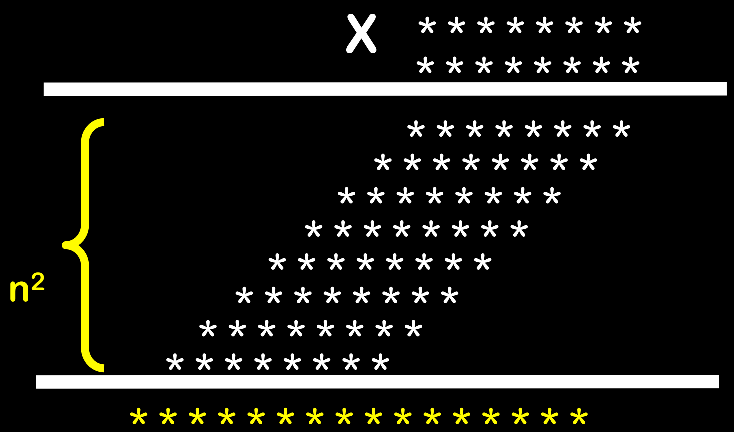
Time complexity of grade school multiplication

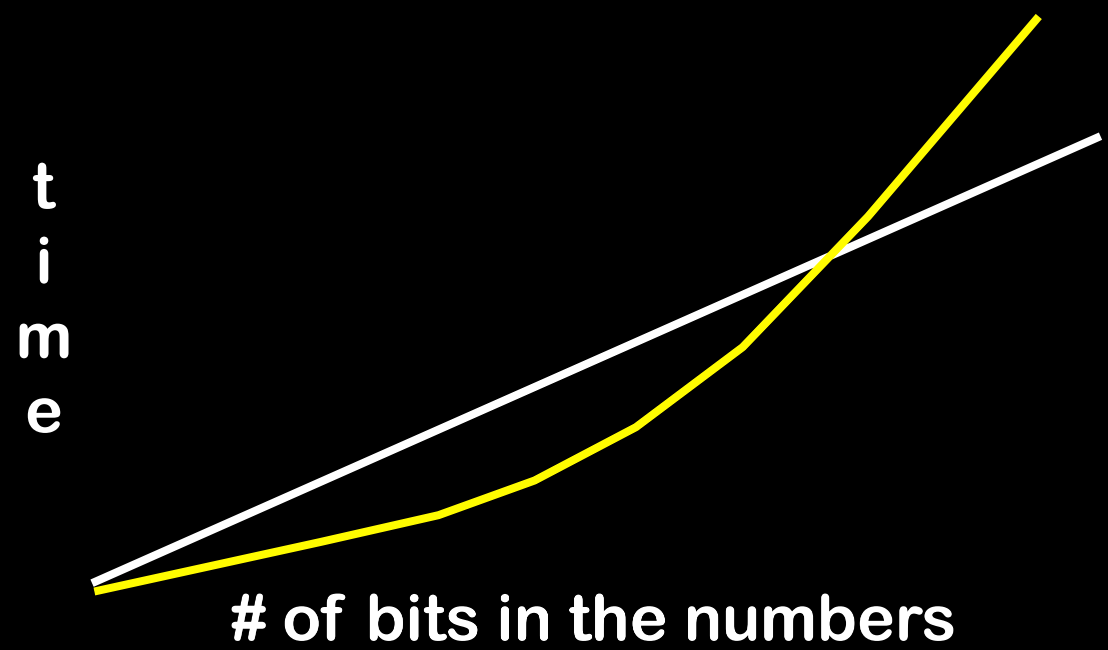


T(n) = The amount of time grade school multiplication uses to add two n-bit numbers

We saw that T(n) was quadratic

$T(n) = \Theta(n^2)$

Grade School Addition: Linear time
Grade School Multiplication: Quadratic time



No matter how dramatic the difference in the constants, the quadratic curve will eventually dominate the linear curve

Is there a sub-linear time method for addition?

Any addition algorithm takes $\Omega(n)$ time

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Claim: Any algorithm for addition must read all of the input bits

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Claim: Any algorithm for addition must read all of the input bits

Proof: Suppose there is a mystery algorithm **A** that does not examine each bit

Any addition algorithm takes Ω(n) time

Claim: Any algorithm for addition must read all of the input bits

Proof: Suppose there is a mystery algorithm A that does not examine each bit

Give A a pair of numbers. There must be some unexamined bit position i in one of the numbers

Any addition algorithm takes Ω(n) time

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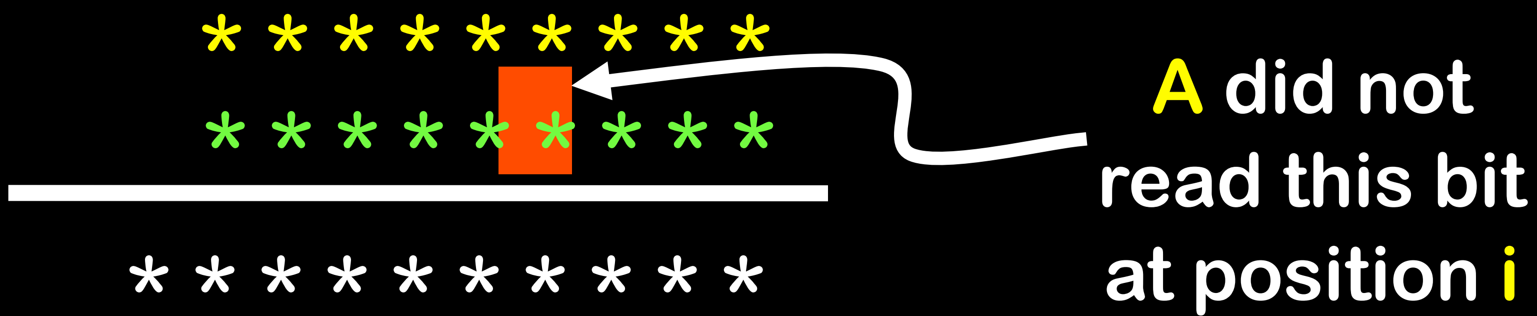
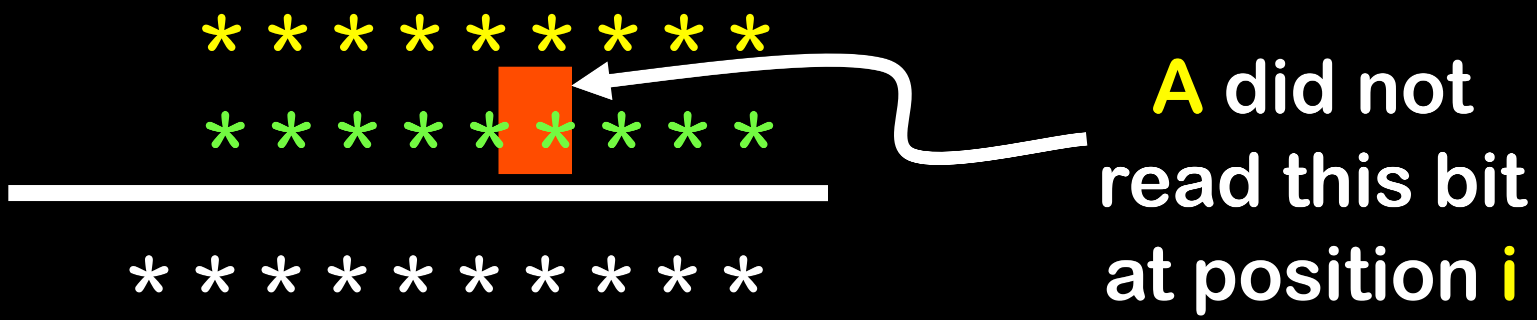
Any addition algorithm takes Ω(n) time

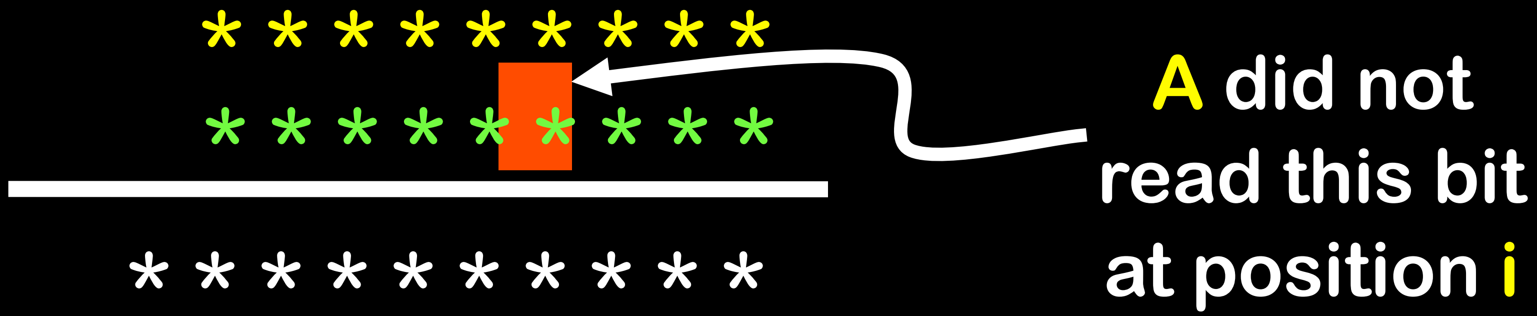


Any addition algorithm takes Ω(n) time



If A is not correct on the inputs, we found a bug

Any addition algorithm takes Ω(n) time



If A is not correct on the inputs, we found a bug

If A is correct, flip the bit at position i and give A the new pair of numbers. A gives the same answer as before, which is now wrong.

Grade school addition can't be improved upon by more than a constant factor

Grade School Addition: $\Theta(n)$ time.
Furthermore, it is optimal

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Grade School Multiplication: $\Theta(n^2)$ time

Grade School Addition: $\Theta(n)$ time. Furthermore, it is optimal

Grade School Multiplication: $\Theta(n^2)$ time

Is there a clever algorithm to multiply two numbers in linear time?

Grade School Addition: $\Theta(n)$ time.
Furthermore, it is optimal

Grade School Multiplication: $\Theta(n^2)$ time

Is there a clever algorithm to multiply two numbers in linear time?

Despite years of research, no one knows! If you resolve this question, Carnegie Mellon will give you a PhD!

Can we even break the quadratic time barrier?

In other words, can we do something very different than grade school multiplication?

Divide And Conquer

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An approach to faster algorithms:

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DIVIDE a problem into smaller subproblems

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An approach to faster algorithms:

DIVIDE a problem into smaller subproblems

CONQUER them recursively

Divide And Conquer

An approach to faster algorithms:

