Introduction to Homogeneous Transformations & Robot Kinematics

Jennifer Kay, Rowan University Computer Science Department
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1. Working with 3 Dimensional Frames in 2 Dimensions

1.1. Multiple 3-D Interpretations of a 2-D drawing

We will be working in 3-D coordinates, and will label the axes x, y, and z. Figure 1 contains a sample 3-D coordinate frame.

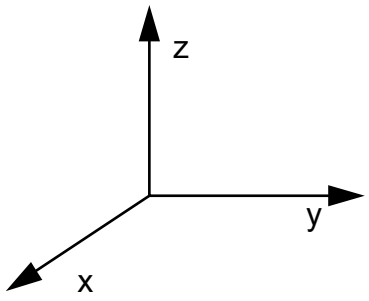


Figure 1. A 3-D Coordinate Frame

Because we are representing 3-D coordinate frames with 2-D drawings, we must agree on what these drawings mean. Clearly the y axis in Figure 1 points to the right, and the z axis points up, but we have to come up with a convention for what direction the x axis is pointing. Since the three axes must be perpendicular to each other, we know that the x axis either points into the paper, or out of the paper.

Most people instantly assume one or the other is the case. To be able to view both cases, it helps to look at the axes overlaid on a cube. Consider the two views of the same cube in Figure 2. In view (a) we are looking at the cube from below, in view (b) we are looking at the cube from above. Let's try and overlay the 3-D coordinate frame from Figure 1 onto these two views.

Before you turn the page, make sure you can see both views of the cube in Figure 2!

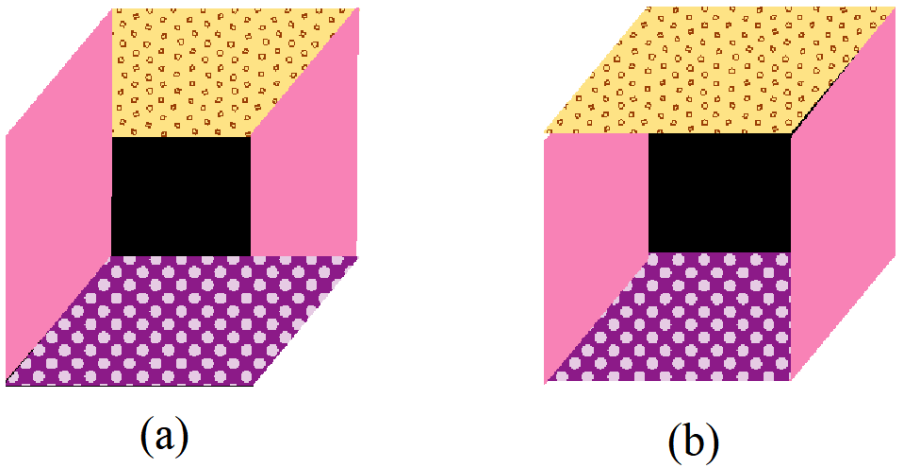


Figure 2. Two views of the same cube. The cube is missing the front side, has a solid back and sides, and patterned top and bottom. In view a we are looking at the cube from below, in view (b) we are looking at the cube from above.

Figure 3 and Figure 4 show the same two views of the cube, this time with the 3-D coordinate frame from Figure 1 overlaid onto the cube. Note that in Figure 3 the x axis points into the paper, away from you, and in Figure 4 the x axis is pointing out of the paper towards you!

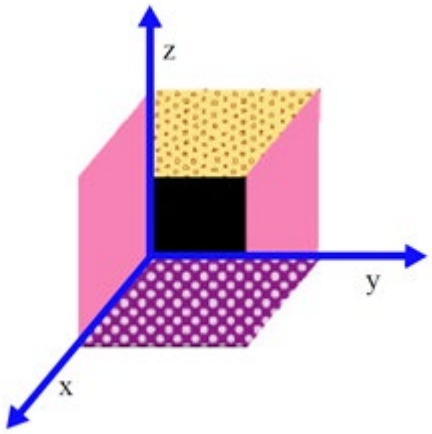


Figure 3. A cube viewed from below. The edge labelled x points into the page away from you.

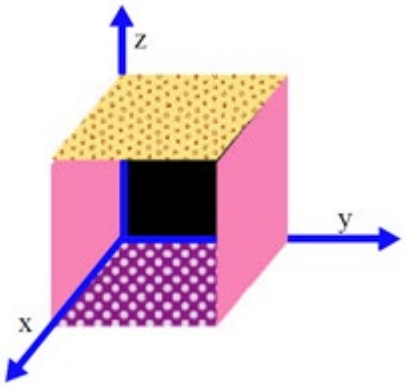


Figure 4. The same cube viewed from above. The edge labelled x points out of the page towards you.

We will use this interpretation throughout our work. An axis pointing down and to the left will be pointing "out of the page." An axis pointing up and to the right will be pointing "into the page."

For the purposes of this document, we will assume that Figure 4 shows the interpretation we will use. In other words, if you see 3 axes drawn as they are in Figure 1, you should assume that the x axis points out of the paper towards you. If you actually wanted the x axis to be pointing into the paper, you should use the illustration shown in Figure 5.

