15-251

Some AWESOME

~~Great Theoretical~~ Ideas

~~in Computer Science~~

about Generating Functions

MORE
Generating Functions

Lecture 7 (February 2, 2010)
Adam Blank

$$\sum_{n=0}^{\infty} x^n$$

Announcements

You are now breathing manually

Homework 3 is due tonight

- Please submit a collaborators graph again.
- It doesn’t matter what you name it.
- You do not also have to list your collaborators in the pdf

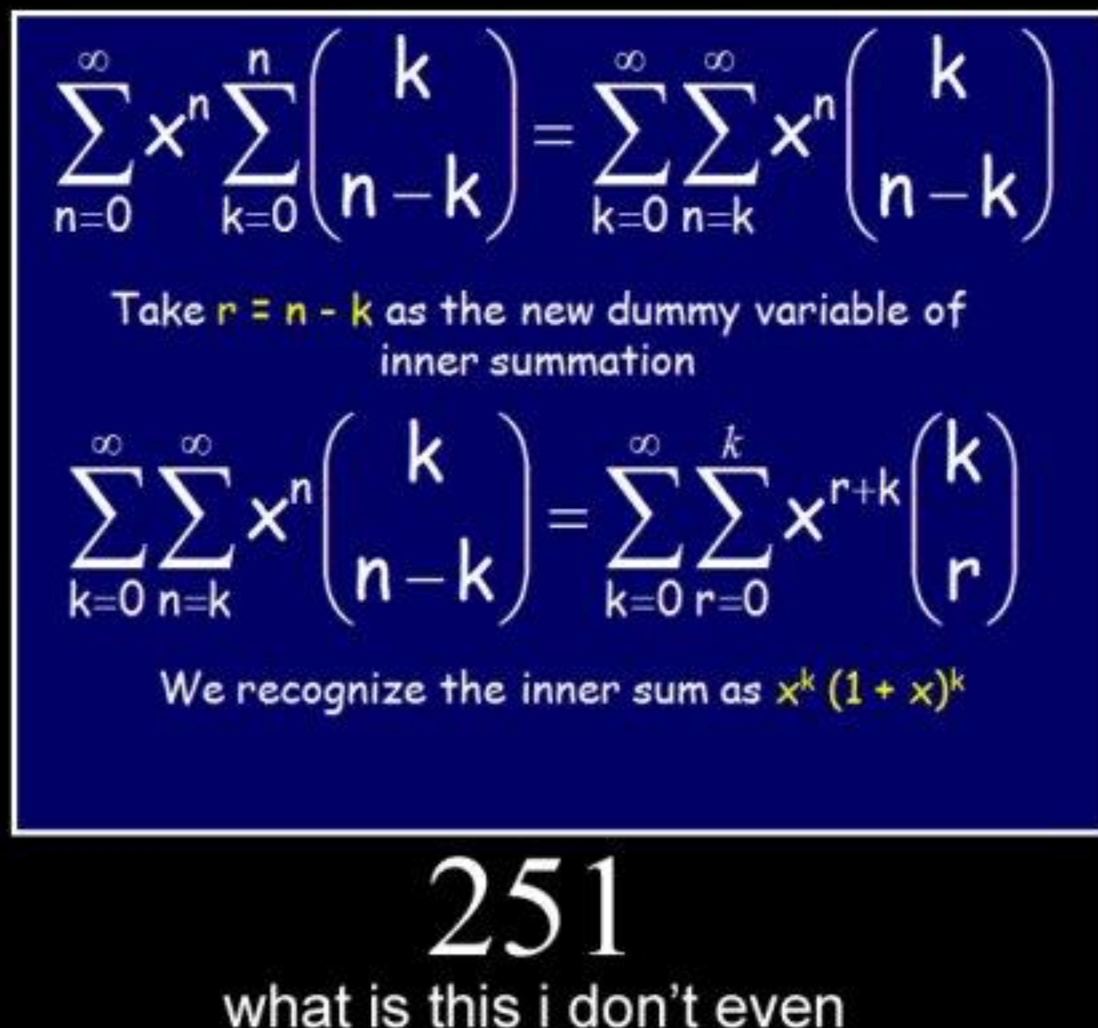
Homework 4 is out

- This assignment is non-trivial
- Start early!