DIVIDE a problem into smaller subproblems

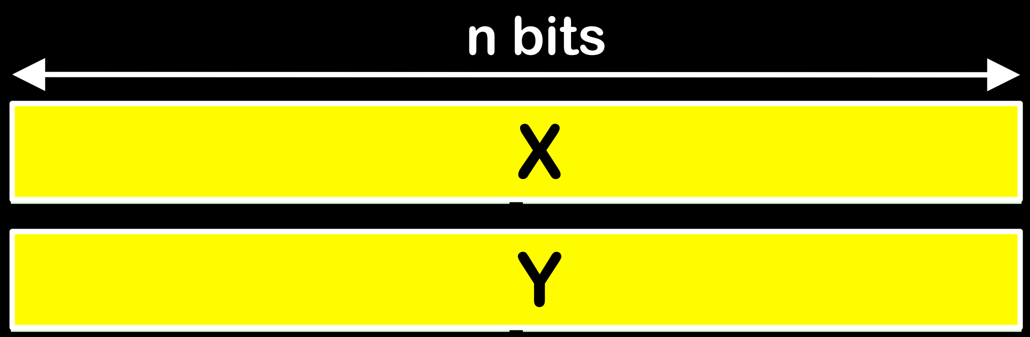
CONQUER them recursively

GLUE the answers together so as to obtain the answer to the larger problem

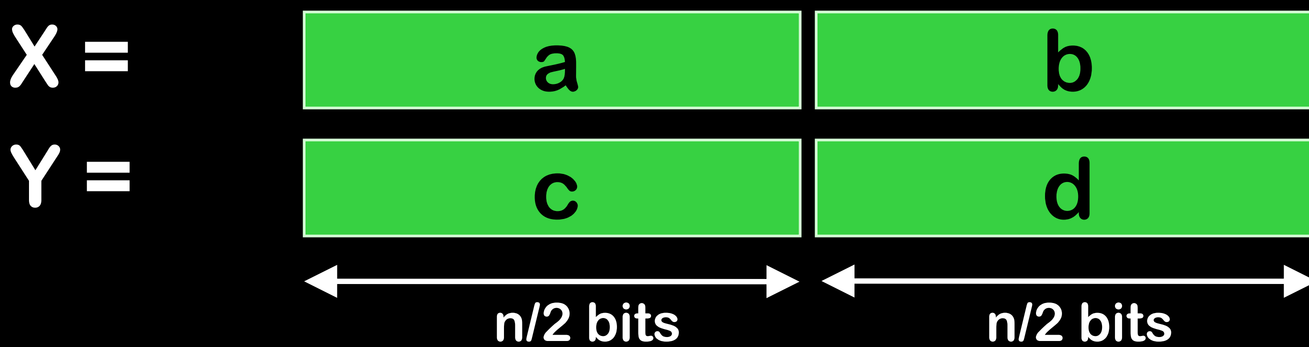
Multiplication of 2 n-bit numbers

X =

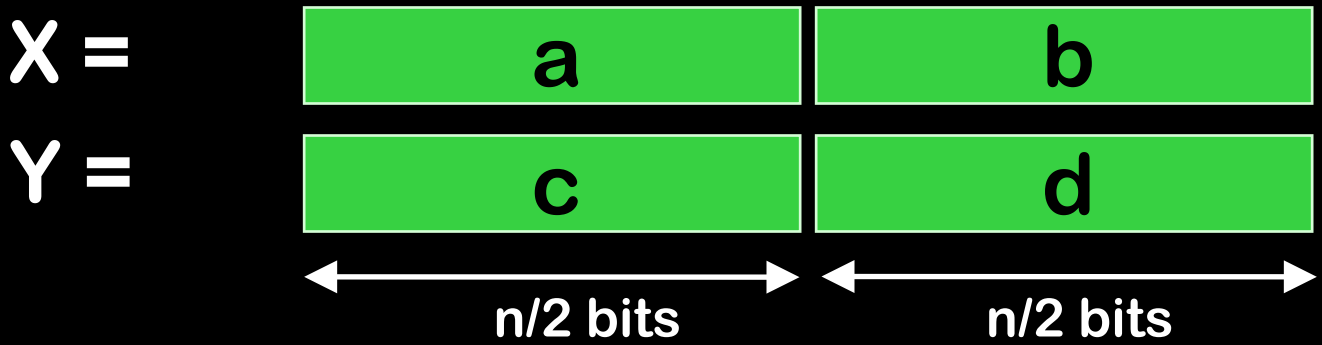
Y =



Multiplication of 2 n-bit numbers

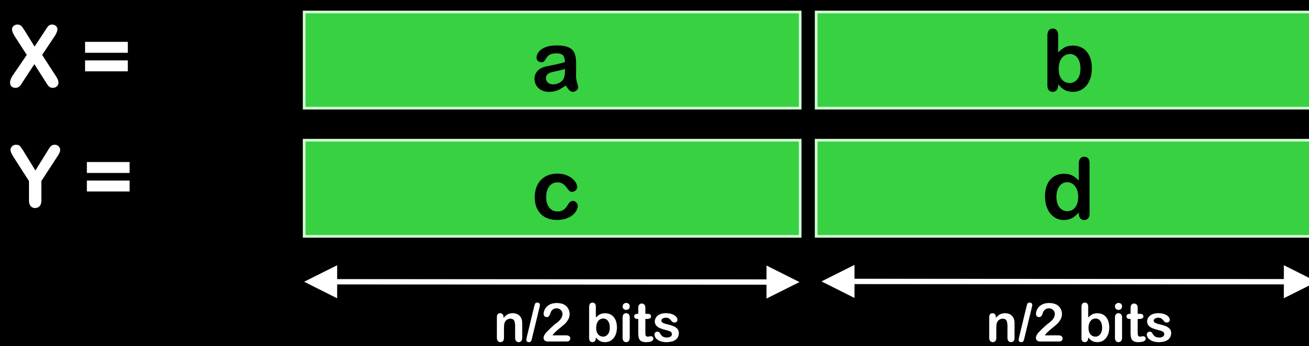


Multiplication of 2 n-bit numbers



$X = a\ 2^{n/2} + b \qquad Y = c\ 2^{n/2} + d$

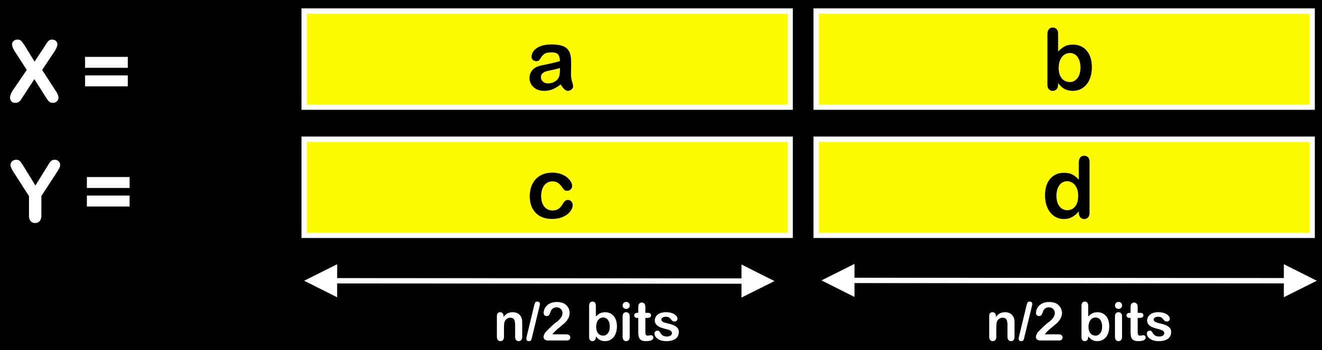
Multiplication of 2 n-bit numbers



$$X = a\,2^{n/2} + b \qquad Y = c\,2^{n/2} + d$$

$$X \times Y = ac\,2^{n} + (ad + bc)\,2^{n/2} + bd$$

Multiplication of 2 n-bit numbers



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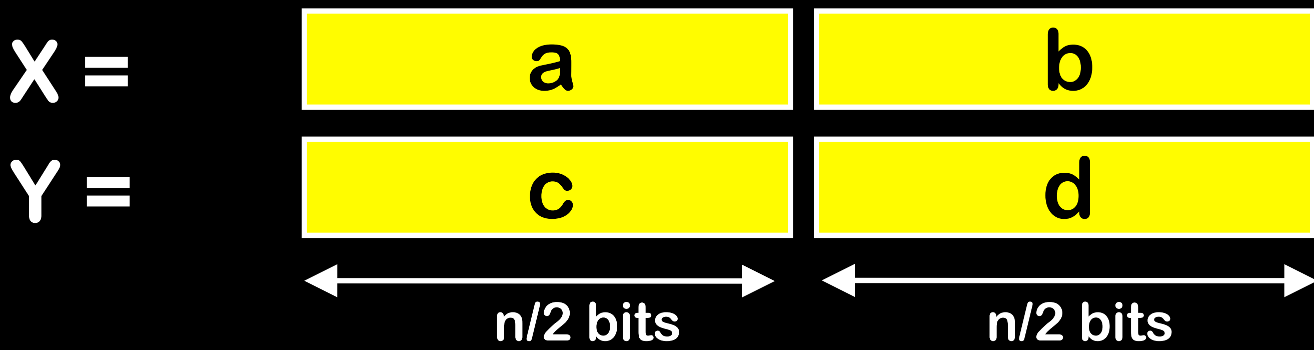
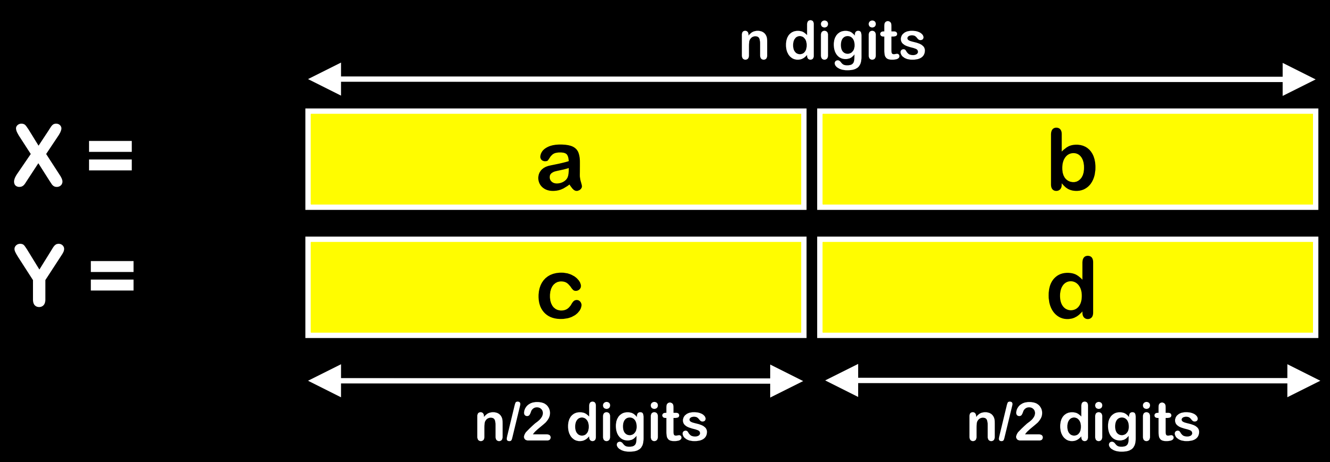
Multiplication of 2 n-bit numbers



$$X \times Y = ac\, 2^n + (ad + bc)\, 2^{n/2} + bd$$

```
MULT(X,Y):
      If |X| = |Y| = 1 then return XY
      else  break X into a;b and Y into c;d
            return MULT(a,c) 2^n + (MULT(a,d)
            + MULT(b,c)) 2^(n/2) + MULT(b,d)
```

Same thing for numbers in decimal!



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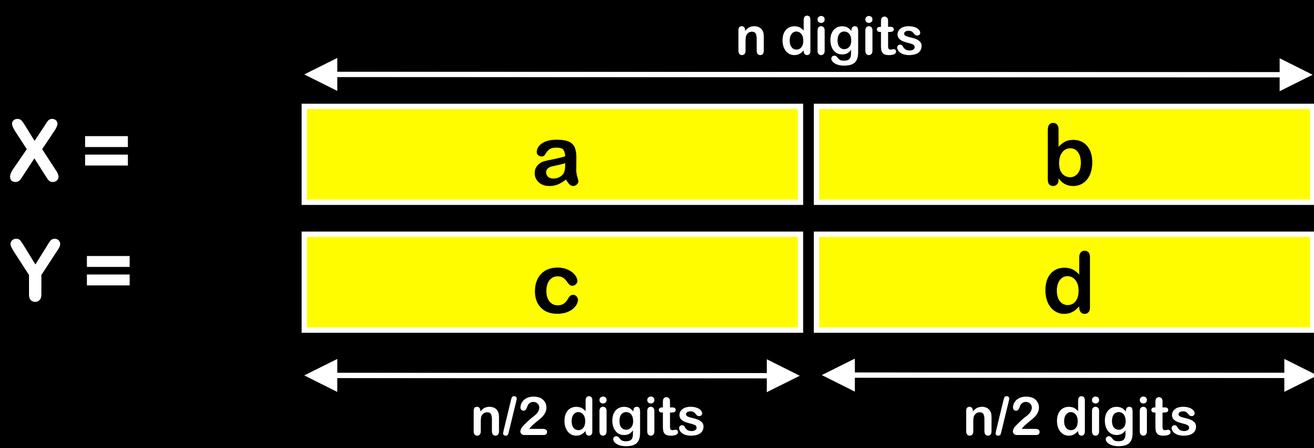

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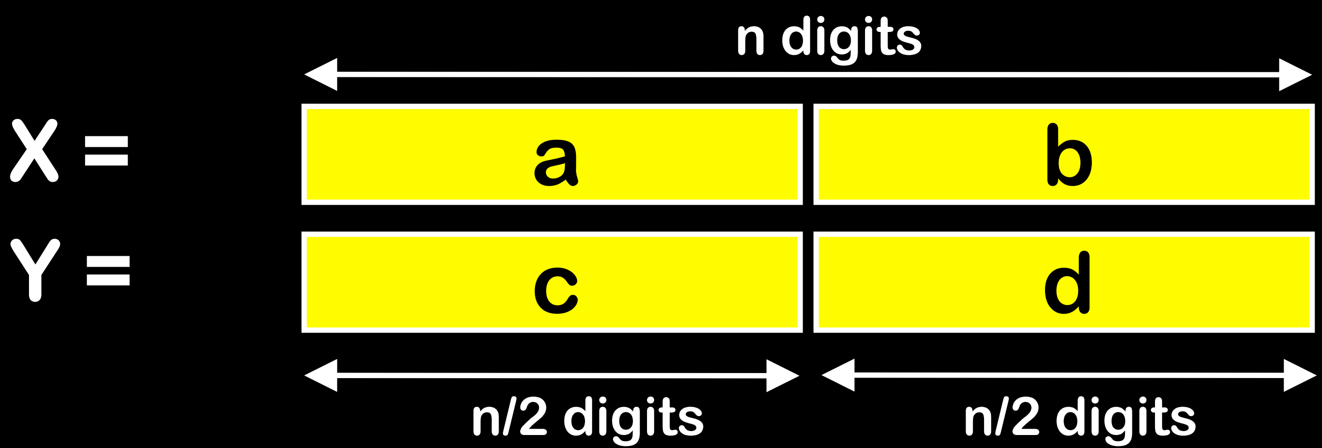

$$X = a\,10^{n/2} + b \qquad Y = c\,10^{n/2} + d$$

$$X \times Y = ac\,10^{n} + (ad + bc)\,10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^{n} + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

X =	a	b
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Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139   1234*4276   5678*2139   5678*4276

X =	a	b
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Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

12*21 12*39 34*21 34*39

X =	a	b
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Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

12*21   12*39   34*21   34*39

1*2 1*1 2*2 2*1

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

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12*21 12*39 34*21 34*39

1*2 1*1 2*2 2*1

2 1 4 2

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

12*21 12*39 34*21 34*39

1*2 1*1 2*2 2*1

2 1 4 2

Hence: $12*21 = 2*10^2 + (1 + 4)10^1 + 2 = 252$

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

2521 12*39 34*21 34*39

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^{n} + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

2521 4689 34*21 34*39

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

2521 4689 7141 34*39

X = | a | b

Y = | c | d

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139   1234*4276   5678*2139   5678*4276

2521   4689   7141   1326

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^{n} + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

2521 4689 7141 1326

$*10^4$ + $*10^2$ + $*10^2$ + $*1$

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

1234*2139 1234*4276 5678*2139 5678*4276

2521 4689 7141 1326

$*10^4$ + $*10^2$ + $*10^2$ + $*1$ = 2639526

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

2639526 9 1234*4276 5678*2139 5678*4276

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^{n} + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

26395269 52765846 5678*2139 5678*4276

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

26395269 52765846 12145242 5678*4276

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

26395269 52765846 12145242 24279128

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^{n} + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

2639526 9 5276584 6 12145242 24279128

$*10^8$ + $*10^4$ + $*10^4$ + $*1$

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Multiplying (Divide & Conquer style)

