(1) Vectors $v_1, \dots v_d \in R^n$ are linearly independent if $\forall \alpha_1, \dots \alpha_d \in R$, we have $\sum_{i=1}^{d} \alpha_i v_i = 0$ if and only if $\alpha_1 = \alpha_2 = \dots = \alpha_d = 0$. This implies, in particular, that if $v_1, \dots v_d$ are linearly independent, we cannot write $v_i = \sum_{j \neq i} \beta_j v_j$ for coefficients $\beta_j \in R$, as otherwise $-v_i + \sum_{j \neq i} \beta_j v_j = 0$.

~~(2) The the span of a set of vectors $v_1, \dots v_d$ is the set $\{y \in R^n$ such that $y = \sum_{i=1}^{d} \alpha_i v_i$ for some coefficients $\alpha_1, \dots \alpha_d \in R\}$.~~

(2) A subspace S of R^n is a set of vectors such that

(1) if $y \in S$ then $\alpha y \in S$ for all $\alpha \in R$, and

(2) if $x, y \in S$ then $x + y \in S$.

(3) The span of a set of vectors $v_1, \dots v_d$ is the set $\{y \in R^n$ such that $y = \sum_{i=1}^{d} \alpha_i v_i$ for some coefficients $\alpha_1, \dots \alpha_d \in R\}$. Note that the span of any set of vectors is a subspace. If $v_1, \dots, v_d$ are linearly independent, then d is called the dimension of the subspace, and $v_1, \dots v_d$ is called a basis.

(4) ~~[illegible]~~ A $d \times d$ matrix R is said to be invertible or non-singular if there exists a matrix R^{-1} so that

$R \cdot R^{-1} = R^{-1} \cdot R = I_d$, where I_d is the identity, so has d ones on the diagonal and is zero otherwise.

(5) If $v_1, \dots v_d$ is a basis of a d-dimensional subspace S, then $(VR)_1, (VR)_2, \dots (VR)_d$ ~~$Rv_1, \dots Rv_d$~~ is also a basis for the same subspace. S, where R is a $d \times d$ invertible matrix.

Indeed $S = \{ y$ such that $y = \sum_{i=1}^{d} \alpha_i v_i$ for coefficients $\alpha_i \in \mathbb{R}\}$. Let V be an $n \times d$ matrix whose columns are $v_1, \dots, v_d$. Then $S = \{ V\alpha$ such that $\alpha \in \mathbb{R}^d\}$. On the other hand the span of ~~$Rv_1, \dots Rv_d$~~ is the set T of vectors $T = \{VR\beta$ such that $\beta \in \mathbb{R}^d\}$. But for any y of the form $y = V\alpha$, if we set $\beta = R^{-1}\alpha$ we have that $y \in T$, and on the other hand if ~~[illegible]~~ $y = VR\beta$, if we set $\alpha = R\beta$, we have $y \in S$.

(6) Let A be an $n \times d$ matrix, $n \geq d$, such that the columns of A are linearly independent. Then $A^T A$ is a $d \times d$ matrix and for all $x \neq 0$, $A^T A x \neq 0$. Indeed, suppose $A^T A x = 0$. Then $x^T (A^T A x) = 0$. But $x^T A^T A x = |Ax|_2^2 = \sum_{i=1}^{n} (Ax)_i^2$, and so this is equal to 0 if and only if $x = 0$.

(7) If a symmetric $d \times d$ matrix B is such that $Bx = 0$ if and only if $x = 0$, then there is a matrix B^{-1} for which $B^{-1}B = I$.

~~It suffices to show for any vector y, there is a solution x to the equation $xB = y$. Then we can apply this to each row of I separately.~~

~~Need simple fact: (1) if you add a multiple of one row to another,~~

Need a simple fact: suppose you start with d linearly independent rows in $\mathbb{R}^d$. Then the following operations preserve linear independence

(1) scaling a row by a non-zero real number α

(2) swapping two rows

(3) adding a multiple of one row to another

Note that if there is no $x \neq 0$ for which $Bx = 0$, then there is also no $x \neq 0$ for which $x^T B = 0$, since B is symmetric.

The proof is an algorithm, called Gauss-Jordan algorithm.

Let's do an example.

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & 1 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad \text{ADD}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & .2 & .2 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad \text{Divide by 5}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & .2 & .2 & 0 \\ 0 & 0 & -2 & -.4 & .6 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad \text{Subtract} \times 2$$

$$\cancel{\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & \end{array}\right]}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & .2 & .2 & 0 \\ 0 & 0 & 1 & .2 & -.3 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad \text{Multiply by } -1/2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & .2 & .2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & .2 & -.3 & 0 \end{array}\right] \quad \text{Swap}$$

Why can you always do this?

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & .2 & .2 & 0 \\ 0 & 1 & 0 & -.2 & .3 & 1 \\ 0 & 0 & 1 & .2 & -.3 & 0 \end{array}\right] \quad \text{Subtract}$$