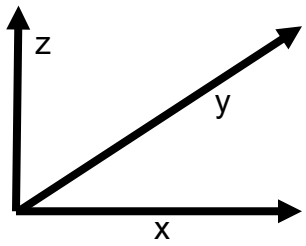


Figure 5. In this document, we will interpret an axis that is pointing up and to the right as pointing into the paper. Thus, in this frame, x points to the right, z points up, and y points away from you into the paper.

1.2. Right-Handed Coordinate Systems

In this document, we will use *right-handed* coordinate systems. In a right-handed coordinate system, if you know the directions of two out of the three axes, you can figure out the direction of the third. Let's suppose that you know the directions of the x and y axes. For example, suppose that x points to the left, and y points out of the paper, as shown in Figure 6. We want to determine the direction of the z axis. To do so, take your **right hand**, and hold it so that your fingers point in the direction of the x axis in such a way that your palm is in the direction of the y axis and you can curl your fingers towards the y axis. When your right hand is in this position, your thumb will point in the direction of the z axis. This process is illustrated in Figure 7. The chart in figure 8 details how to compute the direction of any axis given the directions of the other two.

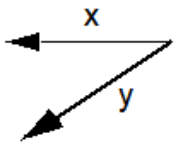


Figure 6. x points to the left, and y points out of the paper towards you.

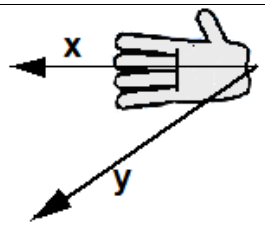
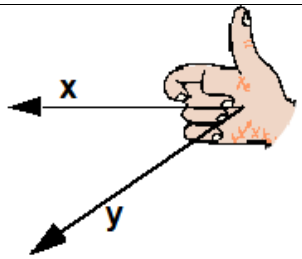
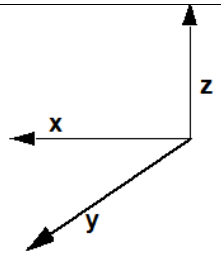
		
Step 1: hold your right hand in such a way that your fingers point in the direction of the x axis and when you curl your fingers, they curl towards the y axis.	Step 2: As you curl your fingers from the x axis towards the y axis, stick your thumb in the air. This will be the direction of the z axis	The final 3-D coordinate system with the z axis shown.

Figure 7. Using the right-hand rule to compute the direction of the z axis.

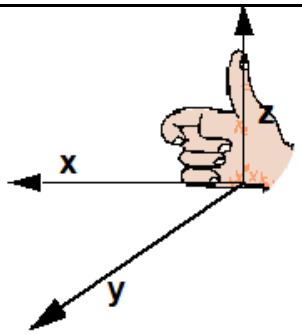
If you know the direction of these axes.	Point the fingers of your right hand in the direction of this axis.	Curl you right fingers towards the direction of this axis.	Your thumb will point in the direction of this axis.
x & y	x	y	z
y & z	y	z	x
x & z	z	x	y

Figure 8. Using the right-hand rule to compute the direction of any axis given the directions of the other two.

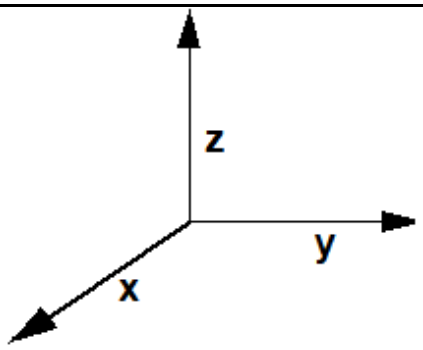
1.3. Direction of Positive Rotation

Sometimes we want to talk about rotating around one of the axes of a coordinate frame by some angle. Of course, if you are looking down an axis and want to spin it, you need to know whether you should spin it clockwise or counterclockwise. We are going to use another right-hand rule to determine the direction of positive rotation.

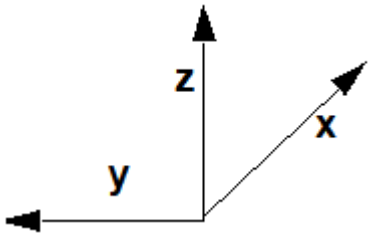
To determine the direction of positive rotation for a given axis, point the thumb or your right hand along the positive direction of the axis you wish to rotate around. The direction that the fingers of your right hand curl is the direction of positive rotation about that axis.



(a) The original axes with a right hand determining the direction of positive rotation around the z axis.



(b) After rotating *the original* axes (a) 90º around the z axis.



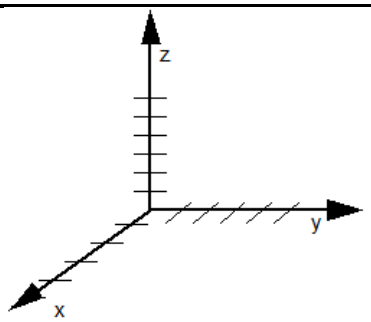
(c) After rotating *the original* axes (a) -90º around the z axis.

Figure 9. Another "right-hand rule" is used to determine the direction of positive angles. Point your right thumb along the positive direction of the axis you wish to rotate around. Curl your fingers. The direction that your fingers curl is the direction of positive rotation.

