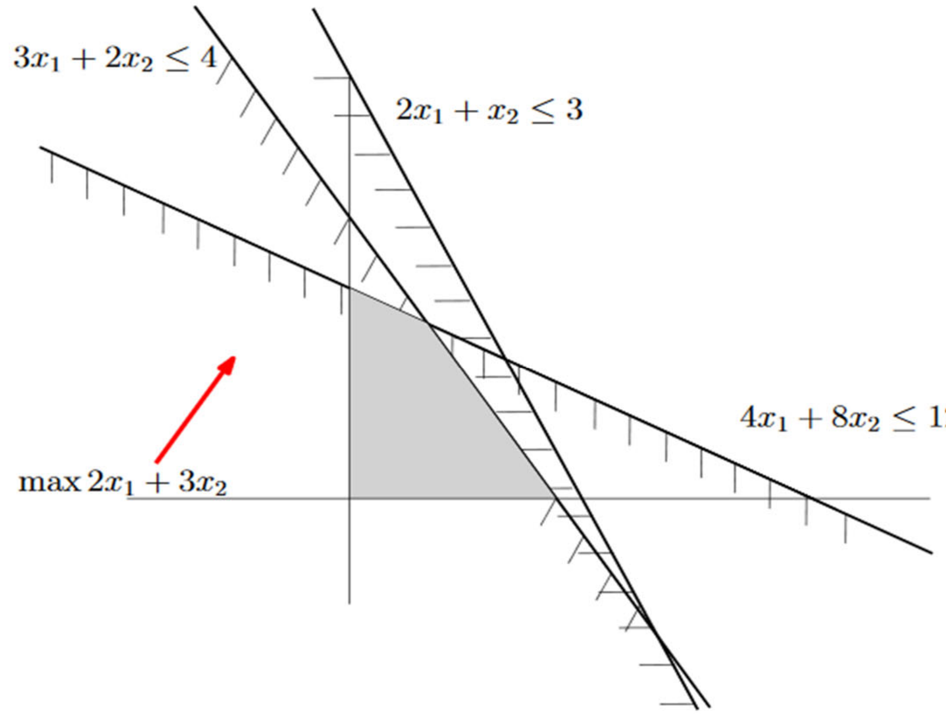
Lecture 15: Linear Programming III

David Woodruff
Carnegie Mellon University

Outline

- Linear Programming Duality
- Application to zero sum games

$$P = \max(2x_1 + 3x_2)$$
$$\text{s.t.} \quad 4x_1 + 8x_2 \leq 12$$
$$2x_1 + x_2 \leq 3$$
$$3x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$



Since $2x_1 + 3x_2 \leq 4x_1 + 8x_2 \leq 12$, we know OPT ≤ 12

Since $2x_1 + 3x_2 \leq \frac{1}{2}(4x_1 + 8x_2) \leq 6$, we know OPT ≤ 6

Since $2x_1 + 3x_2 \leq \frac{1}{3}\big((4x_1 + 8x_2) + (2x_1 + x_2)\big) \leq 5$, we know OPT ≤ 5

Duality

- We took non-negative linear combinations of the constraints
- How do we find the best upper bound on OPT this way?
- Let $y_1, y_2, y_3 \geq 0$ be the coefficients of our linear combination. Then,

$$
\begin{aligned}
4y_1 + 2y_2 + 3y_3 &\geq 2 \\
8y_1 + y_2 + 2y_3 &\geq 3 \\
y_1, y_2, y_3 &\geq 0
\end{aligned}
$$

and we seek $\min(12y_1 + 3y_2 + 4y_3)$

$$
\begin{aligned}
P = \max&(2x_1 + 3x_2) \\
\text{s.t.} \quad 4x_1 + 8x_2 &\leq 12 \\
2x_1 + x_2 &\leq 3 \\
3x_1 + 2x_2 &\leq 4 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Primal LP

$$P = \max(2x_1 + 3x_2)$$
$$\text{s.t.} \quad 4x_1 + 8x_2 \leq 12$$
$$2x_1 + x_2 \leq 3$$
$$3x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Dual LP

$$4y_1 + 2y_2 + 3y_3 \geq 2$$
$$8y_1 + y_2 + 2y_3 \geq 3$$
$$y_1, y_2, y_3 \geq 0$$
$$\text{and we seek } \min(12y_1 + 3y_2 + 4y_3)$$

- If (x_1, x_2) is feasible for the primal, and (y_1, y_2, y_3) feasible for the dual,
$$2x_1 + 3x_2 \leq 12y_1 + 3y_2 + 4y_3$$
- If these are equal, we've found the optimal value for both LPs
- $(x_1, x_2) = (\frac{1}{2}, \frac{5}{4})$ and $(y_1, y_2, y_3) = (\frac{5}{16}, 0, \frac{1}{4})$ give the same value 4.75, so optimal

