Game Theory 15-451 12/04/07
- Zero-sum games
- General-sum games

Shall we play a game?

Game Theory and Computer Science

Plan for Today

- 2-Player Zero-Sum Games (matrix games)
  - Minimax optimal strategies
  - Minimax theorem and proof

test material / not test material

- General-Sum Games (bimatrix games)
  - notion of Nash Equilibrium
- Proof of existence of Nash Equilibria
  - using Brouwer's fixed-point theorem

2-player zero-sum game recap

Consider the following scenario…

- Shooter has a penalty shot. Can choose to shoot left or shoot right.
- Goalie can choose to dive left or dive right.
- If goalie guesses correctly, (s)he saves the day. If not, it's a goooooaaaaall!
- Vice-versa for shooter.

2-Player Zero-Sum games

- Two players R and C. Zero-sum means that what's good for one is bad for the other.
- Game defined by matrix with a row for each of R's options and a column for each of C's options. Matrix tells who wins how much.
  - an entry (x,y) means: x = payoff to row player, y = payoff to column player. "Zero sum" means that y = -x.
- E.g., penalty shot:

shooter \ goalie	Left	Right
Left	(0,0)	(1,-1)
Right	(1,-1)	(0,0)

goalie; shooter; GOAALLL!!!; No goal

Minimax-optimal strategies

- Minimax optimal strategy is a (randomized) strategy that has the best guarantee on its expected gain, over choices of the opponent. [maximizes the minimum]
- I.e., the thing to play if your opponent knows you well.

shooter \ goalie	Left	Right
Left	(0,0)	(1,-1)
Right	(1,-1)	(0,0)

goalie; shooter; GOAALLL!!!; No goal

Minimax-optimal strategies

- In class on Linear Programming, we saw how to solve for this using LP.
  - polynomial time in size of matrix if use poly-time LP alg.
- I.e., the thing to play if your opponent knows you well.

shooter \ goalie	Left	Right
Left	(0,0)	(1,-1)
Right	(1,-1)	(0,0)

goalie

shooter

GOAALLL!!!

No goal

Minimax-optimal strategies

- What are the minimax optimal strategies for this game?

Minimax optimal strategy for both players is 50/50. Gives expected gain of $\frac{1}{2}$ for shooter ($-\frac{1}{2}$ for goalie). Any other is worse.

shooter \ goalie	Left	Right
Left	(0,0)	(1,-1)
Right	(1,-1)	(0,0)

goalie

shooter

GOAALLL!!!

No goal

Minimax-optimal strategies

- How about penalty shot with goalie who's weaker on the left?

Minimax optimal for shooter is (2/3,1/3).
Guarantees expected gain at least 2/3.
Minimax optimal for goalie is also (2/3,1/3).
Guarantees expected loss at most 2/3.

shooter \ goalie	Left	Right
Left	($\frac{1}{2}$,$-\frac{1}{2}$)	(1,-1)
Right	(1,-1)	(0,0)

goalie

shooter

GOAALLL!!!

50/50

Minimax Theorem (von Neumann 1928)

- Every 2-player zero-sum game has a unique value V.
- Minimax optimal strategy for R guarantees R's expected gain at least V.
- Minimax optimal strategy for C guarantees C's expected loss at most V.

Counterintuitive: Means it doesn't hurt to publish your strategy if both players are optimal. (Borel had proved for symmetric 5x5 but thought was false for larger games)

Matrix games and Algorithms

- Gives a useful way of thinking about guarantees on algorithms for a given problem.
- Think of rows as different algorithms, columns as different possible inputs.
- M(i,j) = cost of algorithm i on input j.
- Algorithm design goal: good strategy for row player. Lower bound: good strategy for adversary.

E.g., sorting

One way to think of upper-bounds/lower-bounds: on value of this game

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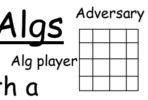
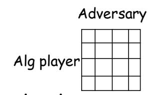
Of course matrix may be HUGE. But helpful conceptually.