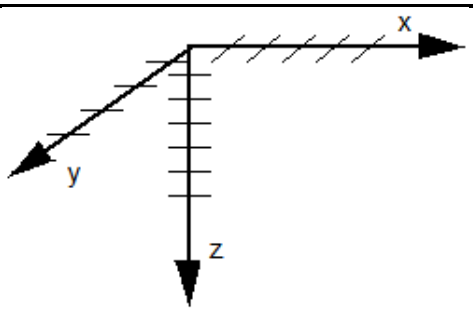
1.4. Plotting Points in 3 Dimensions

All of us have experience in plotting points on 2-D axes. When it comes to plotting points on 3-D axes, things become a bit more difficult.

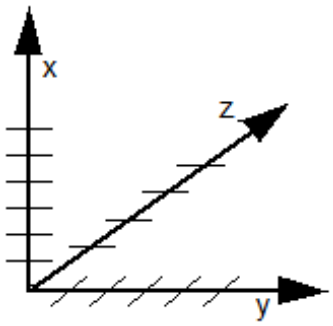
The first step is to draw tick marks on the axes to indicate scale. For the purposes of this document, we will assume that each tick represents one unit. Figure 10 shows several different right-handed coordinate systems with tick marks added. Note that each tick mark is parallel to one of the other axes. This helps the viewer to visualize the 3-D effect.



x: out of page towards you
y: right
z: up



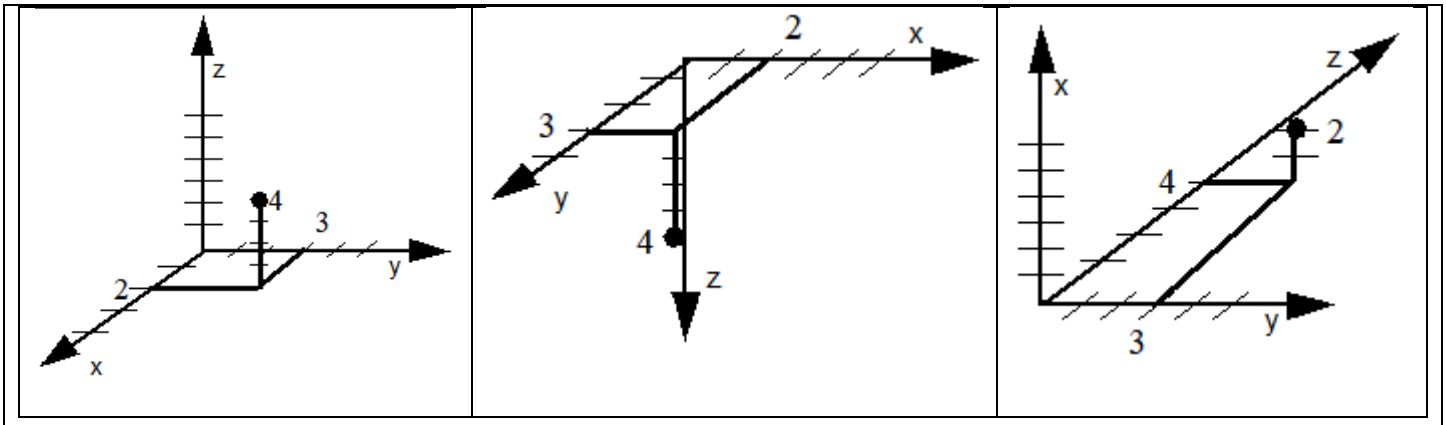
x: right
y: out of page towards you
z: down



x: up
y: right
z: into page away from you

Figure 10. Several different right handed 3-D coordinate frames with tick marks to indicate scale.

Figure 11 shows the point (2,3,4) plotted on the different axes of Figure 10. The technique is quite straightforward if two of your axes form a plane parallel with the ground. First, draw lines to indicate the projection of the point on that plane. Then, draw a line through that point that is parallel to the remaining axis, add tick marks to it, and plot your point.



(a) (b) (c)

Figure 11. The point (2,3,4) plotted on different axes.

For example, in Figure 11(a) the x and y axes form the *groundplane*, and so we draw lines to indicate where (2,3,0) would be. Then, we draw a line through the point (2,3,0) that is parallel to the z axis, add tick marks to it, and finally plot our point. Although Figure 11(b) looks different, the x and y axes still form the groundplane and so the procedure is virtually the same. The only difference is that the tick marks on the z axis have been left out because when they are included, they are difficult to distinguish from the tick marks on the vertical line that connects to the point (2,3,4). In Figure 11 (c), the y and z axes form the groundplane. Thus, we first plot the point (0, 3, 4), then draw a line through the point (0,3,4) parallel to the x axis, add tick marks to it, and again plot our point (2,3,4).

2. Working with Multiple Coordinate Frames

2.1. Converting points from one coordinate frame to another

Our ultimate goal is to be able to move between coordinate frames with ease. Practically speaking, what does that mean? It means that if I know the location of a particular point in one coordinate frame, I can rapidly tell you it's location relative to a different coordinate frame.