12345678 * 21394276

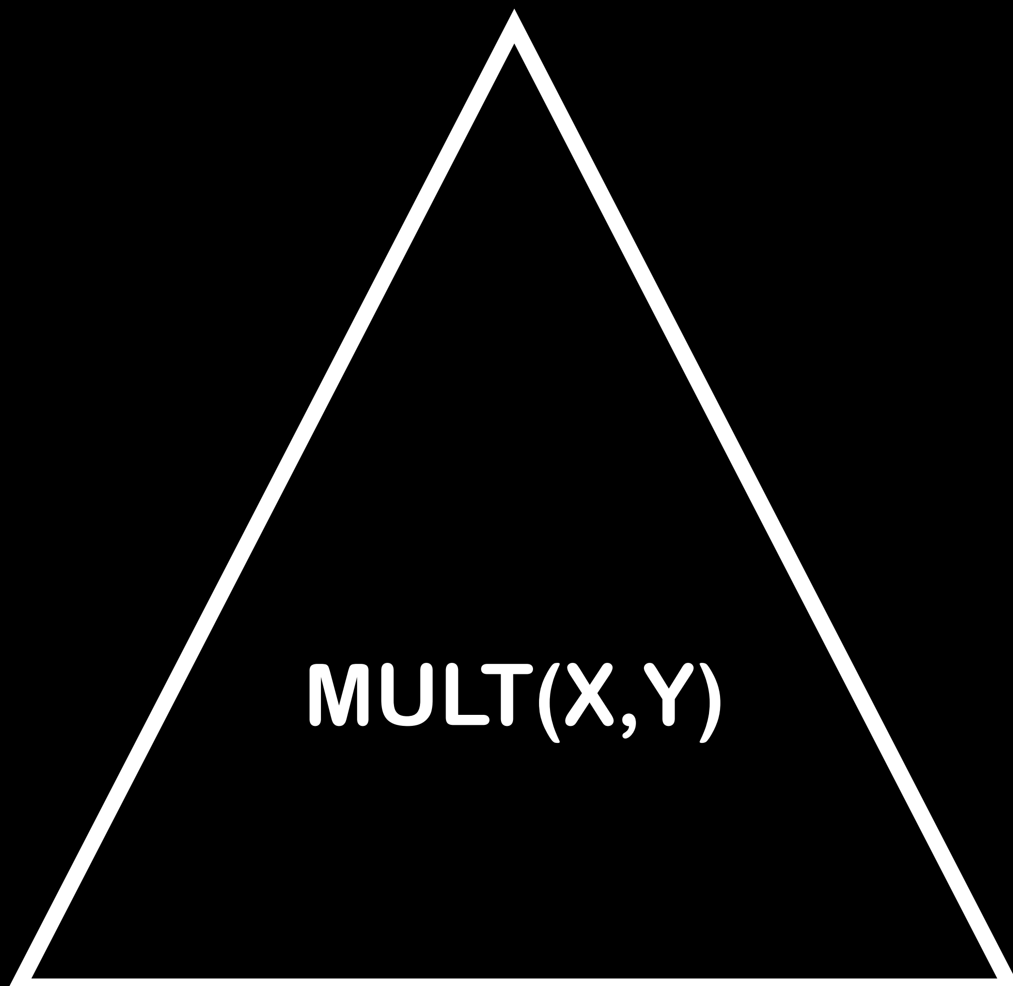
26395269 $*10^8$ + 52765846 $*10^4$ + 12145242 $*10^4$ + 24279128 $*1$

= 264126842539128

X =	a	b
Y =	c	d

$$X \times Y = ac\ 10^n + (ad + bc)\ 10^{n/2} + bd$$

Divide, Conquer, and Glue



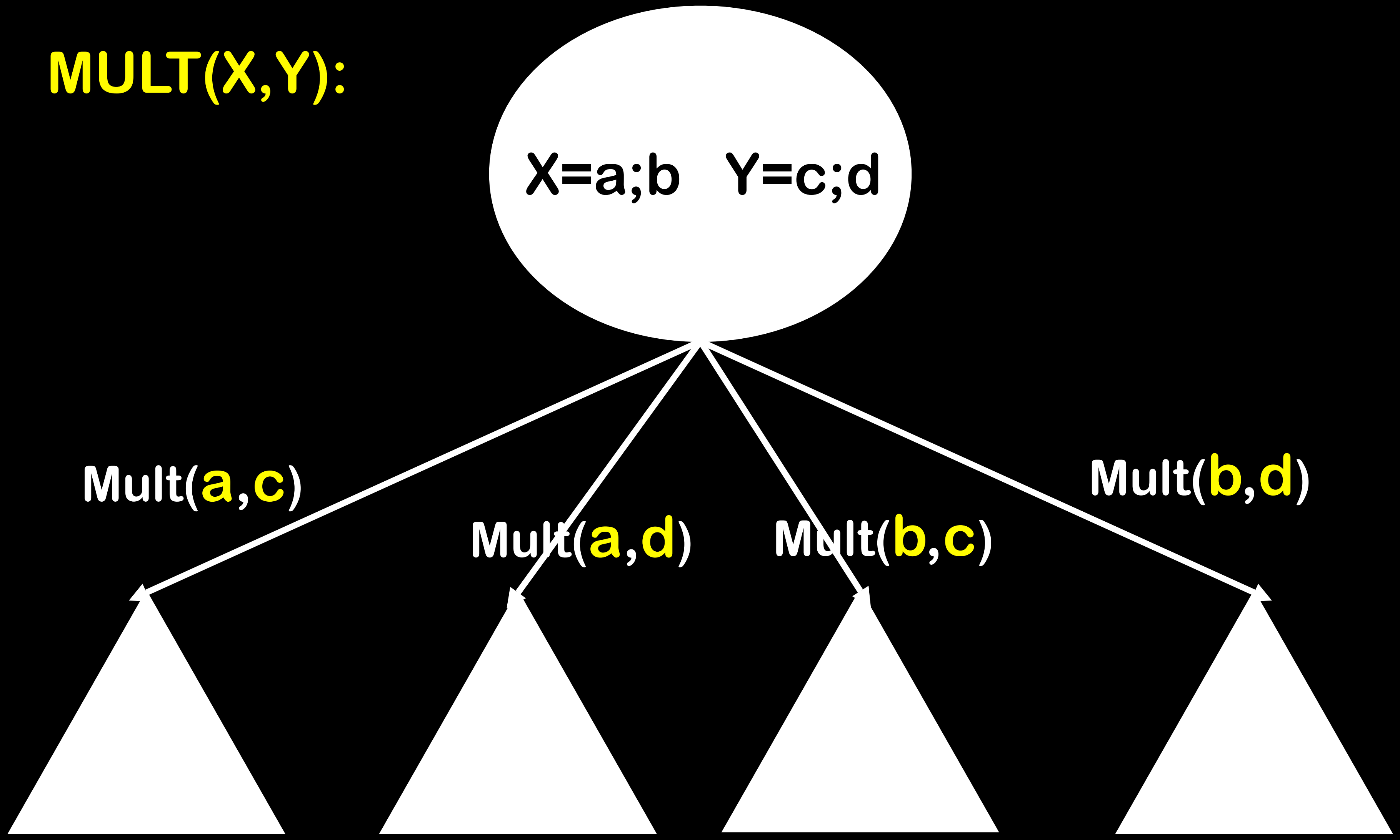
Divide, Conquer, and Glue

MULT(X,Y):

if $|X| = |Y| = 1$
then return XY,
else…

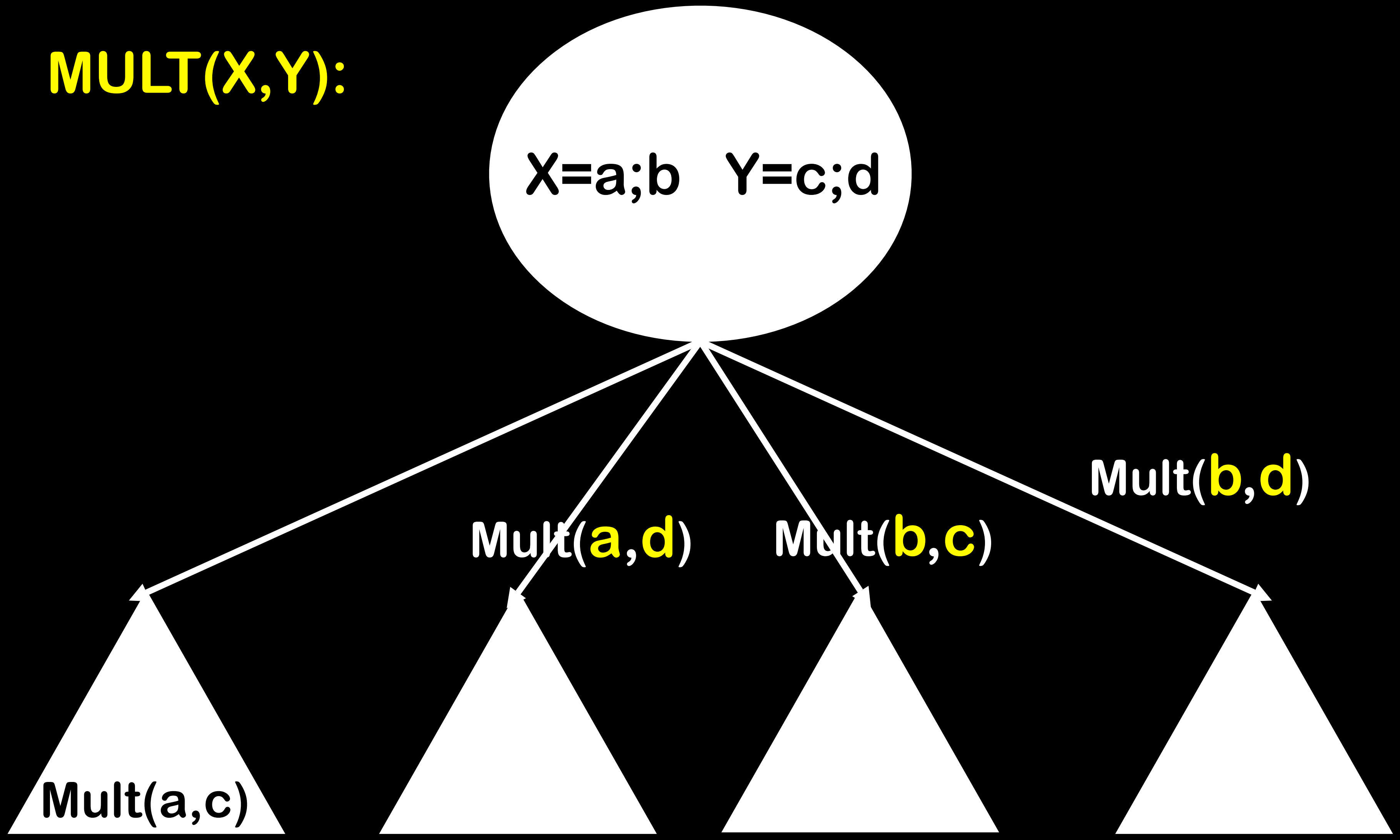
Divide, Conquer, and Glue

MULT(X,Y):



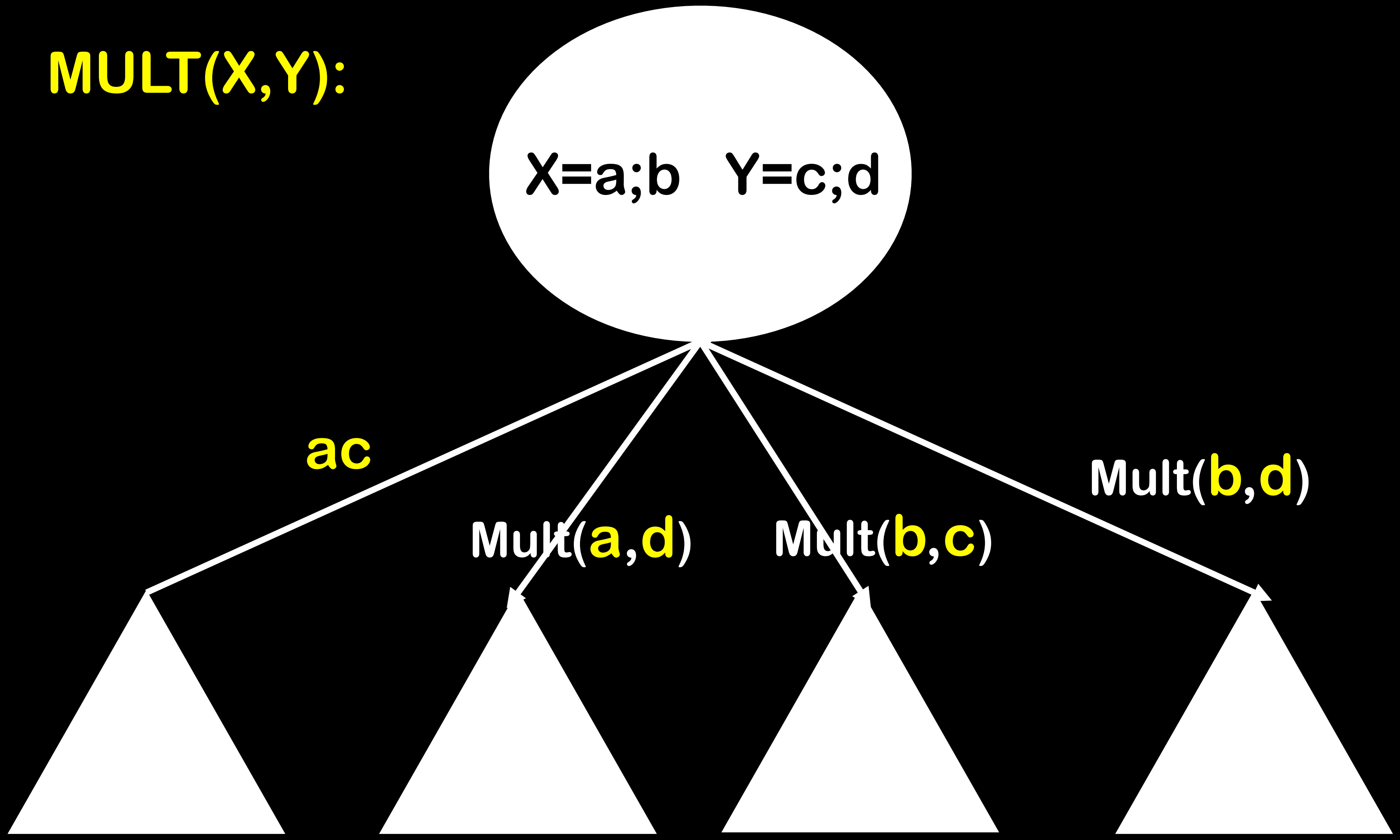
Divide, Conquer, and Glue

MULT(X,Y):



Divide, Conquer, and Glue

MULT(X,Y):