Matrix games and Algs



- What is a deterministic alg with a good worst-case guarantee?
  - A row that does well against all columns.
- What is a lower bound for deterministic algorithms?
  - Showing that for each row i there exists a column j such that M(i,j) is bad.
- How to give lower bound for randomized algs?
  - Give randomized strategy for adversary that is bad for all i. Must also be bad for all distributions over i.

E.g., hashing



- Rows are different hash functions.
- Cols are different sets of n items to hash.
- M(i,j) = #collisions incurred by alg i on set j.

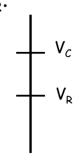
We saw:

- For any row, can reverse-engineer a bad column (if universe of keys is large enough).
- Universal hashing is a randomized strategy for row player that has good behavior for every column.
  - For any set of inputs, if you randomly construct hash function in this way, you won't get many collisions in expectation.

We are now below the red line from slide 2

Nice proof of minimax thm (sketch)

- Suppose for contradiction it was false.
- This means some game G has $V_C > V_R$:
  - If Column player commits first, there exists a row that gets the Row player at least V_C.
  - But if Row player has to commit first, the Column player can make him get only V_R.
- Scale matrix so payoffs to row are in [-1,0]. Say $V_R = V_C - \delta$.



Proof sketch, contd

- Now, consider randomized weighted-majority alg from last lecture as Row, against Col who plays optimally against Row's distrib.
- In T steps,

How can we think of RWM as an alg for repeatedly playing a matrix game???

  - Alg gets $\geq (1-\varepsilon/2)$[best row in hindsight] – $\log(n)/\varepsilon$
  - BRiH $\geq T \cdot V_C$ [Best against opponent's empirical distribution]
  - Alg $\leq T \cdot V_R$ [Each time, opponent knows your randomized strategy]
  - Gap is δT. Contradicts assumption if use $\varepsilon=\delta$, once $T > 2\log(n)/\varepsilon^2$.

Proof sketch, contd

- Consider repeatedly playing game G against some opponent. [think of you as row player]
- Use exponential weighting alg from Nov 16 lecture to do nearly as well as best fixed row in hindsight.
  - Alg gets $\geq (1-\varepsilon/2)$OPT – c*$\log(n)/\varepsilon$
    $> (1-\varepsilon)$OPT [if play long enough]
  - OPT $\geq V_C$ [Best against opponent's empirical distribution]
  - Alg $\leq V_R$ [Each time, opponent knows your randomized strategy]
  - Contradicts assumption.

General-Sum Games

- Zero-sum games are good formalism for design/analysis of algorithms.
- General-sum games are good models for systems with many participants whose behavior affects each other's interests
  - E.g., routing on the internet
  - E.g., online auctions

General-sum games

- In general-sum games, can get win-win and lose-lose situations.
- E.g., "what side of sidewalk to walk on?":

	Left	Right
Left	(1,1)	(-1,-1)
Right	(-1,-1)	(1,1)

you

person walking towards you

General-sum games

- In general-sum games, can get win-win and lose-lose situations.
- E.g., "which movie should we go to?":

	Borat	Harry potter
Borat	(8,2)	(0,0)
Harry potter	(0,0)	(2,8)

No longer a unique "value" to the game.

Nash Equilibrium

- A Nash Equilibrium is a stable pair of strategies (could be randomized).
- Stable means that neither player has incentive to deviate on their own.
- E.g., "what side of sidewalk to walk on":

	Left	Right
Left	(1,1)	(-1,-1)
Right	(-1,-1)	(1,1)

NE are: both left, both right, or both 50/50.

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NE are: both B, both HP, or (80/20,20/80)

Uses

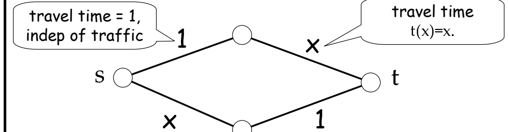
- Economists use games and equilibria as models of interaction.
- E.g., pollution / prisoner's dilemma:
  - (imagine pollution controls cost $4 but improve everyone's environment by $3)

	don't pollute	pollute
don't pollute	(2,2)	(-1,3)
pollute	(3,-1)	(0,0)

Need to add extra incentives to get good overall behavior.