For example, consider an airport scenario. The coordinate frame of the airport might have its origin at the base of the control tower, with the x axis pointing North, the y axis pointing West, and the z axis pointing up. While this is a useful coordinate frame for the air traffic controllers to use, a pilot may be more interested in where objects are relative to her airplane. Thus, we might have two coordinate frames, "tower coordinates" and "plane coordinates" as illustrated in Figure 12. We use subscripts to distinguish between x_t, y_t, and z_t (the tower coordinate frame) and x_p, y_p, and z_p (the plane coordinate frame).

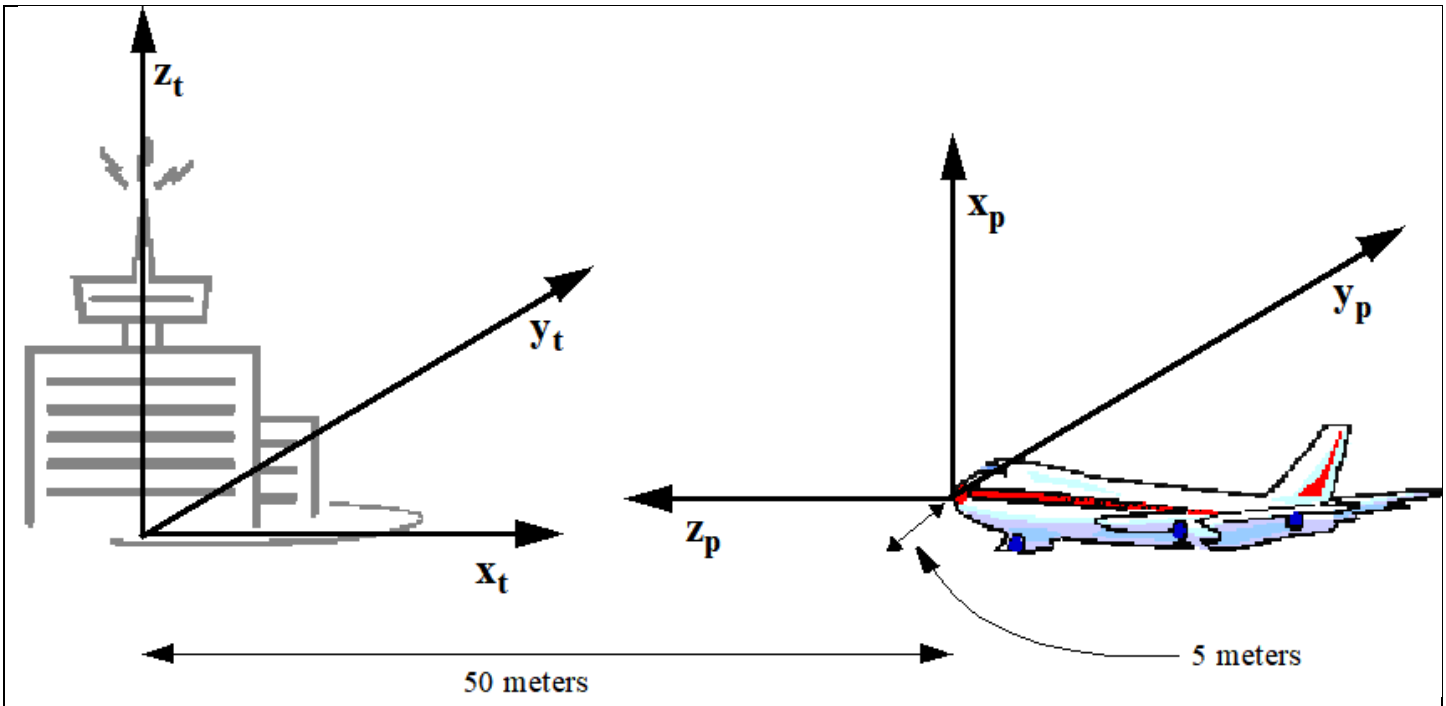


Figure 12. The plane is on the ground preparing for takeoff. Tower coordinates are centered at the base of the air traffic control tower. x points North, y points West, and z points up. The airplane coordinate system is centered at the nose of the plane. The x axis points towards the top of the plane, the y axis points out to the right as you are sitting in the captain's chair, and the z axis points straight out the front of the plane.

When the plane is stopped on the runway as depicted in Figure 12, the nose of the plane might be at location (50, 5, 0) in tower coordinates, but it is at the origin (location (0, 0, 0)) in plane coordinates. Similarly, the base of the tower is at location (0, 0, 0) in tower coordinates, but at location (0, -5, 50) in plane coordinates.

Note that the location of the nose of the plane is fixed with respect to the plane, but not with respect to the tower. When the plane begins to take-off as depicted in Figure 13, its nose is still at location (0,0,0) in plane coordinates, but it is at location (30, 15, 5) in tower coordinates.