Dual LP

$$4y_1 + 2y_2 + 3y_3 \geq 2$$
$$8y_1 + y_2 + 2y_3 \geq 3$$
$$y_1, y_2, y_3 \geq 0$$
and we seek $\min(12y_1 + 3y_2 + 4y_3)$

- Let's try do the same thing to the dual:
- $12y_1 + 3y_2 + 4y_3 \geq 4y_1 + 2y_2 + 3y_2 \geq 2$
- $12y_1 + 3y_2 + 4y_3 \geq 8y_1 + y_2 + 2y_3 \geq 3$
- $12y_1 + 3y_2 + 4y_3 \geq \frac{2}{3}(4y_1 + 2y_2 + 3y_2) + (8y_1 + y_2 + 2y_3) \geq \frac{4}{3} + 3$

Dual LP

$$4y_1 + 2y_2 + 3y_3 \geq 2$$
$$8y_1 + y_2 + 2y_3 \geq 3$$
$$y_1, y_2, y_3 \geq 0$$
$$\text{and we seek } \min(12y_1 + 3y_2 + 4y_3)$$

$$P = \max(2x_1 + 3x_2)$$
$$\text{s.t.} \quad 4x_1 + 8x_2 \leq 12$$
$$2x_1 + x_2 \leq 3$$
$$3x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

- Take non-negative linear combination of the two constraints
- How do we find the best lower bound on OPT this way?
- Let $x_1, x_2 \geq 0$ be the coefficients of our linear combination. Then,
- $4x_1 + 8x_2 \leq 12, \;\; 2x_1 + x_2 \leq 3, \;\; 3x_1 + 2x_2 \leq 4, \;\; x_1 \geq 0, \;\; x_2 \geq 0$ and we seek to maximize $2x_1 + 3x_2$

We got back the primal!

Non-Nice Constraints

$$P = \max(7x_1 - x_2 + 5x_3)$$
$$\text{s.t.} \quad x_1 + x_2 + 4x_3 \leq 8$$
$$3x_1 - x_2 + 2x_3 \geq 3$$
$$x_1, x_2, x_3 \geq 0$$

$$D = \min(8y_1 + 3y_2)$$
$$\text{s.t.} \quad y_1 + 3y_2 \geq 7$$
$$y_1 - y_2 \geq -1$$
$$4y_1 + 2y_2 \geq 5$$
$$y_1 \geq 0, y_2 \leq 0$$

Formal Definition of Duality

<u>Primal</u>

Max $c^T x$

subject to $Ax \leq b$

$x \geq 0$

<u>Dual</u>

Min $b^T y$

subject to $A^T y \geq c$

$y \geq 0$

- Dual of the dual is the primal!
- Can we get better upper/lower bounds by looking at more complicated combinations of the inequalities, not just linear combinations?

Weak Duality

Primal

Max $c^T x$

subject to $Ax \leq b$

$x \geq 0$

Dual

Min $b^T y$

subject to $A^T y \geq c$

$y \geq 0$

- (Weak Duality) If x is a feasible solution of the primal, and y is a feasible solution of the dual, then $c^T x \leq b^T y$

- Proof: Since $x \geq 0$ and $y \geq 0$,

$$c^T x \leq y^T A x \leq y^T b = b^T y$$

Strong Duality

Primal

Max $c^T x$

subject to $Ax \leq b$

$x \geq 0$

Dual

Min $b^T y$

subject to $A^T y \geq c$

$y \geq 0$

- (Strong Duality) If primal is feasible and bounded (i.e., optimal value is not ∞), then dual is feasible and bounded. If x^* is optimal solution to the primal, and y^* is optimal solution to dual, then

$$c^T x^* = b^T y^*$$

- To prove x^* is optimal, I can give you y^* and you can check if x^* is feasible for the primal, y^* is feasible for the dual, and $c^T x^* = b^T y^*$

Consequences of Duality

$P \backslash D$	I	O	U
I	?	?	?
O	?	?	?
U	?	?	?

I means infeasible
O means feasible and bounded
U means unbounded

Which combinations are possible?