NE can do strange things

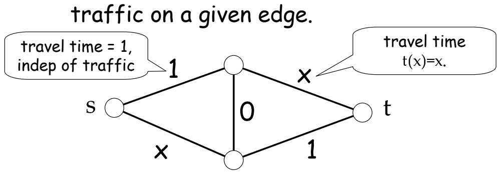
- Braess paradox:
  - Road network, traffic going from s to t.
  - travel time as function of fraction x of traffic on a given edge.



Fine. NE is 50/50. Travel time = 1.5

NE can do strange things

- Braess paradox:
  - Road network, traffic going from s to t.
  - travel time as function of fraction x of traffic on a given edge.



Add new superhighway. NE: everyone uses zig-zag path. Travel time = 2.

Existence of NE

- Nash (1950) proved: any general-sum game must have at least one such equilibrium.
  - Might require randomized strategies (called "mixed strategies")
- This also yields minimax thm as a corollary.
  - Pick some NE and let V = value to row player in that equilibrium.
  - Since it's a NE, neither player can do better even knowing the (randomized) strategy their opponent is playing.
  - So, they're each playing minimax optimal.

Existence of NE

- Proof will be non-constructive.
- Unlike case of zero-sum games, we **do not know any** polynomial-time algorithm for finding Nash Equilibria in $n \times n$ general-sum games. [known to be "PPAD-hard"]
- Notation:
  - Assume an nxn matrix.
  - Use $(p_1,\ldots,p_n)$ to denote mixed strategy for row player, and $(q_1,\ldots,q_n)$ to denote mixed strategy for column player.

Proof

- We'll start with Brouwer's fixed point theorem.
  - Let S be a compact convex region in R^n and let $f:S \to S$ be a continuous function.
  - Then there must exist $x \in S$ such that $f(x)=x$.
  - x is called a "fixed point" of f.
- Simple case: S is the interval [0,1].
- We will care about:
  - S = {(p,q): p,q are legal probability distributions on 1,...,n}. I.e., S = $\text{simplex}_n \times \text{simplex}_n$

Proof (cont)

- S = {(p,q): p,q are mixed strategies}.
- Want to define f(p,q) = (p',q') such that:
  - f is continuous. This means that changing p or q a little bit shouldn't cause p' or q' to change a lot.
  - Any fixed point of f is a Nash Equilibrium.
- Then Brouwer will imply existence of NE.

Try #1

- What about f(p,q) = (p',q') where p' is best response to q, and q' is best response to p?
- Problem: not necessarily well-defined:
  - E.g., penalty shot: if p = (0.5,0.5) then q' could be anything.

	Left	Right
Left	(0,0)	(1,-1)
Right	(1,-1)	(0,0)

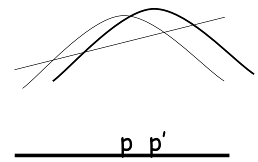
Try #1

- What about f(p,q) = (p',q') where p' is best response to q, and q' is best response to p?
- Problem: also not continuous:
  - E.g., if p = (0.51, 0.49) then q' = (1,0). If p = (0.49,0.51) then q' = (0,1).

	Left	Right
Left	(0,0)	(1,-1)
Right	(1,-1)	(0,0)

Instead we will use…

- f(p,q) = (p',q') such that:
  - q' maximizes [(expected gain wrt p) - $||q-q'||^2$]
  - p' maximizes [(expected gain wrt q) - $||p-p'||^2$]



Note: quadratic + linear = quadratic.

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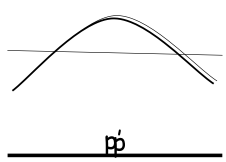

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  - q' maximizes [(expected gain wrt p) - $||q-q'||^2$]
  - p' maximizes [(expected gain wrt q) - $||p-p'||^2$]
- f is well-defined and continuous since quadratic has unique maximum and small change to p,q only moves this a little.
- Also fixed point = NE. (even if tiny incentive to move, will move little bit).
- So, that's it!