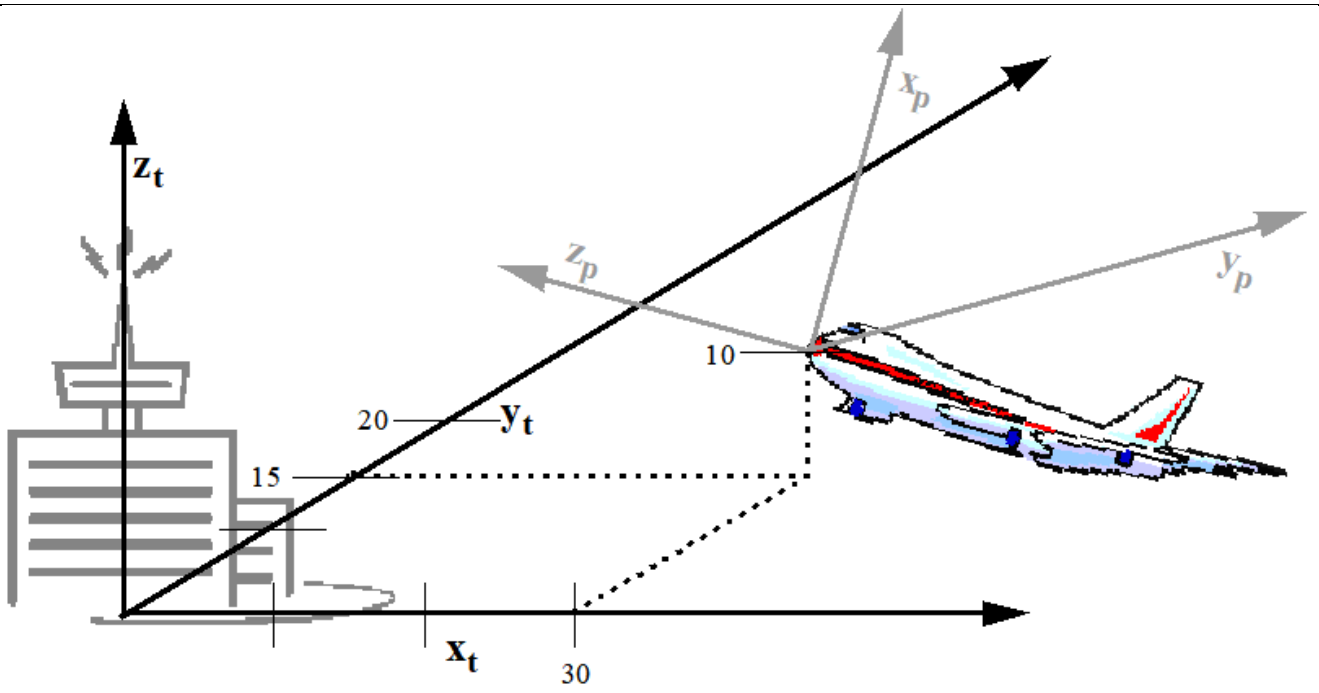


Figure 13. When the plane takes-off, its nose is still at location (0,0,0) in plane coordinates, but it is at a different location in tower coordinates. To determine where it is in tower coordinates, the tower's x and y axes have been extended, and the nose of the plane has been plotted in the manner of Figure 11. Thus we see that the nose of the plane is at location (30, 15, 10) in tower coordinates.

2.2. A very important note

While we may say we are "converting a point" from one coordinate frame to another **our point does not move!** We do all of our conversions at a particular instant in time so that the relative positions of all of the coordinate frames are fixed and the point is fixed for the calculation.

So we CAN ask questions like the following:

- Suppose that at exactly three minutes and 22 seconds after takeoff on January 1, 2020, we know the relative positions of the PHL Air Traffic Control Tower coordinate frame and the Flight ROW222 airplane coordinate frame. Furthermore, we know that the nose of the plane is at location (0,0,0) in plane coordinates. What is the location of the nose of the plane (at this exact point in time) in tower coordinates?

Yes, the nose of the plane may be continuously moving over time, but at exactly 3 minutes and 22 seconds after takeoff on January 1st it was in one particular place and we can talk about the location of that place relative to different coordinate frames.

2.3. The Use of Multiple Coordinate Frames in Robotics

It is very common in robotics to use two or more coordinate frames to solve a problem. Suppose the airplane in Figure 12 were automatically controlled. It would be very useful to keep track of some things in tower coordinates. For example, the altitude of the plane is simply the z coordinate of its location in tower coordinates. It also would be useful to keep track of other things in airplane coordinates. For example, the direction the plane should head to in order to avoid a mountain. Indeed, for many mobile robot applications, it is desirable to know the locations of objects in both "world coordinates" and "robot coordinates."

Multiple coordinate frames are also useful in traditional robotics. For example, consider the simple robot arm depicted in Figure 14. If we want to have the gripper pick a widget up off of a table, then we need to figure out the widget's location. Perhaps we have a camera that we use to initially determine the location of the widget (in camera coordinates). We might need to transform that location into world coordinates to evaluate if it is accessible to the robot at all, and to gripper coordinates to determine when we should close the jaws of the gripper.

