Great Theoretical Ideas In Computer Science
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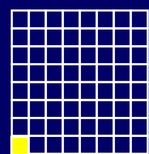
Mathematical Games

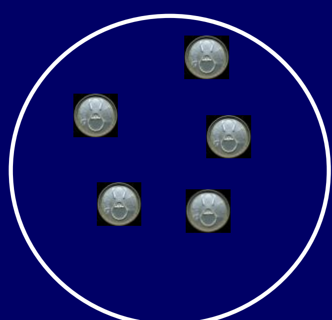


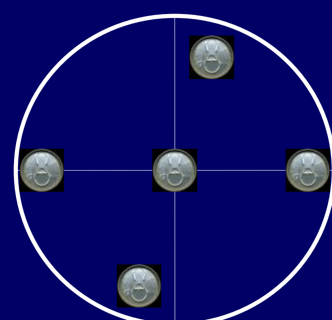
Plan

Introduction to
Impartial Combinatorial Games

Related courses

15-859, (21-801) - Mathematical Games
Look for it in Spring '11





A Take-Away Game

21 chips

Two Players: 1 and 2

A move consists of removing one, two, or three chips from the pile

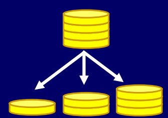
Players alternate moves, with Player 1 starting

Player that removes the last chip wins

Which player would you rather be?

Try Small Examples!

If there are 1, 2, or 3 only, player who moves next wins



If there are 4 chips left, player who moves next must leave 1, 2 or 3 chips, and his opponent will win

With 5, 6 or 7 chips left, the player who moves next can win by leaving 4 chips

21 chips

0, 4, 8, 12, 16, … are target positions; if a player moves to that position, they can win the game

Therefore, with 21 chips, Player 1 can win!

What if the last player to move loses?

If there is 1 chip, the player who moves next loses

If there are 2,3, or 4 chips left, the player who moves next can win by leaving only 1

In this case, 1, 5, 9, 13, … are a win for the second player

Combinatorial Games

- A set of positions
- Two players
- Rules specify for each player and for each position which moves to other positions are legal moves
- The players alternate moving
- A terminal position in one in which there are no moves
- The game ends when a player has no moves
- The game must end in a finite number of moves
- (No draws!)

Normal Versus Misère

Normal Play Rule: The last player to move wins

Misère Play Rule: The last player to move loses

A Terminal Position is one where neither player can move anymore

What is Omitted

No random moves

(This rules out games like poker)

No hidden state

(This rules out games like battleship)

No draws in a finite number of moves

(This rules out tic-tac-toe)

Impartial Versus Partizan

A combinatorial game is impartial if the same set of moves is available to both players in any position.

A combinatorial game is partizan if the move sets may differ for the two players

In this lecture we'll study impartial games. Partizan games will not be discussed

P-Positions and N-Positions

P-Position: Positions that are winning for the Previous player (the player who just moved) (Sometimes called LOSING positions)

N-Position: Positions that are winning for the Next player (the player who is about to move) (Sometimes called WINNING positions)

21 chips

0, 4, 8, 12, 16, … are P-positions; if a player moves to that position, they can win the game

21 chips is an N-position

What's a P-Position?

"Positions that are winning for the Previous player (the player who just moved)"

That means:

For any move that N makes

There exists a move for P such that

For any move that N makes

There exists a move for P such that

⋮

There exists a move for P such that

There are no possible moves for N

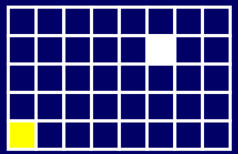
P-positions and N-positions can be defined recursively by the following:

(1) All terminal positions are P-positions (normal winning condition)

(2) From every N-position, there is at least one move to a P-position

(3) From every P-position, every move is to an N-position

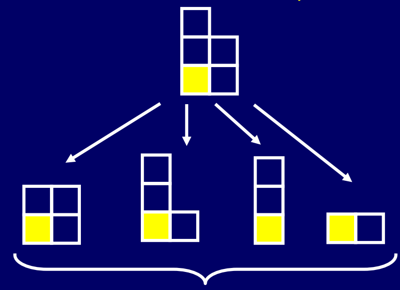
Chomp!



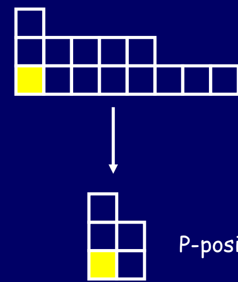
Two-player game, where each move consists of taking a square and removing it and all squares to the right and above.