Exam 1 will be in recitation next week

- Don’t be late!
- We will (tentatively) have a review session on Saturday
- We will email you with updates

What good are Generating Functions anyway?



What good are Generating Functions anyway?

They're fun!

Solving recurrences precisely

They are often easier than the alternative!

Some Terminology

$$[x^n] \sum_{n=0}^{\infty} (2x)^n = 2^n$$

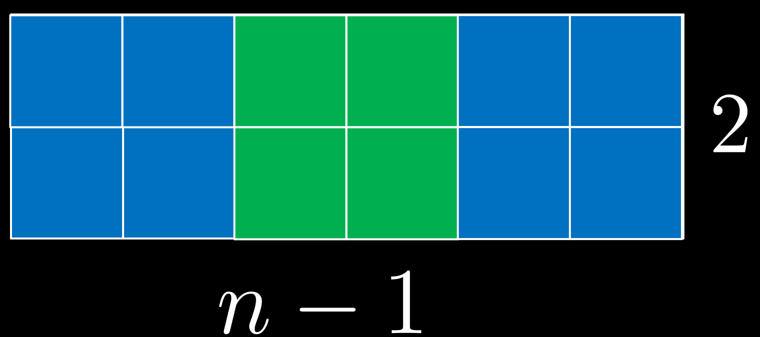
Closed form of a Generating Function

Closed form for a Recurrence

Let’s do some problems!

Domino Domination

We have a $2 \times (n-1)$ board, and we would like to fill it with dominos. We have two colors of dominos: green and blue. The green ones must be used in pairs (so that they don't get blue!), and they must be vertical. How many ways can we tile our board?

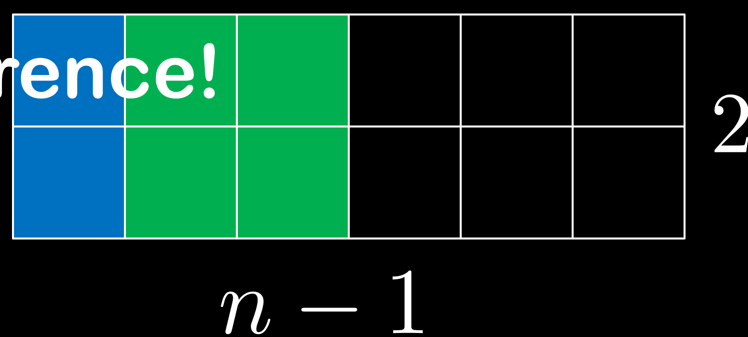


Domino Domination

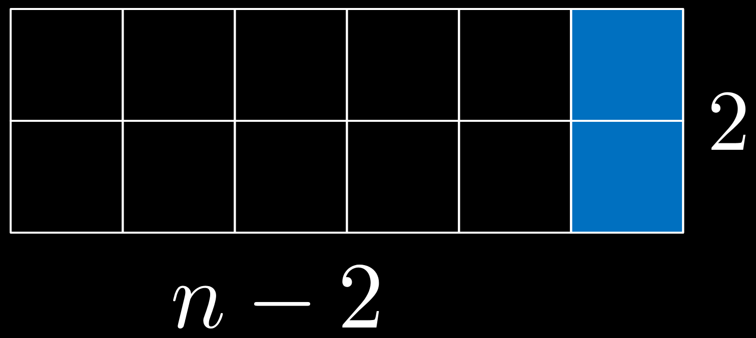
We have a $2 \times (n-1)$ board, and we would like to fill it with dominos. We have two colors of dominos: green and blue. The green ones must be used in pairs (so that they don't get blue!), and they must be vertical. How many ways can we tile our board?

