A Closed Form for Fibonaccis

15-251: Great Theoretical Ideas in Computer Science

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We've just derived the generating function for Fibonaccis:

$$F(z) = \sum_{n=0}^{\infty} F_n z^n = \frac{z}{1 - z - z^2}. \tag{1}$$

If we can expand this into a power series, then we'll be able to read off a closed form for F_n. To do so, we'll first apply the quadratic formula to find the roots of $z^2 + z - 1$; they're just

$$-\phi = \frac{-1 - \sqrt{5}}{2} \tag{2}$$

$$-\widehat{\phi} = \frac{-1 + \sqrt{5}}{2} = \frac{1}{\phi} \tag{3}$$

Thus, we have that

$$F(z) = \frac{-z}{z^2 + z - 1} \tag{4}$$

$$= \frac{-z}{(\phi + z)(\widehat{\phi} - z)} \tag{5}$$

$$= \frac{-z}{\phi\,\widehat{\phi}\,(1 + z/\phi)\left(1 + z/\widehat{\phi}\right)} \tag{6}$$

$$= \frac{z}{(1 - \phi z)\left(1 - \widehat{\phi} z\right)}. \tag{7}$$

Next, let's write this as

$$F(z) = \frac{a}{1 - \phi z} + \frac{b}{1 - \widehat{\phi} z} \tag{8}$$

where a and b are chosen so that

$$a(1 - \widehat{\phi} z) + b(1 - \phi z) = z. \tag{9}$$

Since this must hold for any z, it follows that $a = -b$ and

$$a = \frac{1}{\phi - \widehat{\phi}} = \frac{1}{\sqrt{5}}. \tag{10}$$

Finally, using the form of the geometric series, we see that

$$\begin{aligned} F(z) &= \frac{1}{\sqrt{5}(1-\phi z)} - \frac{1}{\sqrt{5}(1-\widehat{\phi} z)} && (11)\\ &= \frac{1}{\sqrt{5}} \sum_{n\geq 0} \phi^n z^n - \frac{1}{\sqrt{5}} \sum_{n\geq 0} \widehat{\phi}^n z^n && (12)\\ &= \sum_{n\geq 0} \frac{1}{\sqrt{5}} \left(\phi^n - \widehat{\phi}^n\right) z^n. && (13) \end{aligned}$$

We've derived the following closed form for the nth Fibonacci number:

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \widehat{\phi}^n\right) = \frac{1}{\sqrt{5}}\left(\phi^n - \left(\frac{-1}{\phi}\right)^n\right). \tag{14}$$

Now, since $|\widehat{\phi}| \approx 0.618034$, we can see that for large n,

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \widehat{\phi}^n\right) \approx \frac{\phi^n}{\sqrt{5}}. \tag{15}$$

In fact, it holds that

$$F_n = \text{round}\left(\frac{\phi^n}{\sqrt{5}}\right) \tag{16}$$

where $\text{round}(x)$ means the nearest integer to x. It's also easy to see from this closed form that

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \lim_{n\to\infty} \frac{\phi^n - \widehat{\phi}^n}{\phi^{n-1} - \widehat{\phi}^{n-1}} = \lim_{n\to\infty} \frac{\phi^n}{\phi^{n-1}} = \phi. \tag{17}$$