Player who takes position (1,1) loses

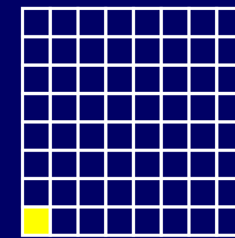
Show That This is a P-position

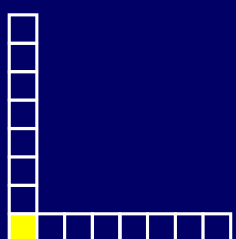


Show That This is an N-position



Let's Play! I'm player 1





No matter what you do, I can mirror it!

Theorem: Player 1 can win in any square starting position of Chomp

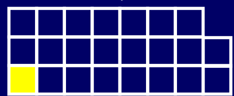
Proof:

The winning strategy for player 1 is to chomp on (2,2), leaving only an "L" shaped position

Then, for any move that Player 2 takes, Player 1 can simply mirror it on the flip side of the "L"

Theorem: Every rectangle is a N-position

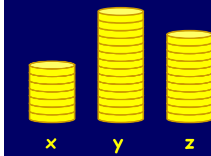
Proof: Consider this position:



This is either a P or an N-position. If it's a P-position, then the original rectangle was N. If it's an N-position, then there exists a move from it to a P-position X.

But by the geometry of the situation, X is also available as a move from the starting rectangle. It follows that the original rectangle is an N-position.

The Game of Nim



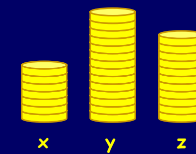
Two players take turns moving

Winner is the last player to remove chips

A move consists of selecting a pile and removing chips from it (you can take as many as you want, but you have to at least take one)

In one move, you cannot remove chips from more than one pile

Analyzing Simple Positions



We use (x,y,z) to denote this position

(0,0,0) is a: P-position

One-Pile Nim

What happens in positions of the form (x,0,0)?

The first player can just take the entire pile, so (x,0,0) is an N-position

Two-Pile Nim

P-positions are those for which the two piles have an equal number of chips

If it is the opponent's turn to move from such a position, he must change to a position in which the two piles have different number of chips

From a position with an unequal number of chips, you can easily go to one with an equal number of chips (perhaps the terminal position)

Nim-Sum

The nim-sum of two non-negative integers is their addition without carry in base 2

We will use $\oplus$ to denote the nim-sum

$2 \oplus 3 = (10)_2 \oplus (11)_2 = (01)_2 = 1$

$5 \oplus 3 = (101)_2 \oplus (011)_2 = (110)_2 = 6$

$7 \oplus 4 = (111)_2 \oplus (100)_2 = (011)_2 = 3$

$\oplus$ is associative: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$

$\oplus$ is commutative: $a \oplus b = b \oplus a$

For any non-negative integer x,

$x \oplus x = 0$

Cancellation Property Holds

If $x \oplus y = x \oplus z$

Then $x \oplus x \oplus y = x \oplus x \oplus z$

So $y = z$

Bouton's Theorem: A position (x,y,z) in Nim is a P-position if and only if $x \oplus y \oplus z = 0$

Proof:

Let Z denote the set of Nim positions with nim-sum zero

Let NZ denote the set of Nim positions with non-zero nim-sum

We prove the theorem by proving that Z and NZ satisfy the three conditions of P-positions and N-positions

(1) All terminal positions are in Z

The only terminal position is (0,0,0)

(2) From each position in NZ, there is a move to a position in Z

001010001	001010001
100010111	100010111
$\oplus$ 111010000	$\oplus$ 101000110
010010110	000000000

Look at leftmost column with an odd # of 1s

Rig any of the numbers with a 1 in that column so that everything adds up to zero

(3) Every move from a position in Z is to a position in NZ

If (x,y,z) is in Z, and x is changed to $x' < x$, then we cannot have

$x \oplus y \oplus z = 0 = x' \oplus y \oplus z$

Because then $x = x'$

Part II - Sums of Games

Consider a game called Boxing Match which was defined in a programming contest

http://potm.tripod.com/BOXINGMATCH/problem.short.html

An n x m rectangular board is initialized with 0 or 1 stone on each cell. Players alternate removing all the stones in any square subarray where all the cells are full. The player taking the last stone wins.