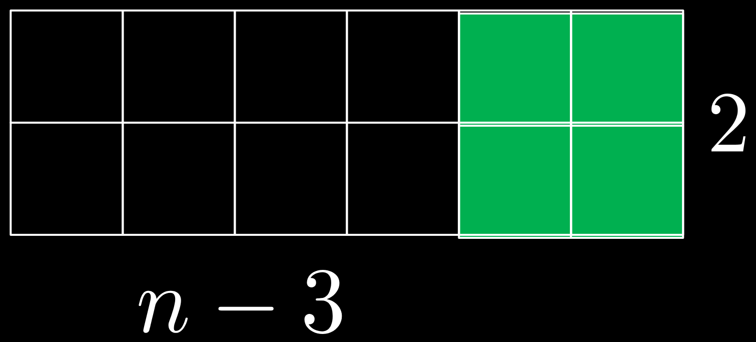
This is a combinatorial question! How should we proceed?!?!

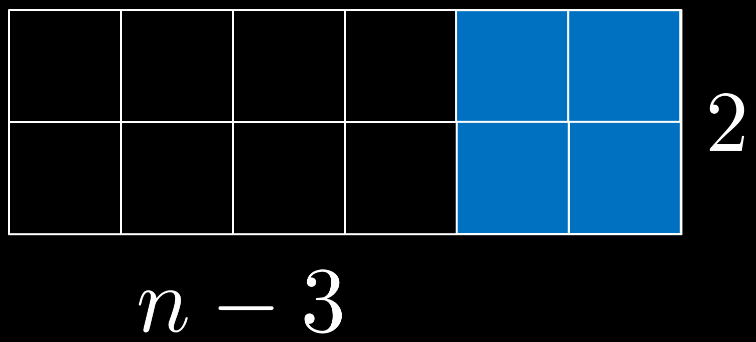
Write a recurrence!



Domino Domination







$$d_n = d_{n-1} + 2d_{n-2}$$

So now we have a recurrence…but now what?

Domino Domination

$$d_n = d_{n-1} + 2d_{n-2}$$

Now we derive a closed form using generating functions!

Let $D(x) = \sum_{n=0}^{\infty} d_n x^n = d_0 + d_1 x + \sum_{n=2}^{\infty} (d_{n-1} + 2d_{n-2}) x^n$

$$= x + \sum_{n=2}^{\infty} d_n x^n$$

We know the base cases: $d_1 = 1$ $d_0 = 0$

Note that these base cases are actually correct.

d_n **is the number of ways to tile a** $2 \times (n-1)$ **board.**

Domino Domination

Now we derive a closed form using generating functions!

Let $D(x) = \sum_{n=0}^{\infty} d_n x^n$

$$= x + \sum_{n=2}^{\infty} (d_{n-1} + 2d_{n-2}) x^n$$

$$= x + \sum_{n=2}^{\infty} d_{n-1} x^n + \sum_{n=2}^{\infty} 2d_{n-2} x^n$$

$$= x + x \sum_{n=2}^{\infty} d_{n-1} x^{n-1} + 2x^2 \sum_{n=2}^{\infty} d_{n-2} x^{n-2}$$

$$= x + x \sum_{n=1}^{\infty} d_n x^n + 2x^2 \sum_{n=0}^{\infty} d_n x^n$$

$$= x + x(D(x) - d_0) + 2x^2 D(x)$$

Domino Domination

Now we derive a closed form using generating functions!

Let $D(x) = \sum_{n=0}^{\infty} d_n x^n \quad = x + x(D(x) - d_0) + 2x^2 D(x)$

$$D(x) = x + x(D(x) - d_0) + 2x^2 D(x)$$

$$(1 - x - 2x^2)D(x) = x$$

$$D(x) = \frac{x}{1-x-2x^2}$$

Domino Domination

Now we have a closed form for the generating function! ...what now?