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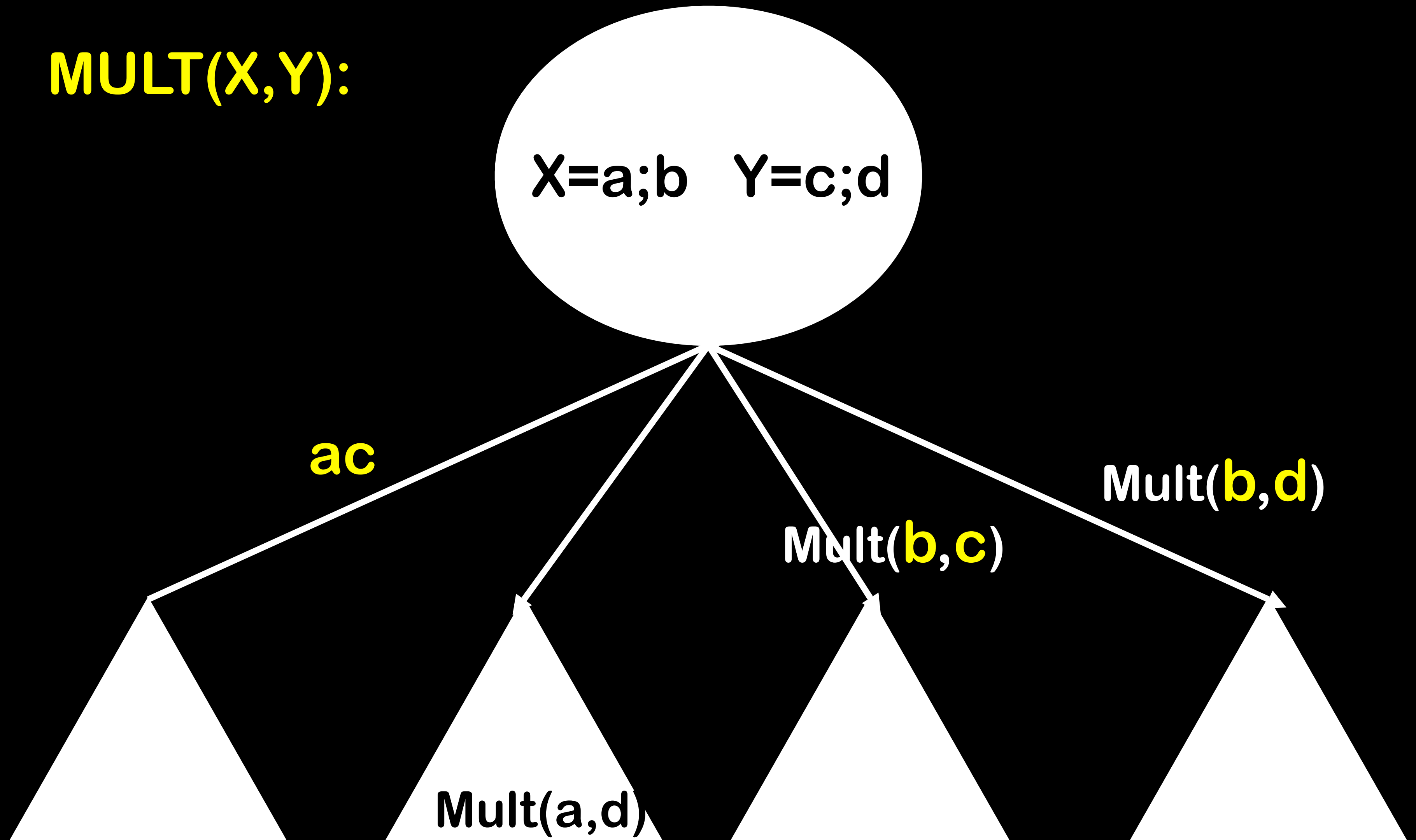
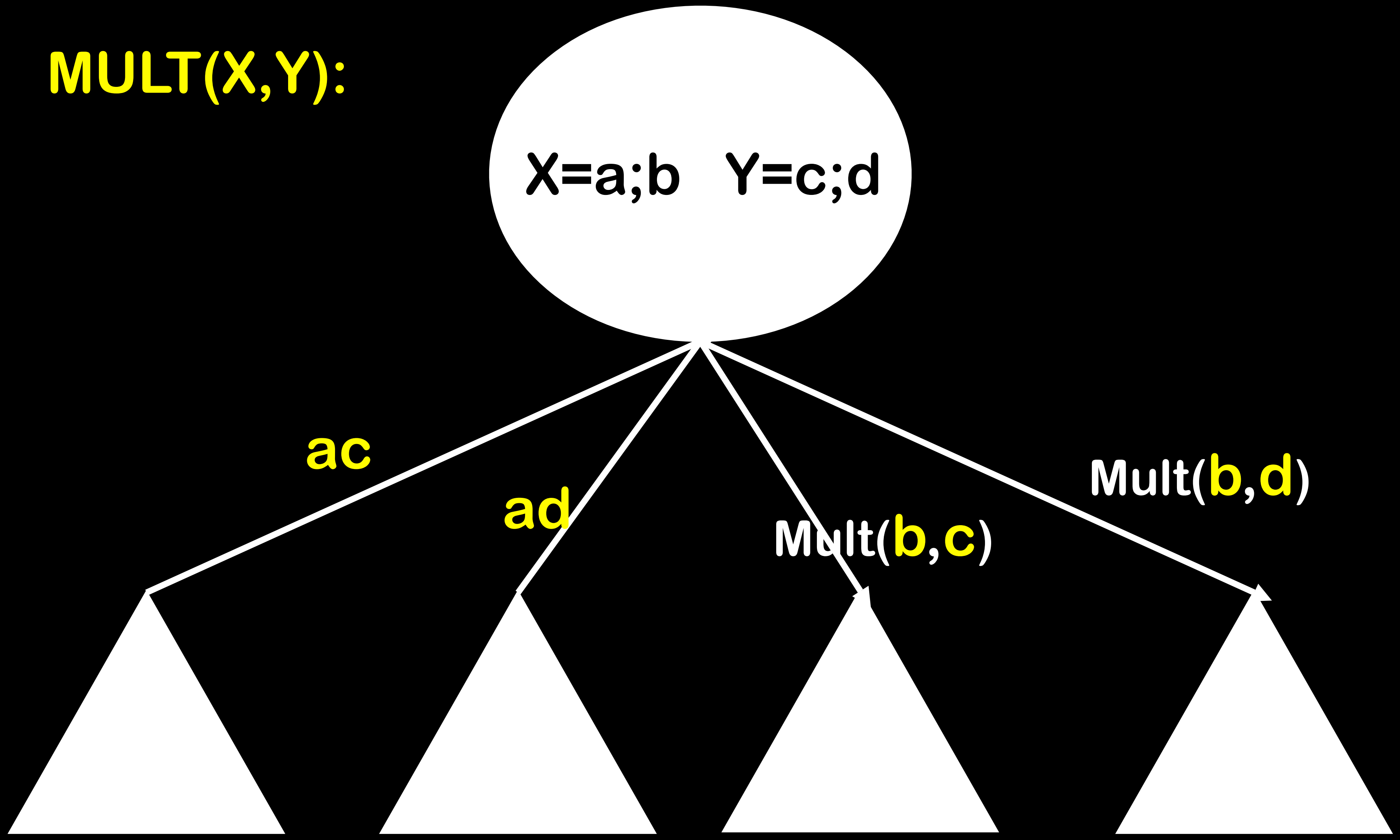


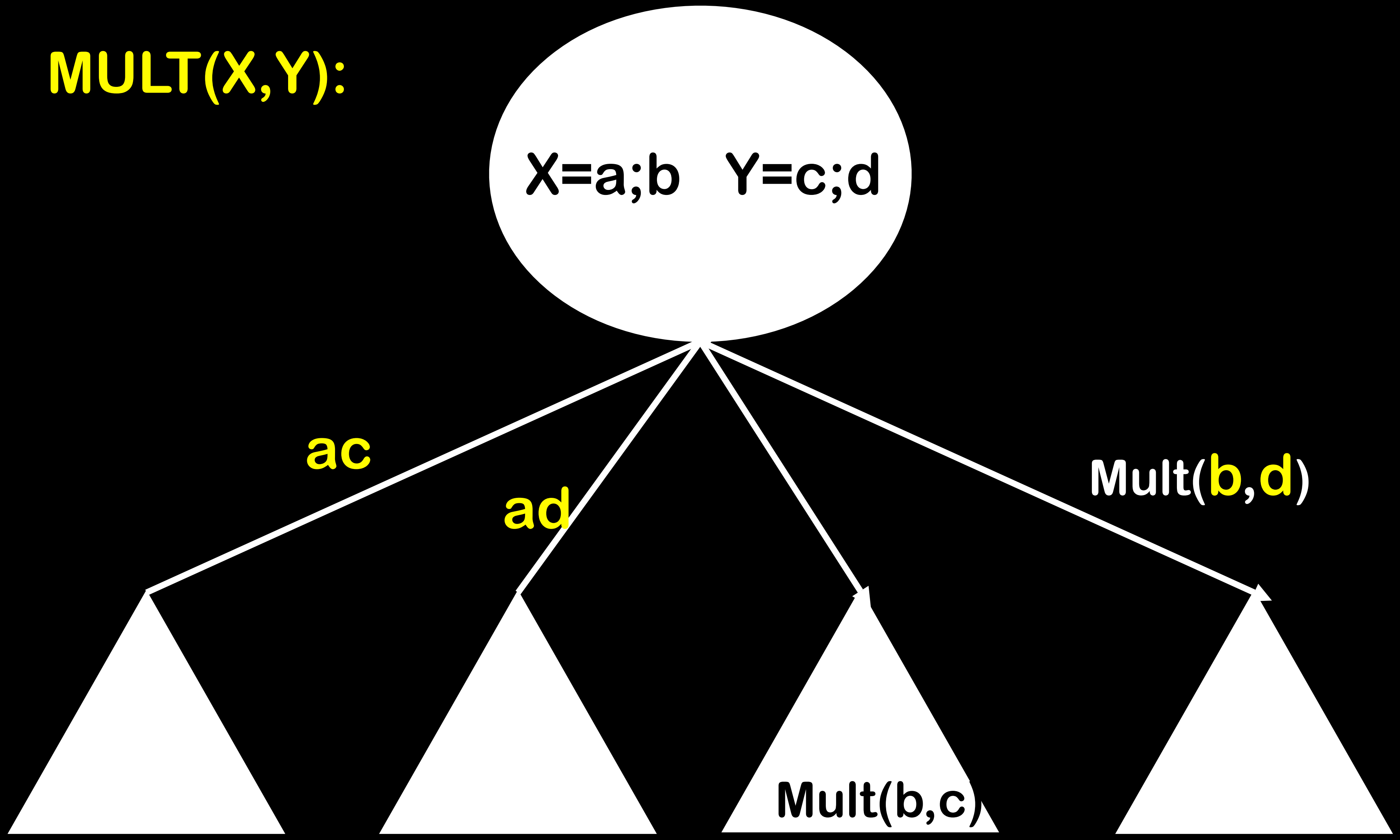
Divide, Conquer, and Glue

MULT(X,Y):



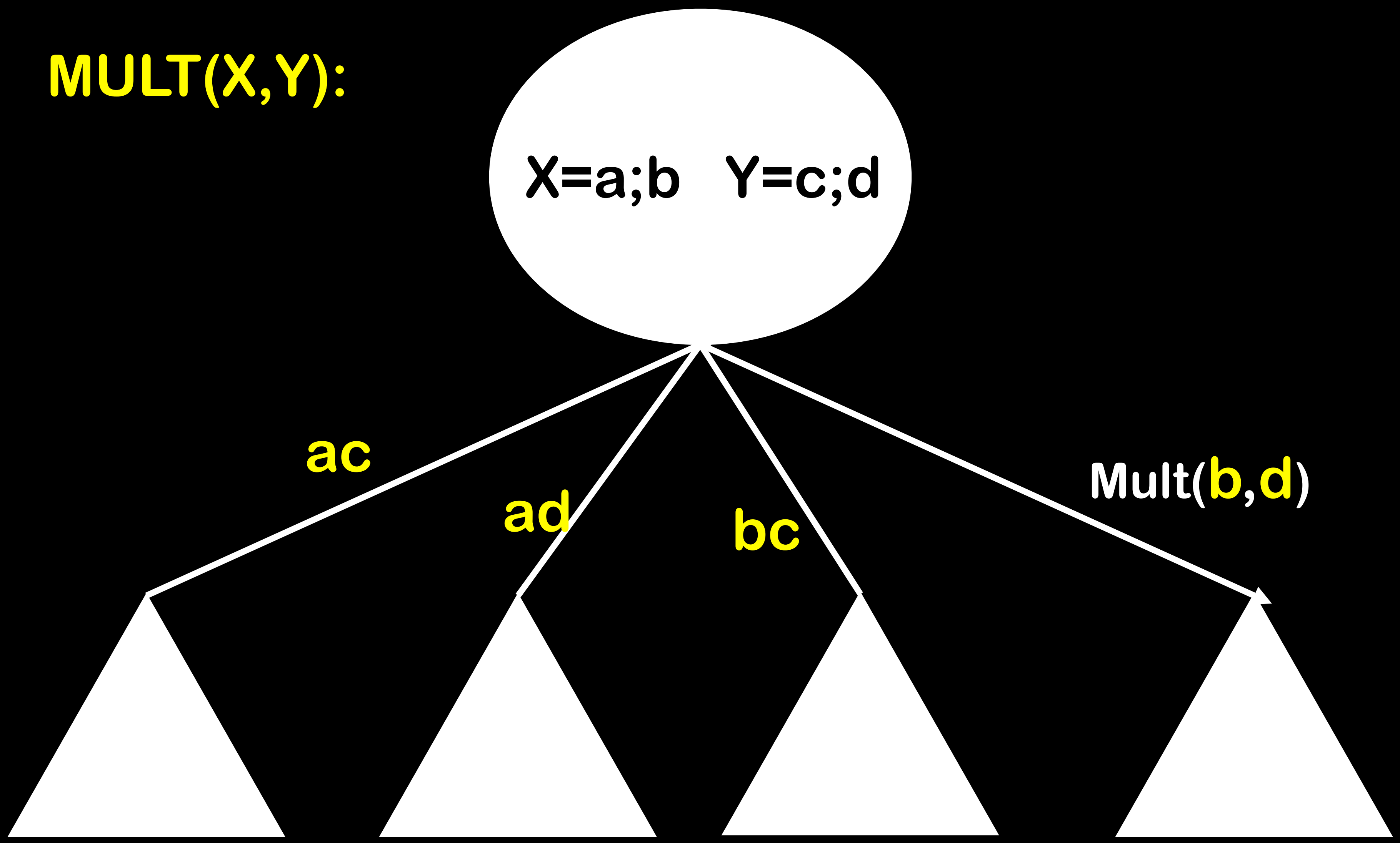
Divide, Conquer, and Glue

MULT(X,Y):