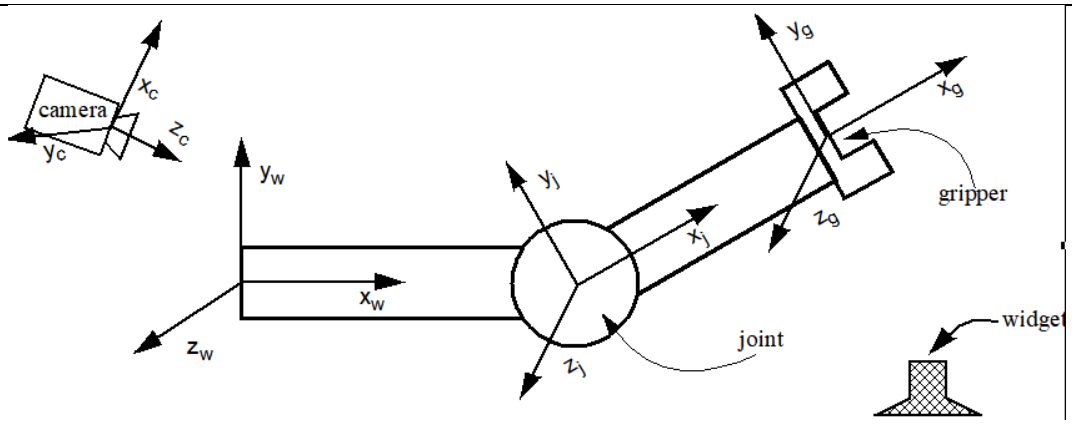


Figure 14. A very simple robot arm with one joint and one gripper.
The world, camera, joint, and gripper coordinate frames are indicated.

3. Introduction to Frame Transformations

3.1. Language / Notation: Moving one frame into alignment with another

- We will call the process that moves frame w into alignment with frame r **"The Frame Transformation From w To r"** .
- We will abbreviate this as:

$$F_w^r$$

- **Important Note**: this F notation was created for use in this document and is not commonly used. But you will find it useful!!

3.2. Notation: Translating a frame by a particular (x, y, z) value

- We will represent the process of translating a frame by some combination of movement in the directions of its x, y, and z axes by writing **Trans (x,y,z)**

3.3. A First Example

In order to figure out the location of a fixed point in a second coordinate frame given its location in another frame, we obviously need to know the relative positions of the two frames. Let's start with a simple example. Consider the world and robot coordinate frames shown in Figure 15. The origin of the robot frame is located at the point (0,3,0) in world coordinates.

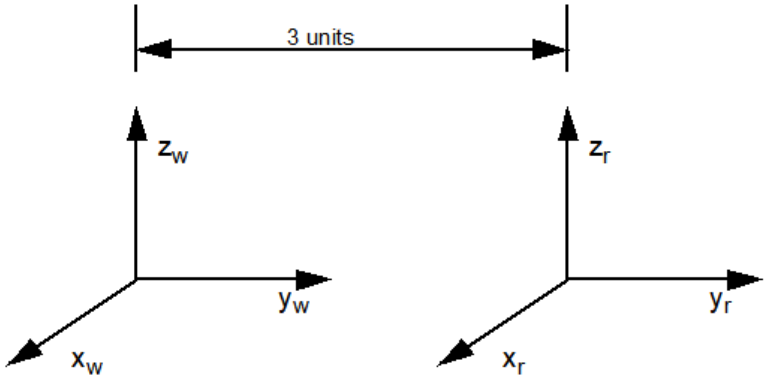


Figure 15. Two coordinate frames, world (w) and robot (r). The origin of the robot frame is located at the point (0,3,0) in world coordinates.

Question: How could I move the w coordinate frame so that it aligns with the r coordinate frame (using w coordinates as a reference?)

Answer: Translate the w frame 3 units in the direction of w's y axis. So we would say that

$$F_w^r = \text{Trans}(0,3,0)$$

Similarly, we could think about how to move the robot coordinate frame into alignment with the world coordinate frame. In this case, we'd have to move 3 units in the negative y direction (because the robot's y axis points off to the right, but we want to move to the left so that's negative) so we would say that:

$$F_r^w = \text{Trans}(0,-3,0)$$

3.4. A second example

The relative positions of the s and t coordinate frames are shown in Figure 16.

- **Question**: What is F_s^t
- **Answer**: F_s^t = Trans (5, -1, 0)

- **Question**: What is F_t^s?
- **Answer**: F_t^s = Trans (-5, 1, 0)