Let $D(x) = \sum_{n=0}^{\infty} d_n x^n = \frac{x}{1 - x - 2x^2} = \frac{-1}{3(1+x)} + \frac{1}{3(1-2x)}$

$$\frac{x}{1-x-2x^2} = \frac{x}{(1+x)(1-2x)} = \frac{A}{1+x} + \frac{B}{1-2x}$$

$$A = \frac{-1}{3}$$

$$x = (1-2x)A + (1+x)B$$

$$B = \frac{1}{3}$$

$$1 = -2A + B$$

$$0 = A + B$$

Domino Domination

Now we have a closed form for the generating function! ...what now?

Let $D(x) = \sum_{n=0}^{\infty} d_n x^n = \dfrac{x}{1 - x - 2x^2}$

$$= \frac{-1}{3(1+x)} + \frac{1}{3(1-2x)}$$

$$= \frac{-1}{3} \sum_{n=0}^{\infty} (-x)^n + \frac{1}{3} \sum_{n=0}^{\infty} (2x)^n$$

$$\frac{1}{1 - (-x)} = \sum_{n=0}^{\infty} (-x)^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1 - (2x)} = \sum_{n=0}^{\infty} (2x)^n$$

Domino Domination

Now we have a closed form for the generating function! ...what now?

Let $D(x) = \sum_{n=0}^{\infty} d_n x^n = \frac{x}{1-x-2x^2} = \frac{-1}{3}\sum_{n=0}^{\infty}(-x)^n + \frac{1}{3}\sum_{n=0}^{\infty}(2x)^n$

$$= \frac{1}{3}\sum_{n=0}^{\infty}(-1)^{n+1}x^n + \frac{1}{3}\sum_{n=0}^{\infty}2^n x^n$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\left((-1)^{n+1}x^n + 2^n x^n\right)$$

$$= \sum_{n=0}^{\infty}\frac{1}{3}\left((-1)^{n+1} + 2^n\right)x^n$$

$$d_n = \frac{1}{3}\left((-1)^{n+1} + 2^n\right)$$

Rogue Recurrence

$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$$ **for n>2**

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 4$$

Solve this recurrence…or else!

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \sum_{n=3}^{\infty} a_n x^n$

$$= x + 4x^2 + \sum_{n=3}^{\infty} (5a_{n-1} - 8a_{n-2} + 4a_{n-3}) x^n$$

Rogue Recurrence

$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$$ **for n>2**

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 4$$

Solve this recurrence…or else!

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$

$$= x + 4x^2 + \sum_{n=3}^{\infty} (5a_{n-1} - 8a_{n-2} + 4a_{n-3})x^n$$

$$= x + 4x^2 + \sum_{n=3}^{\infty} 5a_{n-1}x^n - \sum_{n=3}^{\infty} 8a_{n-2}x^n + \sum_{n=3}^{\infty} 4a_{n-3}x^n$$

$$= x + 4x^2 + 5x\sum_{n=3}^{\infty} a_{n-1}x^{n-1} - 8x^2\sum_{n=3}^{\infty} a_{n-2}x^{n-2} + 4x^3\sum_{n=3}^{\infty} a_{n-3}x^{n-3}$$

$$= x + 4x^2 + 5x\sum_{n=2}^{\infty} a_n x^n - 8x^2\sum_{n=1}^{\infty} a_n x^n + 4x^3\sum_{n=0}^{\infty} a_n x^n$$

$$= x + 4x^2 + 5x(A(x) - a_0 - a_1x^1) - 8x^2(A(x) - a_0) + 4x^3A(x)$$

Rogue Recurrence

$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$ **for n>2**

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 4$$

Solve this recurrence…or else!

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ $= x + 4x^2 + 5x(A(x) - a_0 - a_1x^1) - 8x^2(A(x) - a_0) + 4x^3A(x)$

$$A(x) = x + 4x^2 + 5x(A(x) - x^1) - 8x^2(A(x)) + 4x^3A(x)$$

$$(1 - 5x + 8x^2 - 4x^3)A(x) = x + 4x^2 - 5x^2$$

$$A(x) = \frac{x - x^2}{1 - 5x + 8x^2 - 4x^3}$$

Rogue Recurrence

$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$$ **for n>2**

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 4$$

Solve this recurrence…or else!