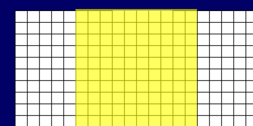
Boxing Match Example

Suppose we start with a 10 x 20 array that is completely full.

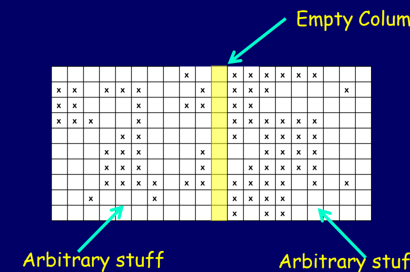
Is this a P or an N-position?

Example Contd.

The 10 x 20 full board is an N-position. A winning move is to take a 10x10 square in the middle. This leaves a 5x10 rectangle on the left and a 5x10 rectangle on the right. This is a P-position via mirroring. QED.



Sums of Games



In this kind of situation, the left and right games are completely independent games that don't interact at all. This naturally leads to the notion of the sum of two games.

A + B

A + B

A and B are games. The game A+B is a new game where the allowed moves are to pick one of the two games A or B (that is non-terminal) and make a move in that game. The position is terminal iff both A and B are terminal.

The sum operator is clearly commutative and associative.

Sums of Games*

We assign a number to any position in any game. This number is called the Nimber of the game.

(It's also called the "Nim Sum" or the "Sprague-Grundy" number of a game. But we'll call it the Nimber.)

*Only applies to normal, impartial games.

The MEX

The "MEX" of a finite set of natural numbers is the Minimum EXcluded element.

MEX {0, 1, 2, 4, 5, 6} = 3

MEX {1, 3, 5, 7, 9} = 0

Definition of Nimber

The Nimber of a game G (denoted N(G)) is defined inductively as follows:

N(G) = 0 if G is terminal

N(G) = MEX{N(G_1), N(G_2), … N(G_n)}

Where G_1, G_2, … Gn are the successor positions of game G. (I.e. the positions resulting from all the allowed moves.)

Another look at Nim

Let P_k denote the game that is a pile of k stones in the game of Nim.

Theorem: $N(P_k) = k$

Theorem: $N(P_k) = k$

Proof: Use induction.
Base case is when k=0. Trivial.
When k>0 the set of moves is
$P_{k-1}, P_{k-2}, \ldots P_0$.

By induction these positions have nimbers k-1, k-2, … 0.

The MEX of these is k. QED.

Nimber = 0 iff P-position

Theorem: A game G is a P-position if and only if N(G)=0.

Proof: Induction.
Trivially true if G is a terminal position.
Suppose G is non-terminal.

If N(G)≠0, then by the MEX rule there must be a move G' in G that has N(G')=0. By induction this is a P-position. Thus G is an N position.

Nimber = 0 iff P-position (contd)

If N(G)=0, then by the MEX rule none of the successors of G have N(G')=0. By induction all of them are N-positions. Therefore G is a P-position.

QED.

The Nimber Theorem

Theorem: Let A and B be two impartial normal games. Then:

$$N(A+B) = N(A) \oplus N(B)$$

Proof: You will prove it on your homework.

Application to Nim

Note that the game of Nim is just the sum of several games. If the piles are of size a, b, and c, then the nim game for these piles is just $P_a + P_b + P_c$.

The nimber of this position, by the nimber theorem, is just $a \oplus b \oplus c$.

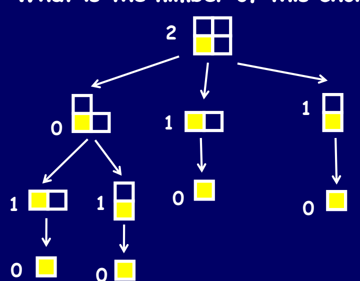
So it's a P-position if and only if $a \oplus b \oplus c = 0$, which is what we proved before.

Application to Boxing Match

The beauty of Nimbers is that they completely capture what you need to know about a game in order to add it to another game. This can speed up game search exponentially.

My friends Guy Jacobson and David Applegate used this to cream all the other players in the Boxing Match contest.

What is the nimber of this chomp game?



What if we add this to a nim pile of size 4?

If we remove two chips from the nim pile, then the nim-sum is 0, giving a P-position. This is the unique winning move in this position.

- P-positions versus N-positions
- Nim-sum
- Bouton's Theorem
- Definition of Nimbers
- The MEX operator
- Sum of Games
- The Nimber Theorem

Study Bee