Consequences of Duality

$P \backslash D$	I	O	U
I	✓	X	✓
O	X	✓	X
U	✓	X	X

I means infeasible
O means feasible and bounded
U means unbounded

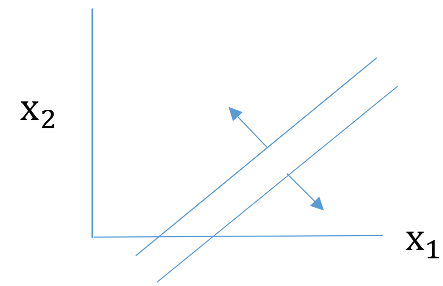
Check means possible
X means impossible

Possible Scenarios

$P \backslash D$	I	O	U
I	✓	X	✓
O	X	✓	X
U	✓	X	X

- Suppose primal is feasible and bounded
- By strong duality, dual is feasible and bounded
- If primal (maximization) is unbounded, by weak duality, $c^T x \leq b^T y$, so no feasible dual solution e.g., $\max x_1$ subject to $x_1 \geq 1$ and $x_1 \geq 0$

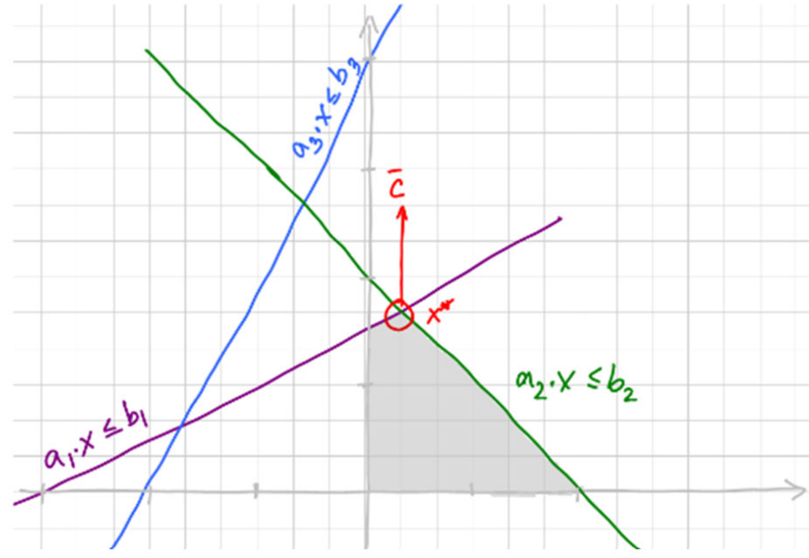
  dual will have $y_1 \leq 0$ and $y_1 \geq 1$

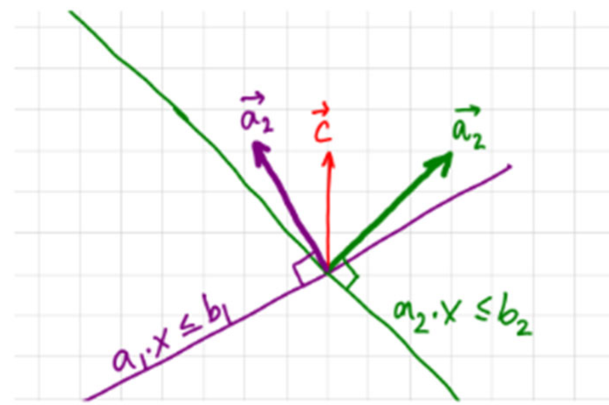


- Can primal and dual both be infeasible?
- Primal: max $2x_1 - x_2$ subject to $x_1 - x_2 \leq 1$ and $-x_1 + x_2 \leq -2$ and $x_1 \geq 0, x_2 \geq 0$
- Dual: $y_1 \geq 0$, $y_2 \geq 0$, and $y_1 - y_2 \geq 2$ and $-y_1 + y_2 \geq -1$, and $\min y_1 - 2y_2$
- Constraints are same for primal and dual, and both infeasible

Strong Duality Intuition

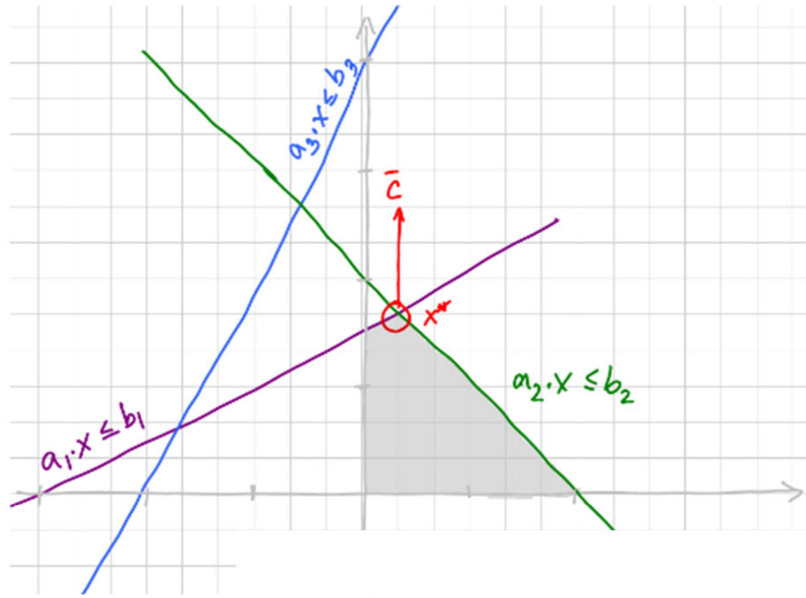
Suppose x^* satisfies $a_1 x = b_1$ and $a_2 x = b_2$