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$

$$= \frac{x - x^2}{1 - 5x + 8x^2 - 4x^3}$$

$$= \frac{x(1-x)}{(1-2x)^2(1-x)}$$

$$= \frac{x}{(1-2x)^2}$$

What next? Partial fractions?

Rogue Recurrence

No. Let's be sneaky instead!

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n$$

$$\frac{d}{dx}\left(\frac{1}{1-2x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} (2x)^n\right)$$

$$\frac{2}{(1-2x)^2} = \sum_{n=0}^{\infty} n2^n x^{n-1}$$

$$\frac{x}{(1-2x)^2} = \sum_{n=0}^{\infty} n2^{n-1} x^n$$

$$\frac{x}{2}\frac{2}{(1-2x)^2} = \frac{x}{2}\sum_{n=0}^{\infty} n2^n x^{n-1}$$

Rogue Recurrence

$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$$ **for n>2**

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = 4$$

Now back to the recurrence…

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n = \dfrac{x}{(1-2x)^2}$

$$\frac{x}{(1-2x)^2} = \sum_{n=0}^{\infty} n2^{n-1}x^n$$

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n2^{n-1}x^n$$

$$a_n = n2^{n-1}$$

Double Sums OMGWTFBBQ!

Let's revisit the last lecture…

$$\sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \binom{k}{n-k}$$

We would like to swap the summations.

$$\sum_{k=0}^{\infty} \sum_{n=k}^{\infty} x^n \binom{k}{n-k}$$

All we have done here is re-group the addition.

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} x^{n+k} \binom{k}{n}$$

Double Sums OMGWTFBBQ!

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} x^{n+k}\binom{k}{n}$$

$$\sum_{k=0}^{\infty} x^k \sum_{n=0}^{\infty} x^n \binom{k}{n}$$

$$\sum_{k=0}^{\infty} x^k \left(\sum_{n=0}^{k} x^n \binom{k}{n} + \sum_{n=k+1}^{\infty} x^n \binom{k}{n}\right)$$

$$\sum_{k=0}^{\infty} x^k \sum_{n=0}^{k} x^n \binom{k}{n}$$

We know that…

$$\sum_{i=0}^{p} \binom{p}{i} x^i = (1+x)^p$$

$$\sum_{k=0}^{\infty} x^k (1+x)^k = \frac{1}{1-x(1+x)} = \frac{1}{1-x-x^2}$$

Double Sums OMGWTFBBQ!

$$\sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \binom{k}{n-k} = \sum_{k=0}^{\infty} x^k (1+x)^k = \frac{1}{1-x-x^2}$$

We know that... $$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2}$$

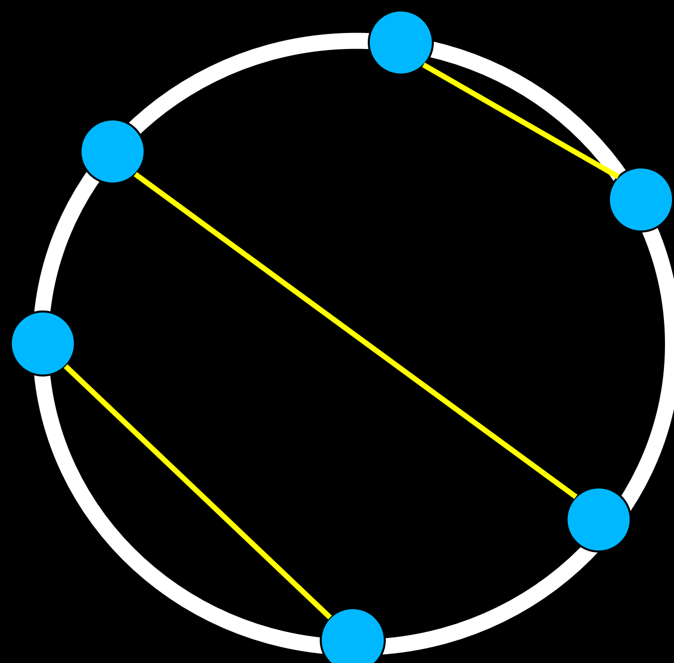
So... $$\sum_{n=0}^{\infty} F_{n+1} x^n = \frac{1}{1-x-x^2}$$