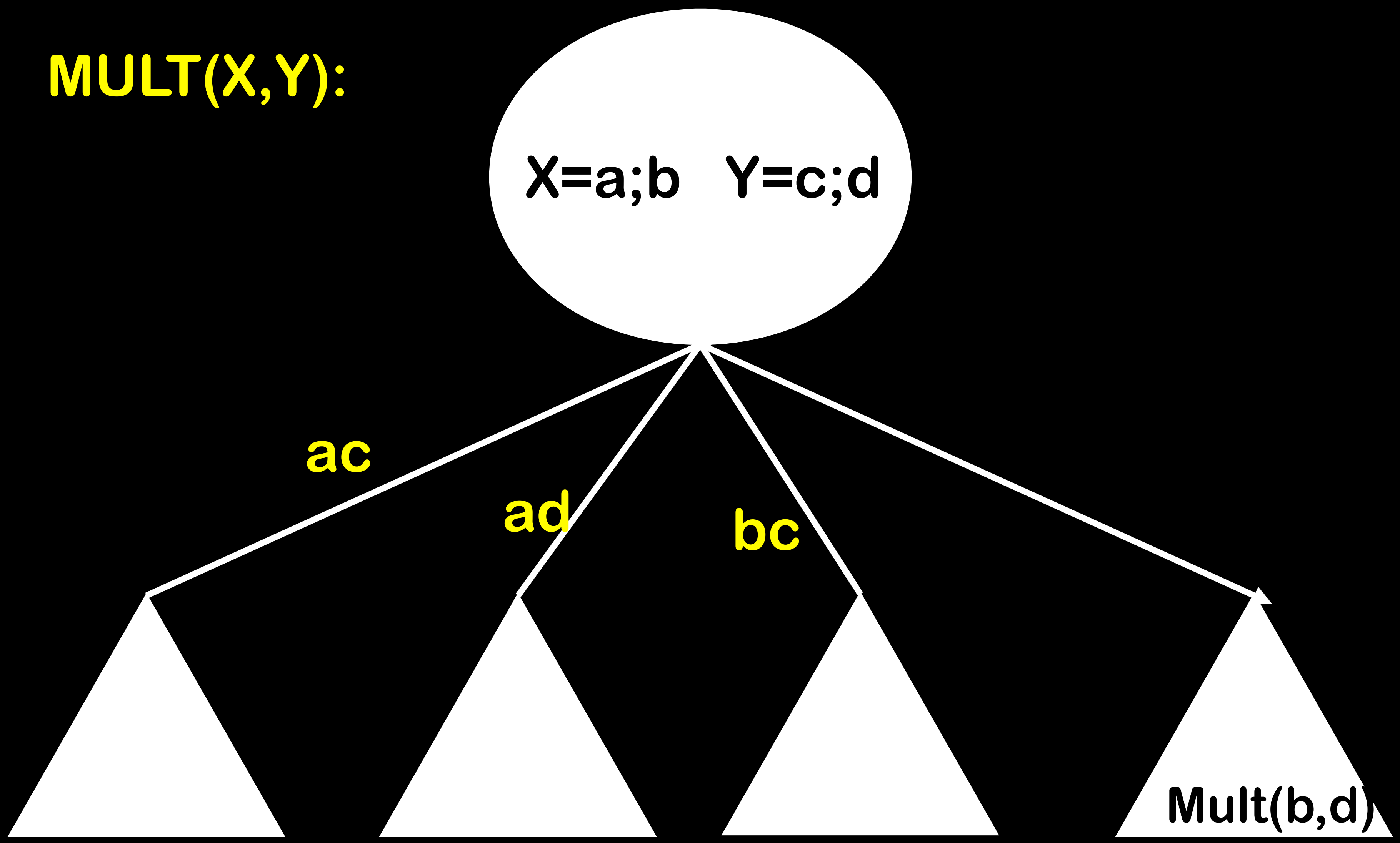
Divide, Conquer, and Glue

MULT(X,Y):



Divide, Conquer, and Glue

MULT(X,Y):



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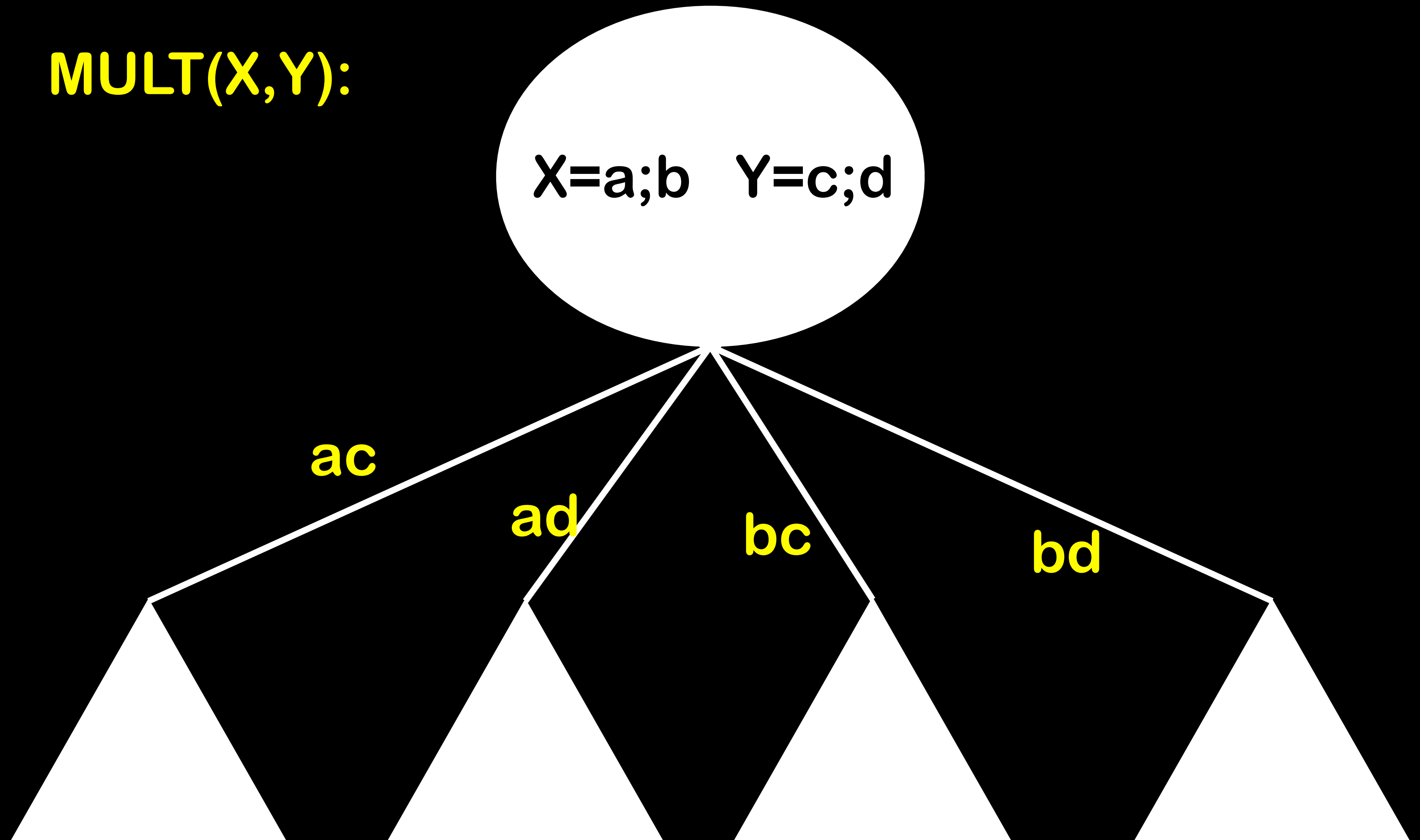

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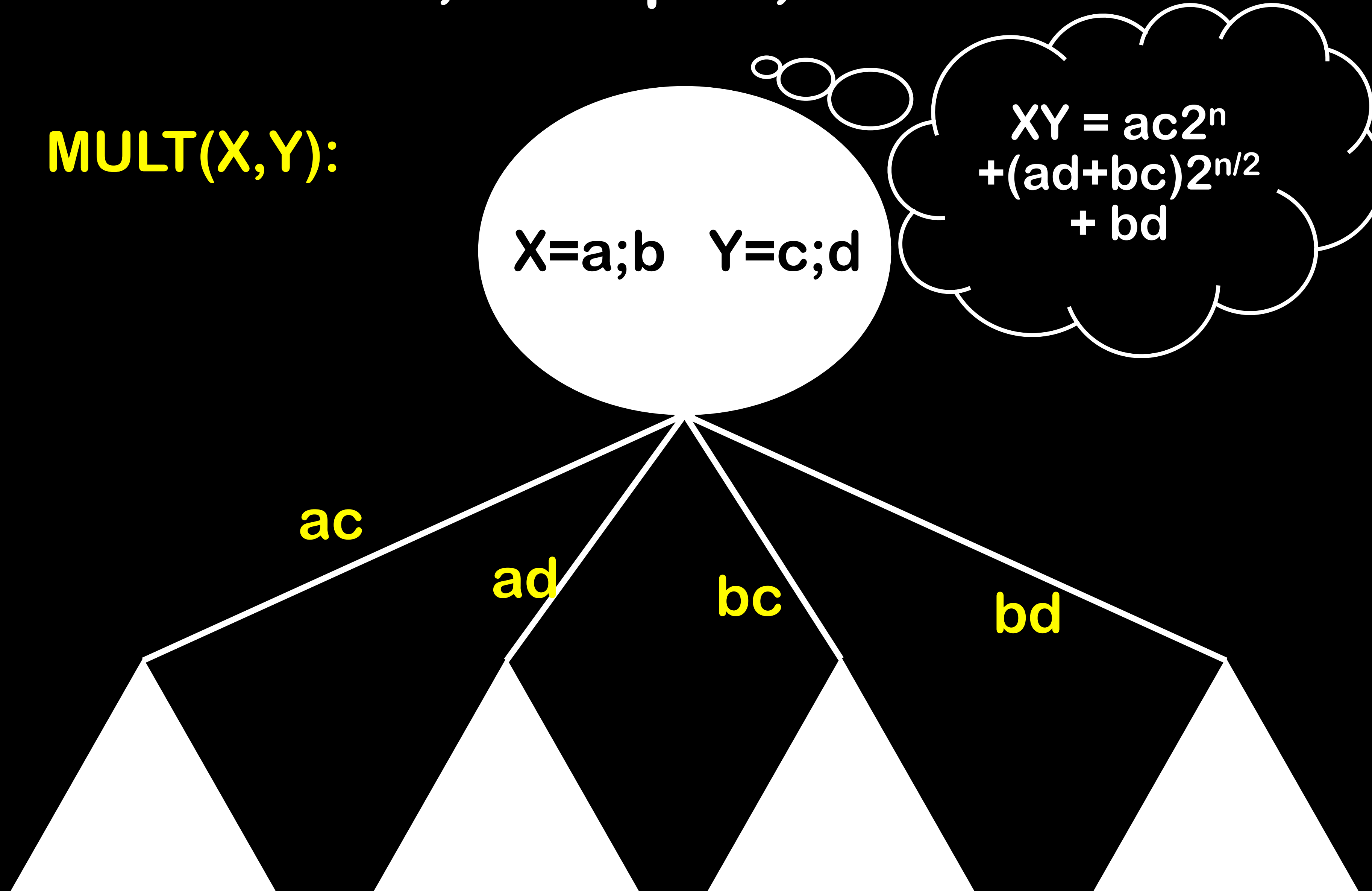

Time required by MULT

$T(n)$ = time taken by MULT on two n-bit numbers

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conquering time

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conquering time

divide and glue

Recurrence Relation

T(1) = k   for some constant k

T(n) = 4 T(n/2) + k'n + k'' for constants k' and k''

```
MULT(X,Y):
    If |X| = |Y| = 1 then return XY
    else  break X into a;b and Y into c;d
          return MULT(a,c) 2^n + (MULT(a,d)
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```

Recurrence Relation

$T(1) = 1$

$T(n) = 4\, T(n/2) + n$

```
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    If |X| = |Y| = 1 then return XY
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```

Technique: Labeled Tree Representation

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$$T(n) = n + 4\,T(n/2)$$

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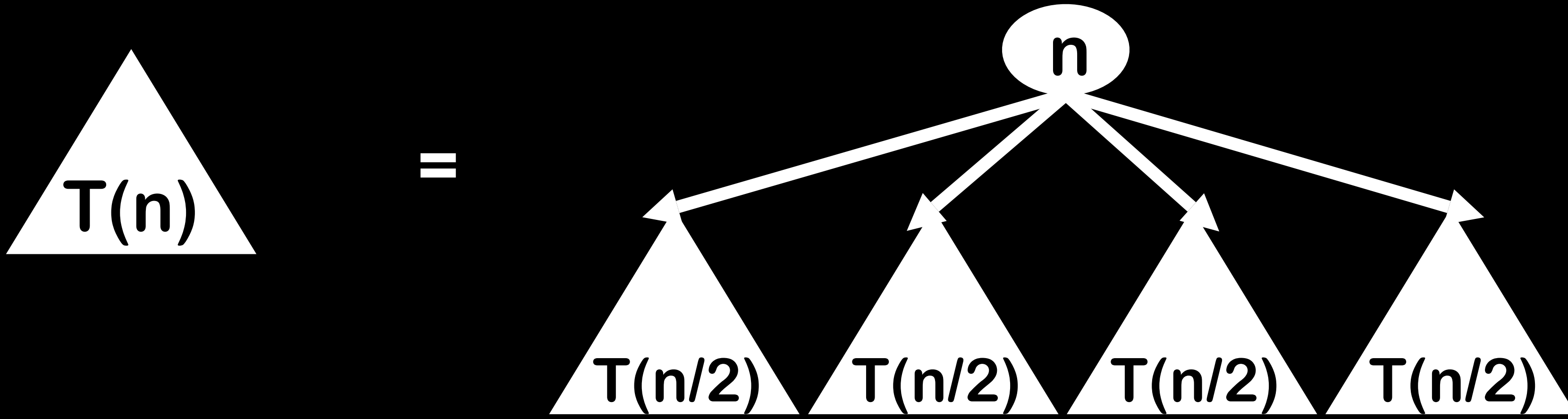
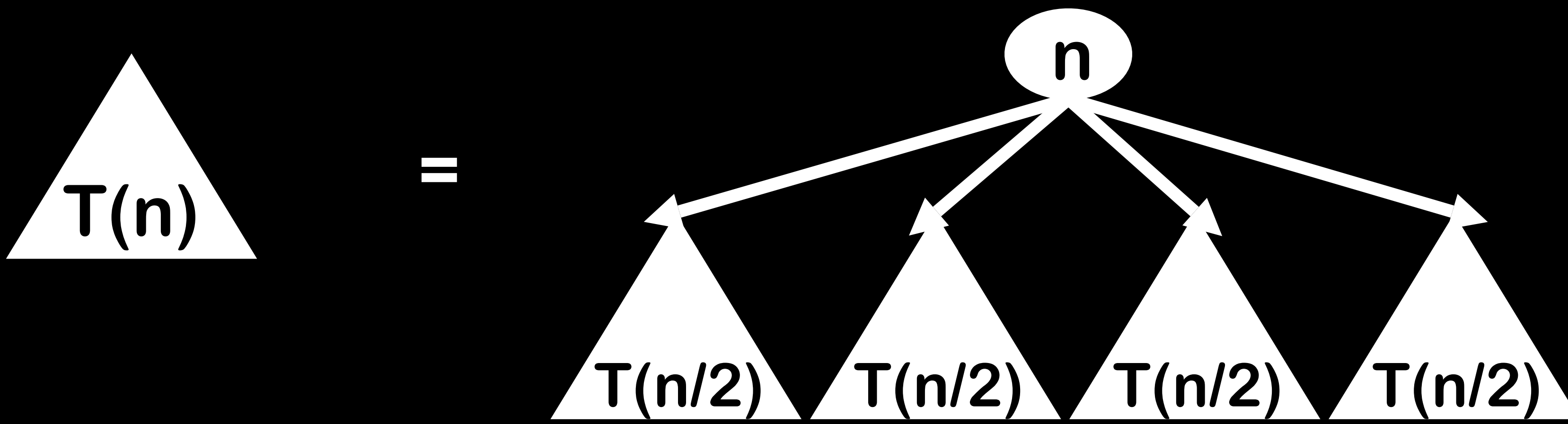