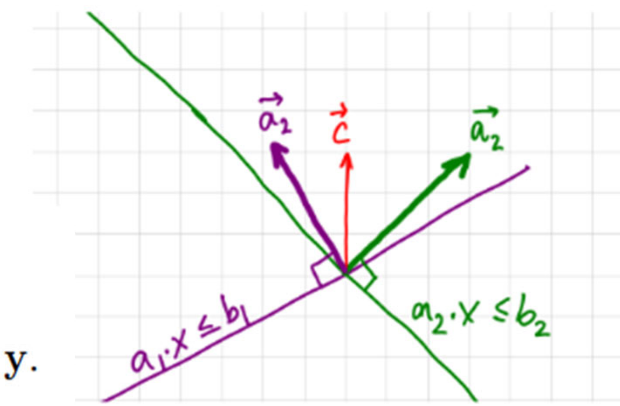
Strong Duality Intuition

- For non-negative y_1 and y_2



$$\mathbf{c} = y_1\mathbf{a}_1 + y_2\mathbf{a}_2.$$

$$\begin{aligned}\mathbf{c}^{\mathsf{T}}\cdot \mathbf{x}^* &= (y_1\,\mathbf{a}_1 + y_2\,\mathbf{a}_2)\cdot \mathbf{x}^* \\ &= y_1(\mathbf{a}_1\cdot \mathbf{x}^*) + y_2(\mathbf{a}_2\cdot \mathbf{x}^*) \\ &= y_1 b_1 + y_2 b_2\end{aligned}$$



Defining $\mathbf{y} = (y_1, y_2, 0, \ldots, 0)$, we get

optimal value of primal $= \mathbf{c}^{\mathsf{T}}\mathbf{x}^* = \mathbf{b}^{\mathsf{T}}\mathbf{y}$ = value of dual solution $\mathbf{y}$.

the $\mathbf{y}$ we found satisfies $\mathbf{c} = y_1\mathbf{a}_1 + y_2\mathbf{a}_2 = \sum_i y_i\mathbf{a}_i = A^T\mathbf{y}$, and hence $\mathbf{y}$ satisfies the dual constraints $\mathbf{y}^T A \geq \mathbf{c}^T$ by construction.

But $b^T y \geq c^T x^*$ by weak duality, so y is optimal!

Duality in Zero-Sum Games

- R is an n x m row payoff matrix
- W.l.o.g. R has all non-negative entries
- Variables: $\mathrm{v}, \mathrm{p}_1, \dots, \mathrm{p}_\mathrm{n}$
- Max v

  subject to $\mathrm{p}_\mathrm{i} \geq 0$ for all rows i, $\sum_\mathrm{i} \mathrm{p}_\mathrm{i} = 1$, $\sum_\mathrm{i} \mathrm{p}_\mathrm{i} \mathrm{R}_{\mathrm{i,j}} \geq \mathrm{v}$ for all columns j

- Replace $\sum_\mathrm{i} \mathrm{p}_\mathrm{i} = 1$ with $\sum_\mathrm{i} \mathrm{p}_\mathrm{i} \leq 1$.
- Include $\mathrm{v} \geq 0$
- Write $\sum_\mathrm{i} \mathrm{p}_\mathrm{i} \mathrm{R}_{\mathrm{i,j}} \geq \mathrm{v}$ as $\mathrm{v} - \sum_\mathrm{i} \mathrm{p}_\mathrm{i} \mathrm{R}_{\mathrm{i,j}} \leq 0$

Duality in Zero-Sum Games

max $c^T x$ subject to $Ax \le b$ and $x \ge 0$

$$\mathbf{x} = \begin{bmatrix} v \\ p_1 \\ p_2 \\ \dots \\ p_n \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{bmatrix}, \text{and } A = \begin{bmatrix} 1 & \\ 1 & -R^T \\ \dots & \\ 1 & \\ 0 & 1 \quad \dots \quad 1 \end{bmatrix}.$$

- Dual: min $y^T b$ subject to $y^T A \ge c^T$ and $y \ge 0$ for $y = (y_1, \dots, y_{m+1})$
- Dual constraints say $y_1 + \cdots + y_m \ge 1$ and $\sum_j y_j R_{ij} \le y_{m+1}$ for all rows i
  - Since we're minimizing y_{m+1} and $R_{i,j}$ all non-negative, $y_1 + \dots + y_m = 1$
- y_{m+1} is value to the row player and $y_1, \dots, y_m$ is column player's strategy
- Strong duality: $\max_p \min_j \sum_i p_i R_{ij} = \min_{y_1,\dots,y_m} \max_i \sum_j y_j R_{ij}$