$$\sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \binom{k}{n-k} = \frac{1}{1-x-x^2} = \sum_{n=0}^{\infty} F_{n+1} x^n$$

$$\sum_{k=0}^{n} \binom{k}{n-k} = F_{k+1}$$

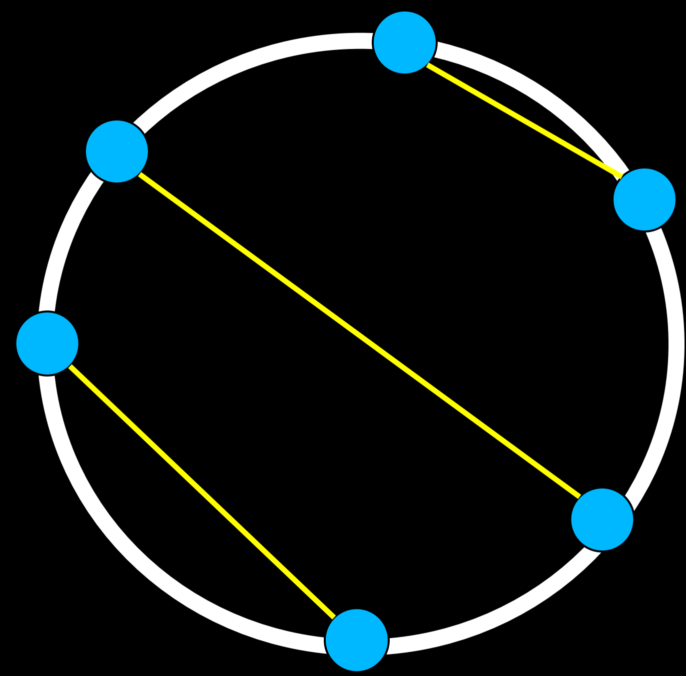
Dainty Diners

2n nobles want to sit down to dinner at a table and shake hands. They are picky and will not shake hands in any way, in which one or more noble reaches over another. How many ways can the nobles shake hands?



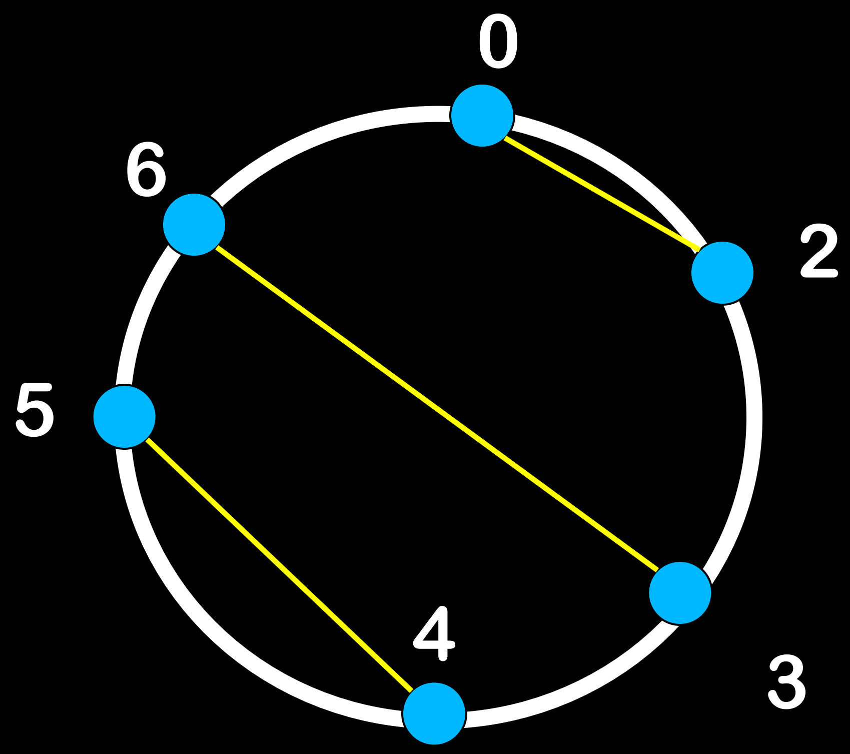
Dainty Diners

Look at an arbitrary person, and think about who he could shake hands with.