Technique: Labeled Tree Representation

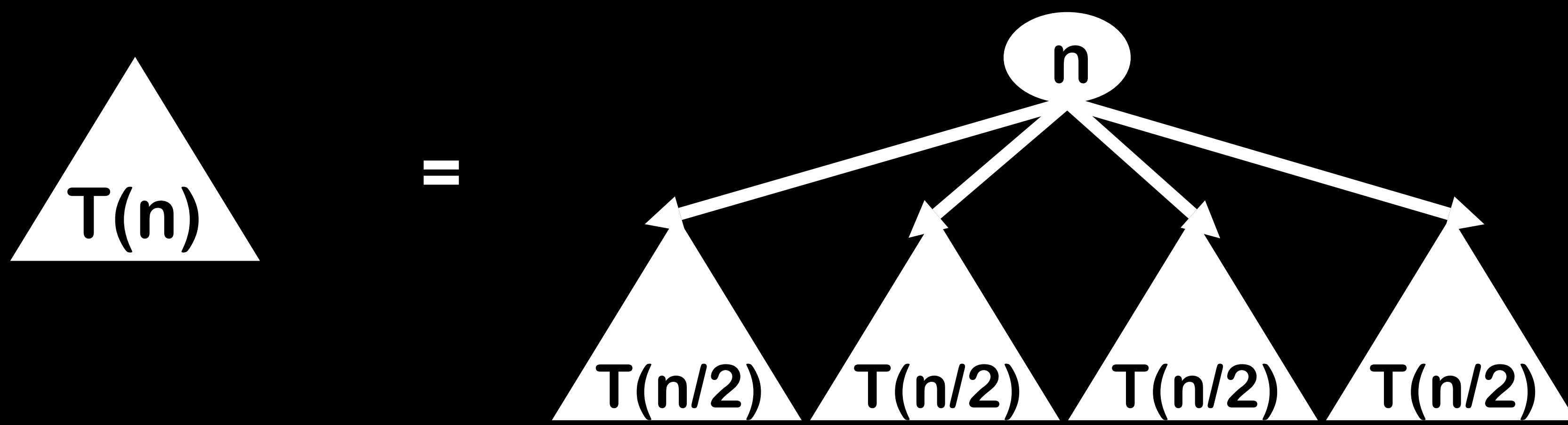
$$T(n) = n + 4\,T(n/2)$$



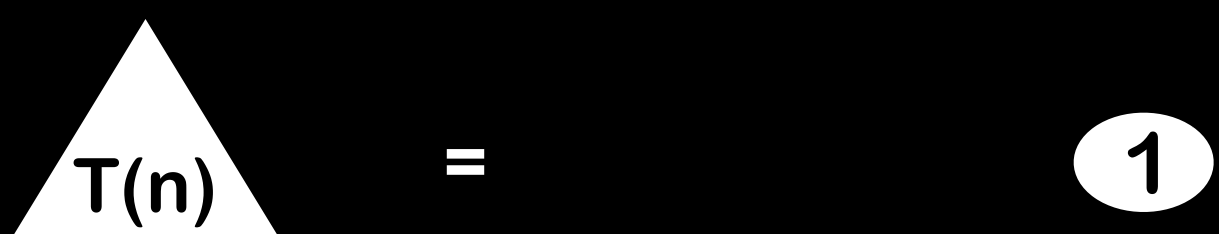
$$T(1) = 1$$

Technique: Labeled Tree Representation

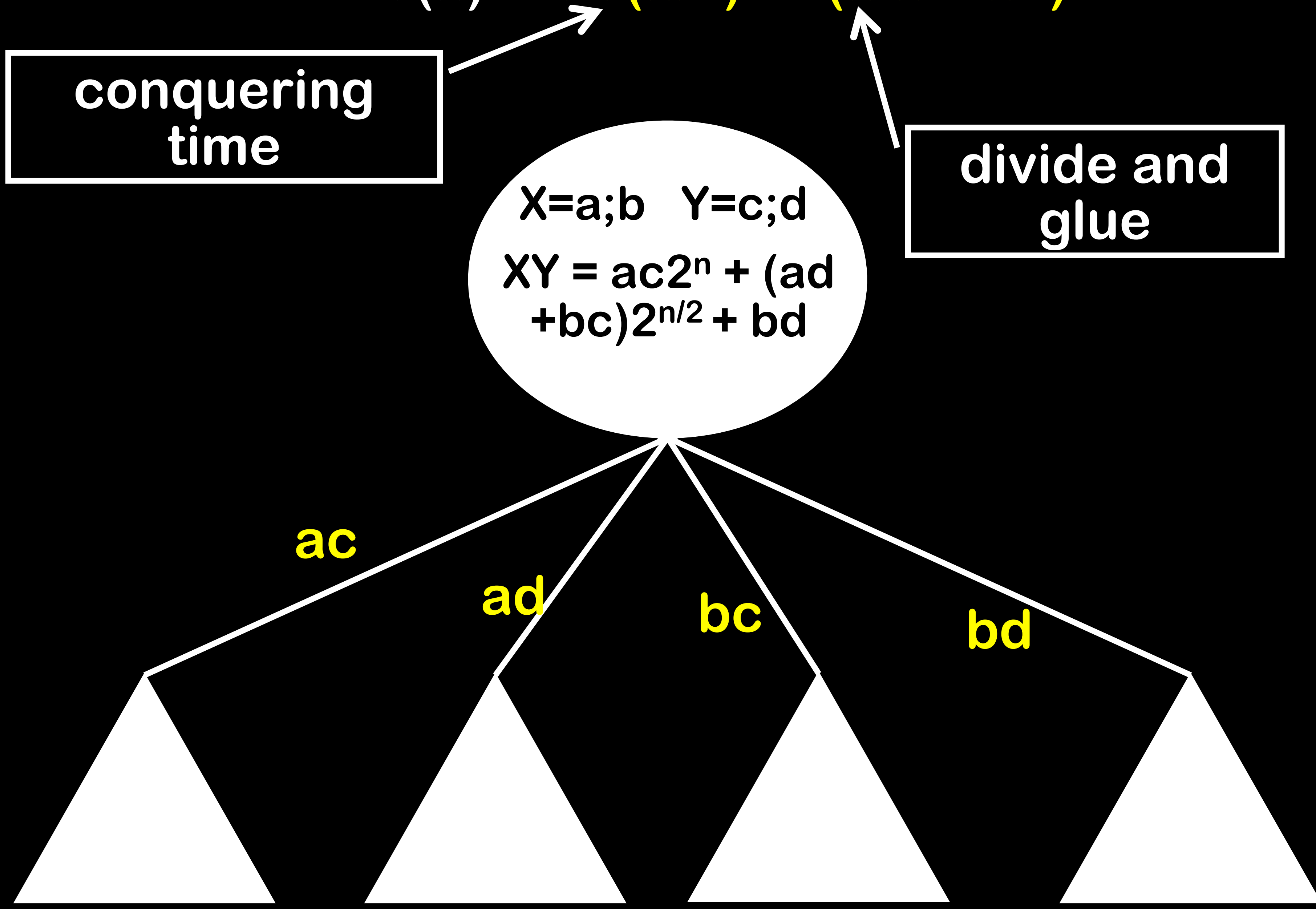
T(n)	=	n + 4 T(n/2)