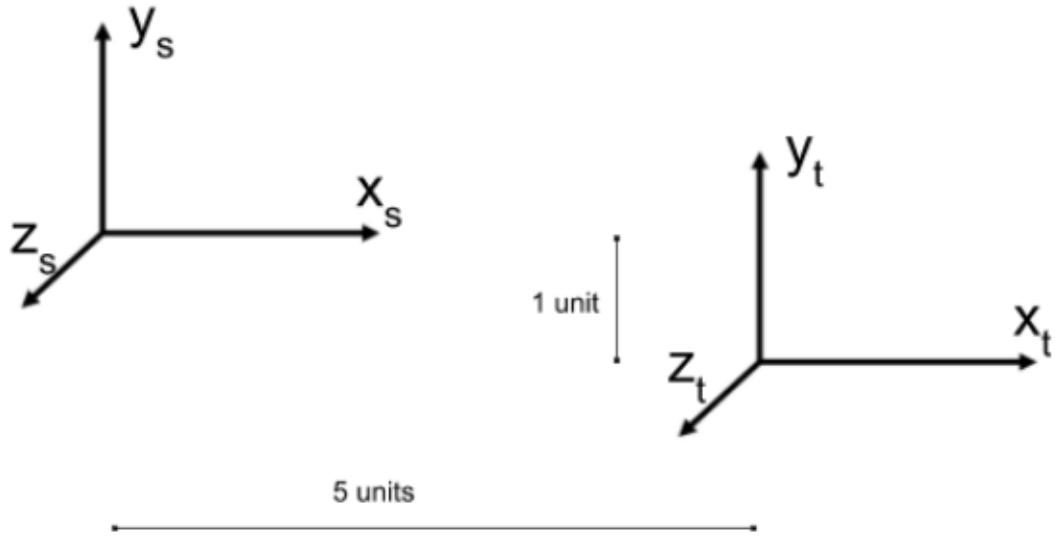


Figure 16. The s and t coordinate frames differ by a translation in both the x and y directions

3.5. A Third Example

The relative positions of the c and m coordinate frames are shown in Figure 17. What are F_c^m and F_m^c ?

- $F_c^m =$ Trans (5, -4, -1)
- $F_m^c =$ Trans (-5, 4, 1)

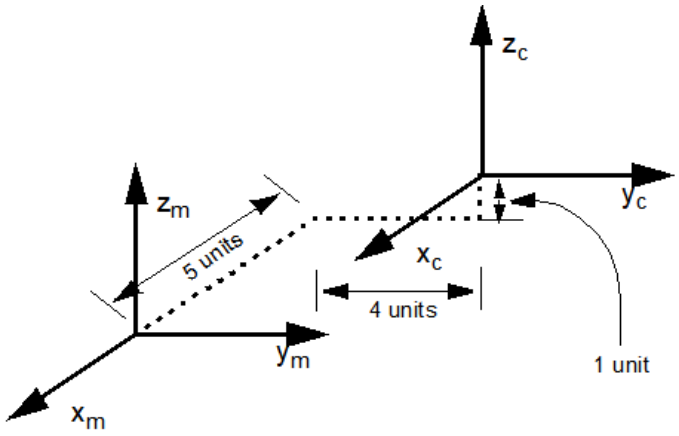


Figure 17. Two coordinate frames that differ by only a translation. To get from car coordinates (c) to mountain coordinates (m) you must translate 5 units along the car's x axis, -4 units along the car's y axis, and -1 unit along the car's z axis.

4. Representing Frame Transformations as Matrices

For reasons that will be made clear shortly, we are going to choose to represent Frame Transformations as 4x4 matrices. We will represent Trans(a,b,c) as follows:

$$\text{Trans(a,b,c)} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It's worth noting that if we changed the a, b, and c to be zeros, we would have a 4x4 identity matrix.

4.1. Examples

Just to offer a few concrete examples: For figure 15 we said:

$$F_r^w = \text{Trans(0,-3,0)}$$

So we can now say:

$$F_r^w = \text{Trans(0,-3,0)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Or consider the last example in Figure 17 – we said that

- $F_c^m =$ Trans (5, -4, -1)

So we now can say:

- $F_c^m =$ Trans (5, -4, -1) $= \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

5. Representing Points as Vectors

Up to this point, we have been using the traditional (x,y,z) notation to represent points in 3-D. However, for the remainder of this document, we are going to use a vector notation to represent points. The point (x,y,z) is represented as the vector

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

The 1 is a weighting factor. So the vectors

$$\begin{bmatrix} 2x \\ 2y \\ 2z \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 257x \\ 257y \\ 257z \\ 257 \end{bmatrix}$$

both represent that same point (x, y, z). Most of the time we will simply use a weighting factor of 1. Consider the more concrete example depicted in Figure 13. The plane is at location (30, 15, 10) in tower coordinates. We will normally represent that location as

$$\begin{bmatrix} 30 \\ 15 \\ 10 \\ 1 \end{bmatrix}$$

however it can also be represented as any of the following:

$$\begin{bmatrix} 60 \\ 30 \\ 20 \\ 2 \end{bmatrix} \quad \begin{bmatrix} -30 \\ -15 \\ -10 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 75 \\ 37.5 \\ 25 \\ 2.5 \end{bmatrix}$$

As well as an infinite number of other vectors. In order to save space in this document, we will often write the point (x,y,z) horizontally like this: $[x \quad y \quad z \quad 1]^T$, i.e. as the transpose of our vertical vector.

6. Rapidly Computing the location of points in different Coordinate Frames

At this point, if we are given two coordinate frames that only differ by a translation, we can easily compute the frame transformation between them, for example, consider the world and robot coordinates example from Figure 15, redrawn in Figure 18 below)

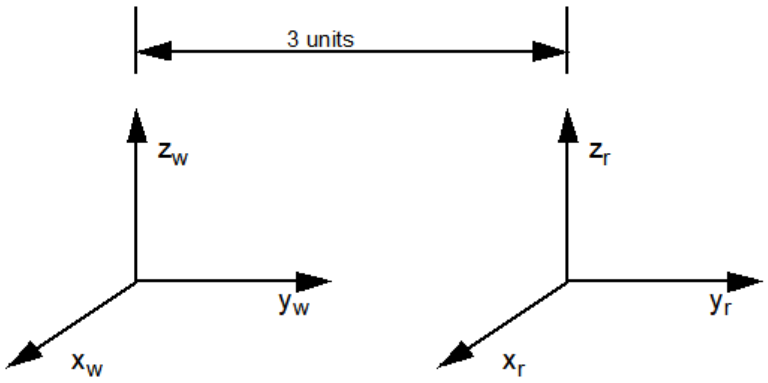


Figure 18 (the same as Figure 15). Two coordinate frames, world (w) and robot (r). The origin of the robot frame is located at the point (0,3,0) in world coordinates.