Dainty Diners

Noble 0 can shake hands with 2, 4, or 6.
He can shake hands with even numbered people! When he shakes hands with someone, it splits the remaining people into two groups.



Dainty Diners

First, we choose which person the 0th person shakes hands with, then we choose the two sub-groups created by this choice

$$p_n = \sum_{k=0}^{n-1} p_k p_{n-k-1}$$

# of ways 2n nobles can shake hands

Choose which person the 0th person shakes hands with

# of ways 2k nobles can shake hands

# of ways 2(n-k-1) nobles can shake hands

Dainty Diners

n=1: $p_0 = 1$

n>1: $p_n = \sum_{k=0}^{n-1} p_k p_{n-k-1}$

Let $P(x) = \sum_{n=0}^{\infty} p_n x^n = 1 + \sum_{n=1}^{\infty} p_n x^n$

$$= 1 + \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} p_k p_{n-k-1} \right) x^n$$

$$= 1 + x \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} p_k p_{n-k-1} \right) x^{n-1}$$

$$= 1 + x \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} p_k p_{n-k} \right) x^n$$

Dainty Diners

$$P(x) = 1 + x \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} p_k p_{n-k} \right) x^n$$

What now? Our usual trick of putting the terms in terms of P(x) doesn't seem to be possible!

Dainty Diners

$$P(x) = 1 + x \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} p_k p_{n-k} \right) x^n$$

Idea! Isolate the double summation and list out some terms.

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} p_k p_{n-k} \right) x^n = (p_0 p_0) + (p_0 p_1 + p_1 p_0) + p_2 p_0 + p_1 p_1 + p_2 p_0 + \ldots$$

Is there a pattern??

Dainty Diners

Let C(x) be an arbitrary generating function!

$$C(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$C^2(x) = \left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=0}^{\infty} c_n x^n\right)$$

$$= (c_0 + c_1 x^1 + c_2 x^2 + \dots)(c_0 + c_1 x^1 + c_2 x^2 + \dots)$$

Suppose we take k terms from the left. We need n-k from the second to make n total powers of k.

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} c_k c_{n-k}\right) x^n$$

Dainty Diners

$$P(x) = 1 + x\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} p_k p_{n-k}\right)x^n$$

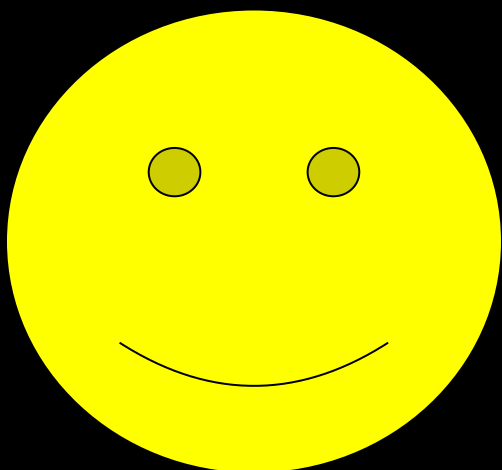
$$P^2(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} p_k p_{n-k}\right)x^n$$

$$P(x) = 1 + xP^2(x)$$

$$P(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

After some “magic” (see Newton’s Binomial Theorem):

$$p_n = \frac{1}{n+1}\binom{2n}{n}$$



Here's What You Need to Know...

Generating Functions

- How to solve recurrences
- Summation techniques
- Simple partial fractions
- Simple differentiation

(mostly remember how to use generating functions to solve recurrences)