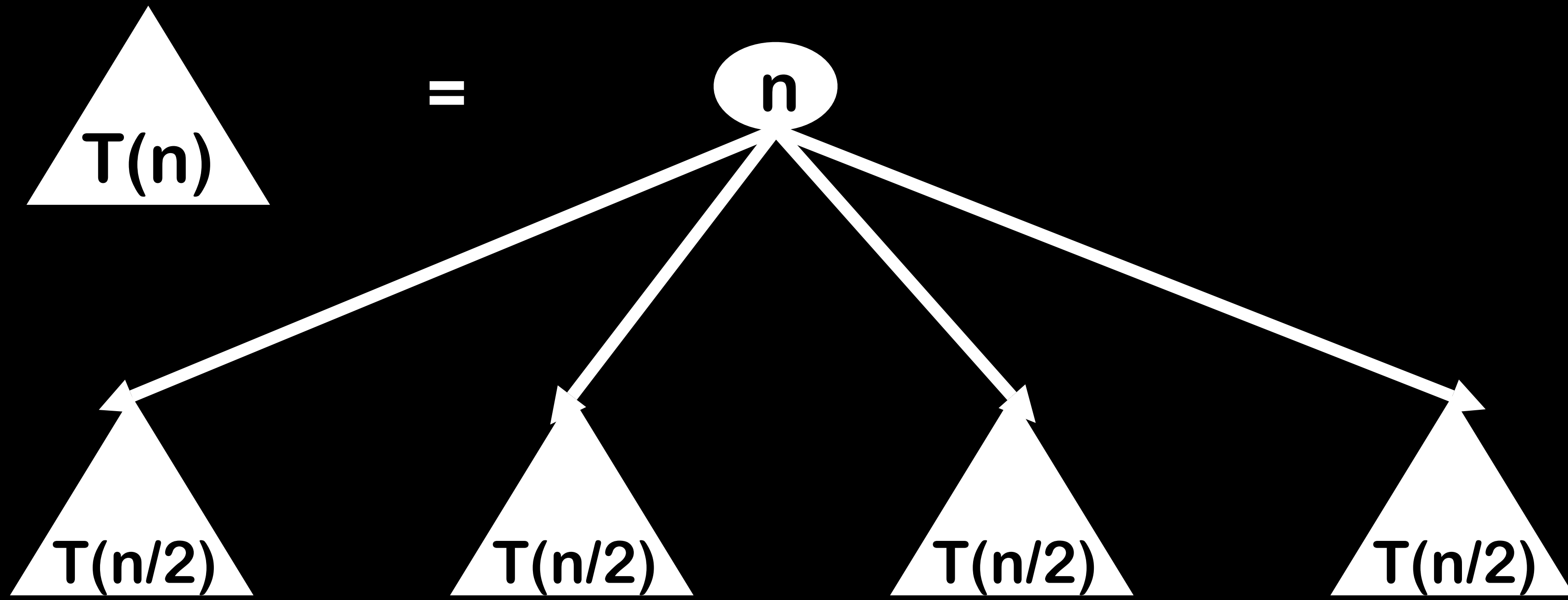
T(1)	=	1



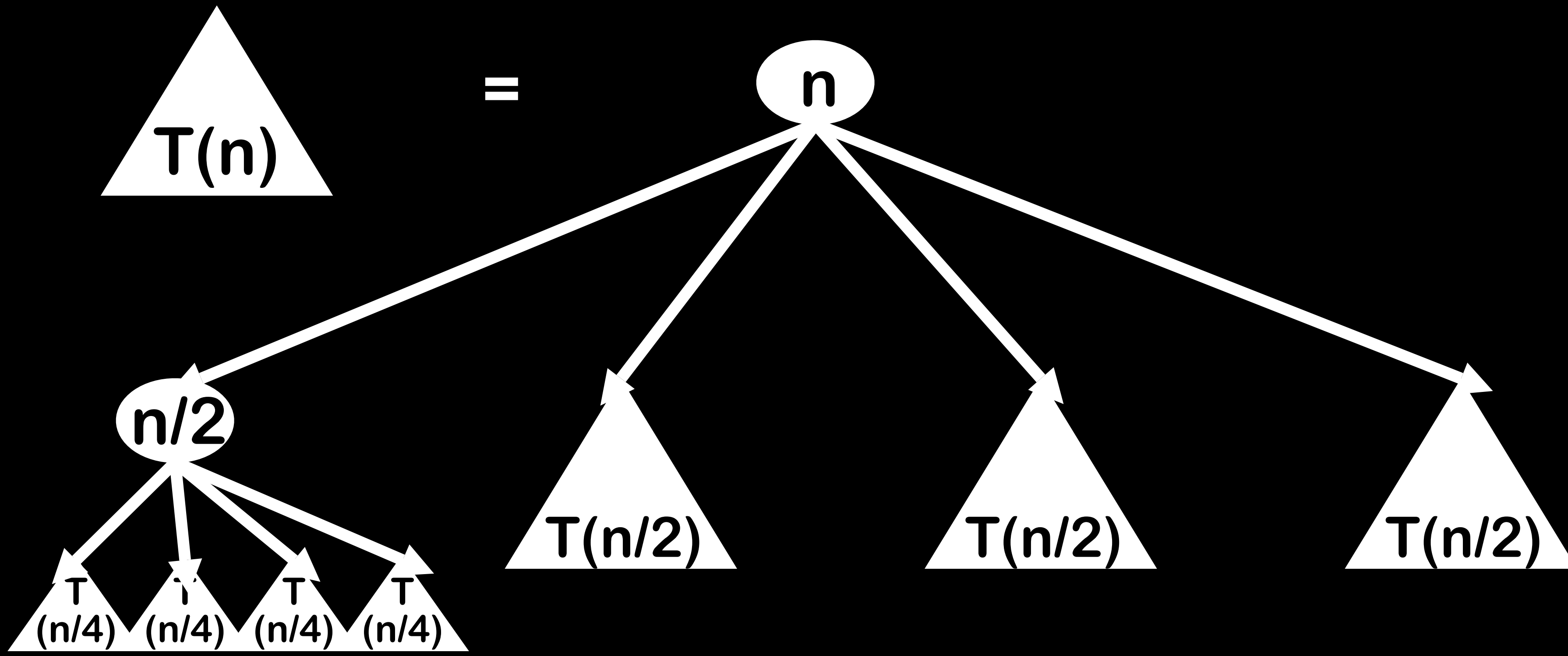
$$T(n) = 4\,T(n/2) + (k'n + k'')$$



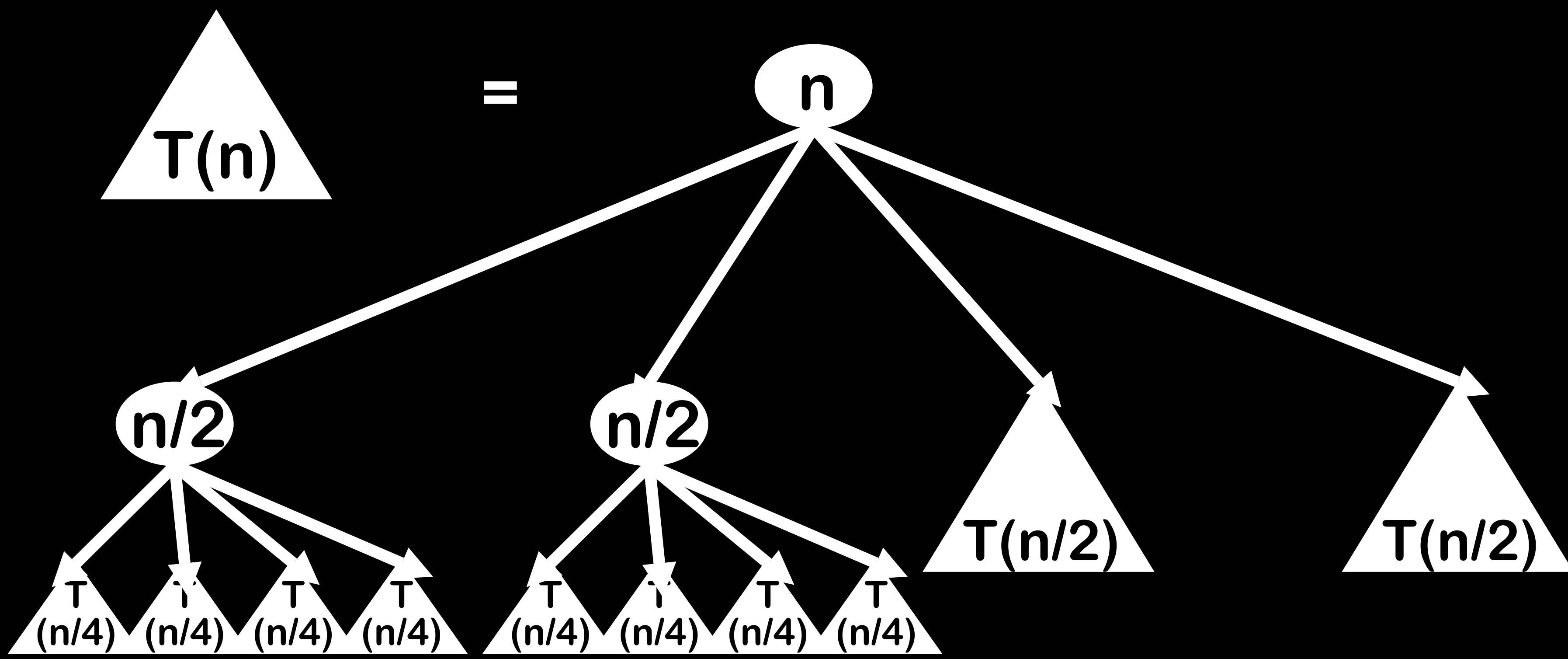
conquering time
divide and glue
X=a;b Y=c;d
$XY = ac2^n + (ad+bc)2^{n/2} + bd$
ac
ad
bc
bd



T(n)
=
n
T(n/2)
T(n/2)
T(n/2)
T(n/2)



T(n)
=
n
n/2
T(n/2)
T(n/2)
T(n/2)
T
(n/4)
T
(n/4)
T
(n/4)
T
(n/4)


T(n)
=
n
n/2
n/2
T(n/2)
T(n/2)
T (n/4)
T (n/4)
T (n/4)
T (n/4)
T (n/4)
T (n/4)
T (n/4)
T (n/4)

0	n
1	n/2 + n/2 + n/2 + n/2
2	
i	
	. . . .
	1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

0	n
1	n/2 + n/2 + n/2 + n/2
2	
i	
	. . . .
$\log_2(n)$	1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

0	n
1	n/2 + n/2 + n/2 + n/2
2	
i	Level i is the sum of 4^i copies of $n/2^i$
	. . . .
$\log_2(n)$	1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

n

n/2 + n/2 + n/2 + n/2

Level i is the sum of 4^i copies of $n/2^i$

. . . .

1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

1n =

n
n/2 + n/2 + n/2 + n/2
Level i is the sum of 4^i copies of $n/2^i$
. . . .
1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

1n =	n
2n =	n/2 + n/2 + n/2 + n/2
	Level i is the sum of 4^i copies of $n/2^i$
	. . . .
	1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

1n =	n
2n =	n/2 + n/2 + n/2 + n/2
4n =	
	Level i is the sum of 4^i copies of $n/2^i$
	. . . .
	1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

$1n =$	n
$2n =$	$n/2 \quad + \quad n/2 \quad + \quad n/2 \quad + \quad n/2$
$4n =$	
$2^i n =$	Level i is the sum of 4^i copies of $n/2^i$
	. . . .
	1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

$1n =$	n
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$4n =$	
$2^i n =$	Level i is the sum of 4^i copies of $n/2^i$
	. . . .
$(n)n =$	1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

1n =	n
2n =	n/2 + n/2 + n/2 + n/2
4n =	
2^in =	Level i is the sum of 4^i copies of $n/2^i$
	. . . .
(n)n =	1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

$$n(1+2+4+8+\ .\ .\ .\ +n) = n(2n-1) = 2n^2-n$$

Divide and Conquer MULT: $\Theta(n^2)$ time

Grade School Multiplication: $\Theta(n^2)$ time

MULT revisited

```
MULT(X,Y):
    If |X| = |Y| = 1 then return XY
    else  break X into a;b and Y into c;d
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MULT revisited

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    If $|X| = |Y| = 1$ then return XY
    else break X into a;b and Y into c;d
        return MULT(a,c) 2^n + (MULT(a,d) + MULT(b,c)) $2^{n/2}$ + MULT(b,d)

MULT calls itself 4 times. Can you see a way to reduce the number of calls?

Gauss' optimization

Input: a,b,c,d
Output: ac-bd, ad+bc

c	$X_1 = a + b$	
c	$X_2 = c + d$	
\$	$X_3 = X_1 X_2$	$= ac + ad + bc + bd$
\$	$X_4 = ac$	
\$	$X_5 = bd$	
c	$X_6 = X_4 - X_5$	$= ac - bd$
cc	$X_7 = X_3 - X_4 - X_5$	$= bc + ad$

Karatsuba, Anatolii Alexeevich (1937-)

Sometime in the late 1950's Karatsuba had formulated the first algorithm to break the n^2 barrier!

Gaussified MULT (Karatsuba 1962)

```
MULT(X,Y):
    If |X| = |Y| = 1 then return XY
    else  break X into a;b and Y into c;d
                e : = MULT(a,c)
                f  := MULT(b,d)
    return
    e 2^n + (MULT(a+b,c+d) – e - f) 2^(n/2) + f
```

$$e\,2^n + (\text{MULT}(a+b,c+d) - e - f)\,2^{n/2} + f$$

Gaussified MULT
(Karatsuba 1962)

MULT(X,Y):

If $|X| = |Y| = 1$ then return XY

else break X into a;b and Y into c;d

e : = MULT(a,c)

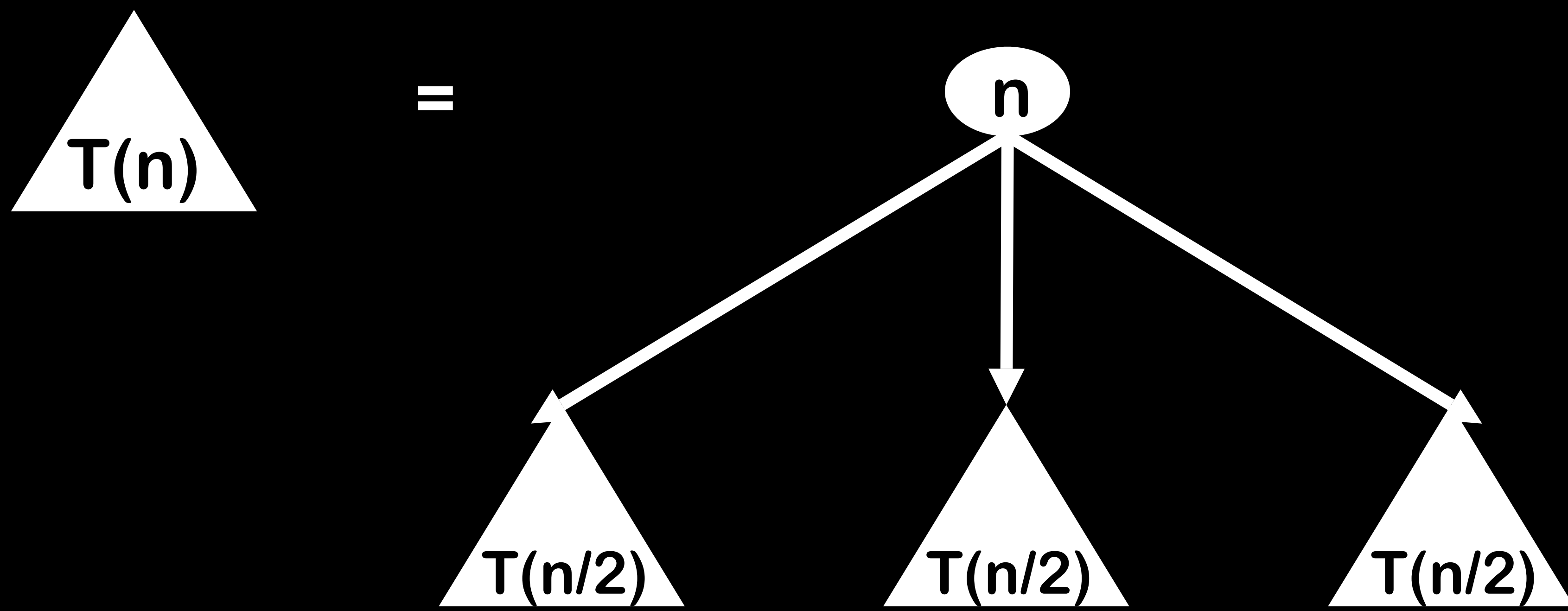
f := MULT(b,d)

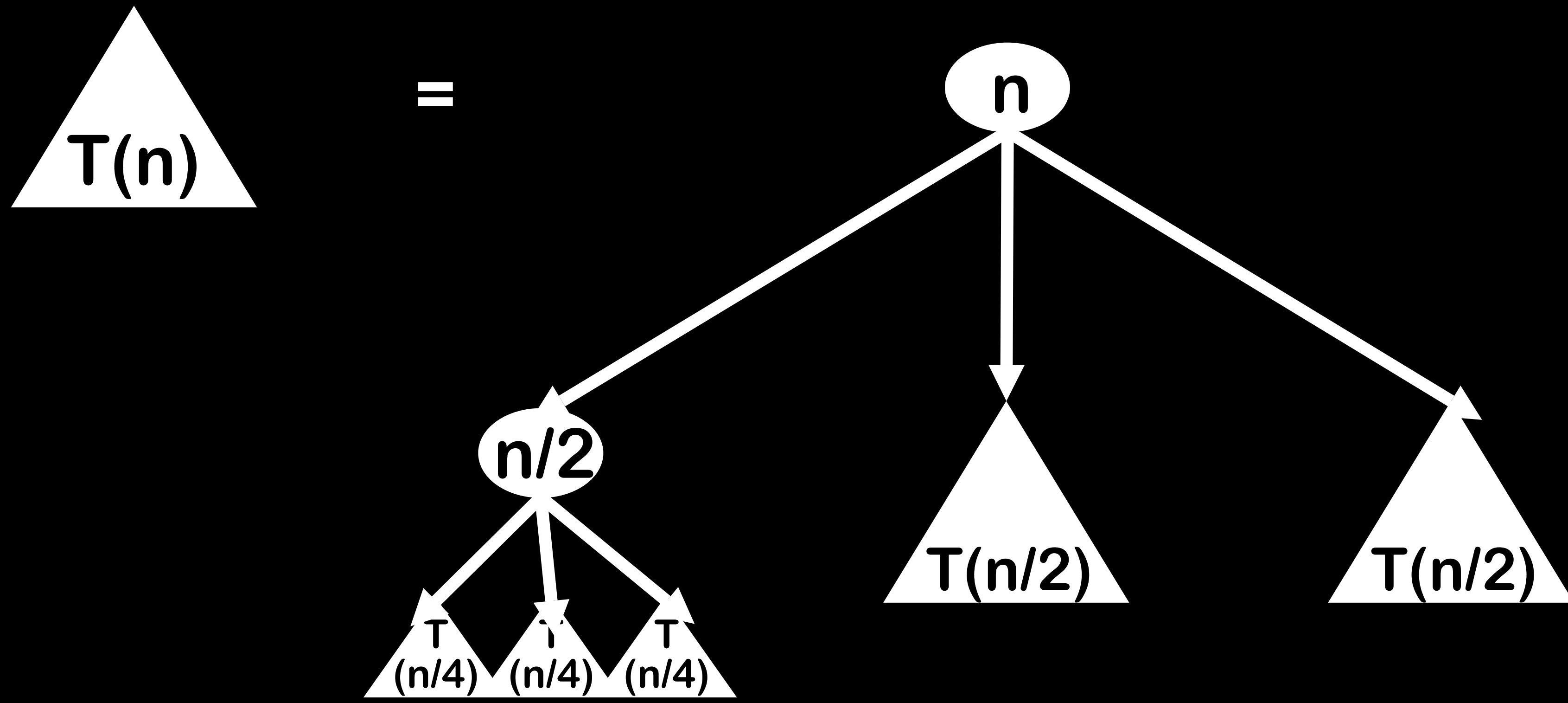
return

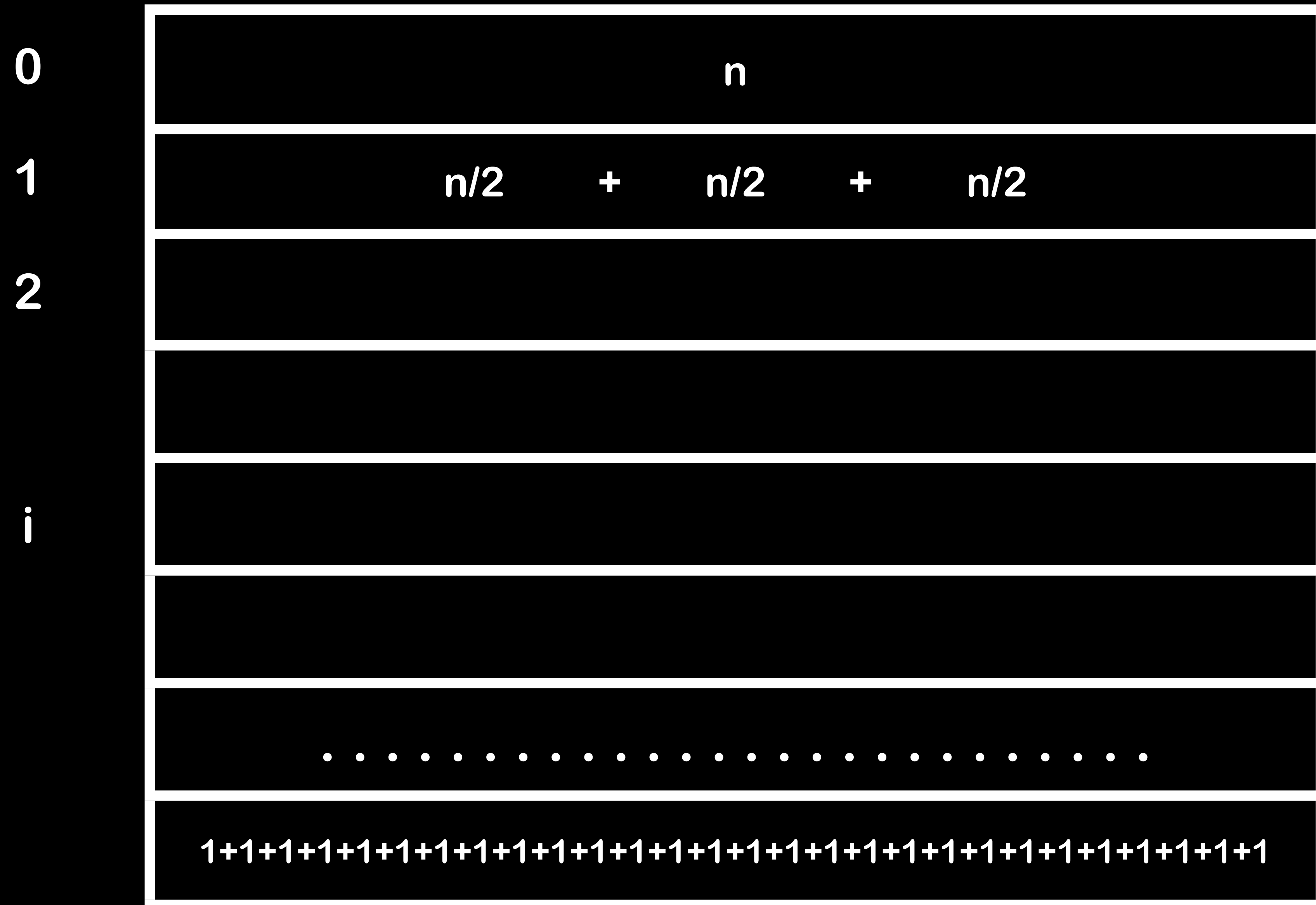
$e\ 2^n + (\text{MULT}(a+b,c+d) - e - f)\ 2^{n/2} + f$

$T(n) = 3\ T(n/2) + n$

Actually: $T(n) = 2\ T(n/2) + T(n/2 + 1) + kn$


T(n)
=
n
T(n/2)
T(n/2)
T(n/2)


T(n)
=
n
n/2
T(n/2)
T(n/2)
T
(n/4)
T
(n/4)
T
(n/4)


0
n
1
n/2 + n/2 + n/2
2
i
. . . .
1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

0	n
1	n/2 + n/2 + n/2
2	
i	
	. . . .
$\log_2(n)$	1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

0	n
1	n/2 + n/2 + n/2
2	
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	. . . .
$\log_2(n)$	1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

n

n/2 + n/2 + n/2

Level i is the sum of 3^i copies of $n/2^i$

. . . .

1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

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n
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Level i is the sum of 3^i copies of $n/2^i$
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1n =	n
3/2n =	n/2 + n/2 + n/2
	Level i is the sum of 3^i copies of $n/2^i$
	. . . .
	1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

1n =	n
3/2n =	n/2 + n/2 + n/2
9/4n =	
	Level i is the sum of 3^i copies of $n/2^i$
	. . . .
	1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

1n =	n
3/2n =	n/2 + n/2 + n/2
9/4n =	
$(3/2)^i n$ =	Level i is the sum of 3^i copies of $n/2^i$
	. . . .
	1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

$1n =$	n
$3/2n =$	n/2 + n/2 + n/2
$9/4n =$	
$(3/2)^i n =$	Level i is the sum of 3^i copies of $n/2^i$
	. . . .
$(3/2)^{\log n} n =$	1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

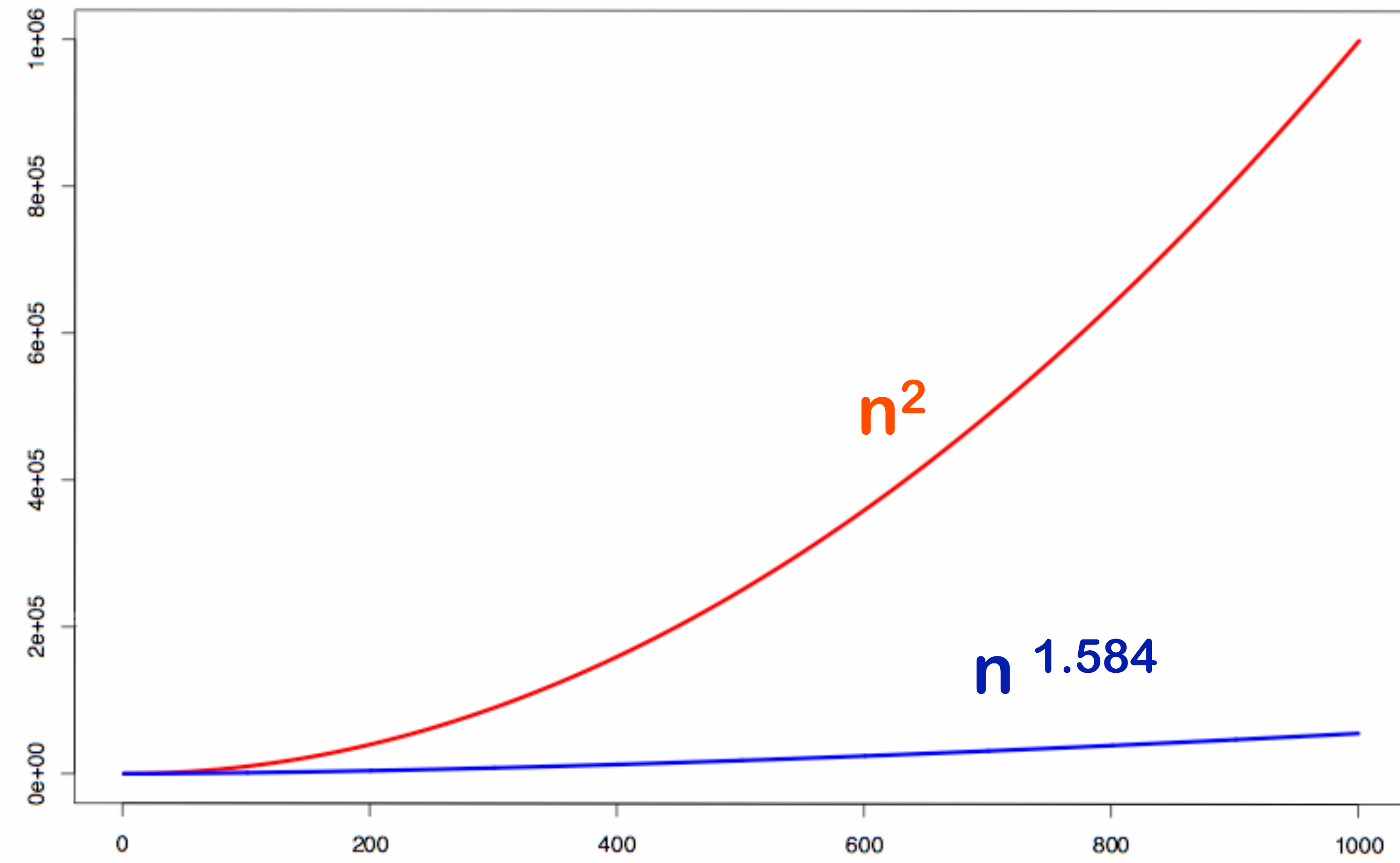
$1n =$	n
$3/2n =$	$n/2 \quad + \quad n/2 \quad + \quad n/2$
$9/4n =$	
$(3/2)^i n =$	Level i is the sum of 3^i copies of $n/2^i$
	. . . .
$(3/2)^{\log n} n =$	1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1

$$n(1+3/2+(3/2)^2+ \ldots + (3/2)^{\log_2 n}) \quad = \quad 3n^{1.58\ldots} - 2n$$

Dramatic Improvement for Large n

$$T(n) = 3n^{\log_2 3} - 2n$$
$$= \Theta(n^{\log_2 3})$$
$$= \Theta(n^{1.58\ldots})$$

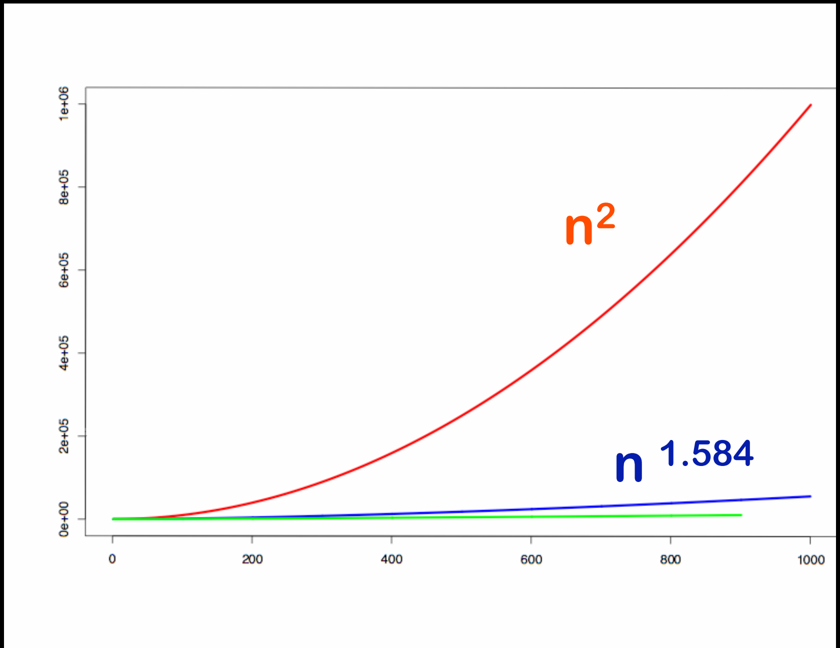
A huge savings over $\Theta(n^2)$ when n gets large.



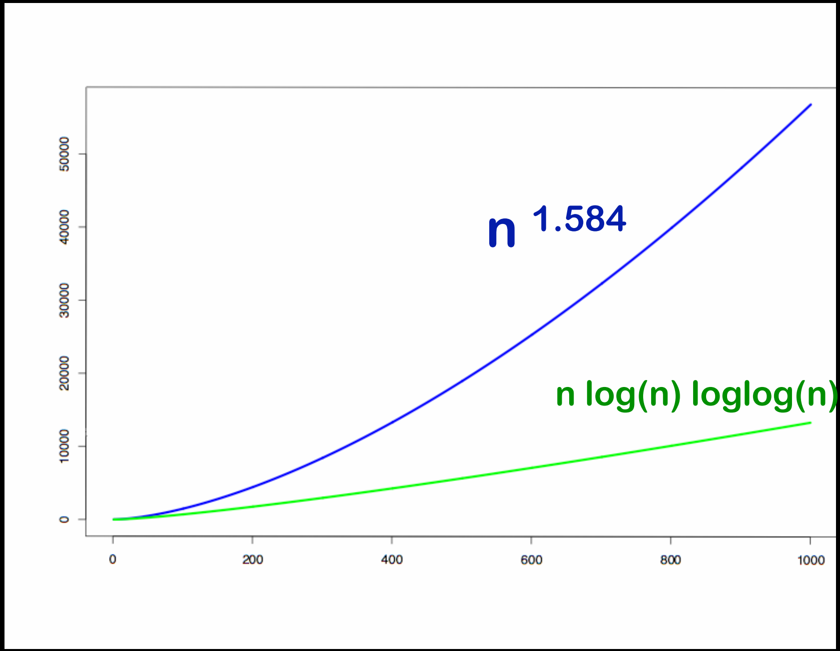
1e+06
8e+05
6e+05
4e+05
2e+05
0e+00
0
200
400
600
800
1000
n2
n 1.584

Multiplication Algorithms

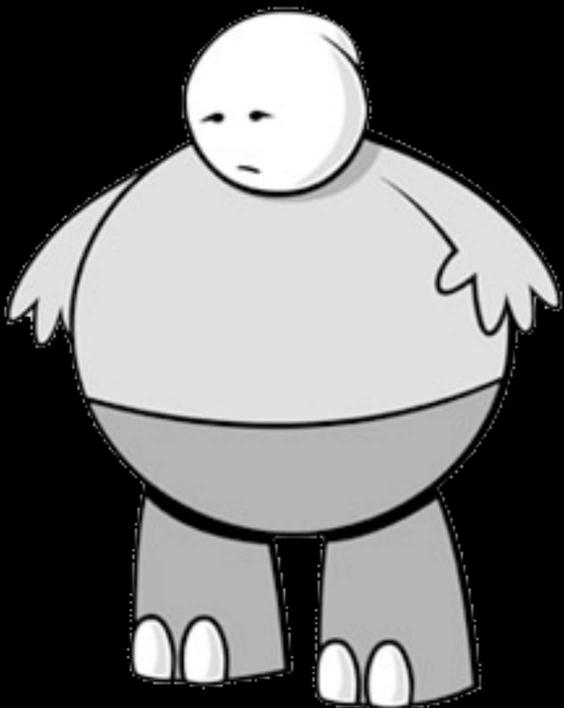
Kindergarten	$n2^n$
Grade School	n^2
Karatsuba	$n^{1.58...}$
Fastest Known	n logn loglogn



n2
n 1.584
0e+00
2e+05
4e+05
6e+05
8e+05
1e+06
0
200
400
600
800
1000



n 1.584
n log(n) loglog(n)
0
10000
20000
30000
40000
50000
0
200
400
600
800
1000



Here's What You Need to Know…

- Gauss's Multiplication Trick
- Proof of Lower bound for addition
- Divide and Conquer
- Solving Recurrences
- Karatsuba Multiplication