The table in Figure 19 shows some sample points in world coordinates, and their corresponding values in robot coordinates. For the moment, ignore the third column of Figure 19, and just look at the first two columns. Notice that any point $[a \;\; b \;\; c \;\; 1]^T$ in world coordinates is the same as the point $[a \;\; (b-3) \;\; c \;\; 1]^T$ in robot coordinates.[1]

One way to compute the location of a point in robot coordinates given its location in world coordinates for the system in Figure 15 is to subtract 3 from the y value in world coordinates.

More surprisingly, another way to compute the location of a point in robot coordinates given its location in world coordinates to pre-multiply the point's location in world by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The third column of Figure 19 does exactly this and results in the same answer!

[1] Similarly, any point $[d \;\; e \;\; f \;\; 1]^T$ in robot coordinates is the same as the point $[d \;\; (e+3) \;\; f \;\; 1]^T$ in world coordinates.

Location of a Point in World Co-ordinates of Figure 15	Location of the Same Point in Robot Coordinates of Figure 15	Pre-multiplying the point in world coordinates by $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$
$\begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$
$\begin{bmatrix} 5 \\ 10 \\ 15 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 84 \\ 81 \\ 84 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \\ 15 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 15 \\ 1 \end{bmatrix}$
$\begin{bmatrix} 84 \\ 84 \\ 84 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 84 \\ 81 \\ 84 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 84 \\ 84 \\ 84 \\ 1 \end{bmatrix} = \begin{bmatrix} 84 \\ 81 \\ 84 \\ 1 \end{bmatrix}$
$\begin{bmatrix} 4 \\ -4 \\ 4 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 4 \\ -7 \\ 4 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -4 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ 4 \\ 1 \end{bmatrix}$
$\begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix}$	$\begin{bmatrix} a \\ b-3 \\ c \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b-3 \\ c \\ 1 \end{bmatrix}$

Figure 19. Converting a point between world and robot coordinates as depicted in Figure 15.

7. The Homogeneous Transformation Matrix

We call the 4x4 matrix that we can use to transform a point in **foo** coordinates into **baz** coordinates the **Homogeneous Transformation from foo to baz Coordinates** and we use the following notation to represent this transformation:

$$T_{foo}^{baz}$$

8. Review

It is very important to note that we have been considering two distinct concepts in our previous discussion:

1. How do move frame **a** in such a way that it aligns with frame **b**

   i.e. what is F_a^b? (This is something we can often compute by just looking at the two frames)

2. How do we compute the location of a given point relative to frame **a** coordinates if we already know its location relative to frame **b** coordinates?

   i.e. what is T_b^a?

Make sure this makes sense before continuing on!!!

8.1. Strange Coincidence? Or something more …

We said the Homogeneous Transformation from world to robot coordinates for the example in Figure 19 was:

$$T_w^r = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

But wait!!!! Go back a couple of pages. Doesn't that matrix look familiar?? Earlier we said that:

$$F_r^w = \text{Trans}(0,-3,0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So, at least for this case, it seems that T_w^r is the same as F_r^w

In other words for this case, the Homogeneous transformation from world to robot coordinates is the same as the Frame transformation from robot to world coordinates

8.2. Let's see if this works on another example.

Figure 20 is a duplicate of figure 17

$F_m^c =$ Trans (-5, 4, 1)

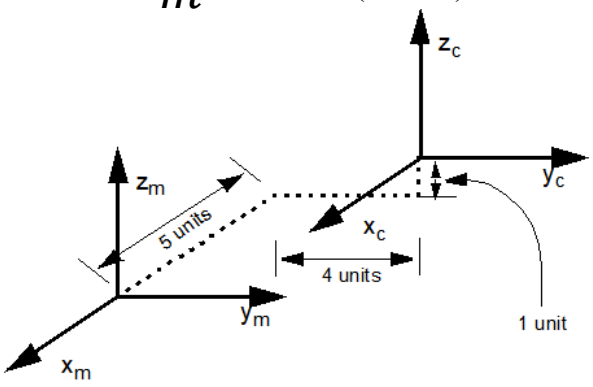


Figure 20. (the same as Figure 17) Two coordinate frames that differ by only a translation. To get from car coordinates (c) to mountain coordinates (m) you must translate 5 units along the car's x axis, -4 units along the car's y axis, and -1 unit along the car's z axis.

Suppose we want to convert points from mountain coordinates to car coordinates. We're starting to suspect that the matrix that allows us to do that would be the same one as the Frame transformation from car to mountain coordinates. What did we say that was?

- $F_c^m =$ Trans (5, -4, -1) $= \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Let's try premultiplying some points in mountain coordinates by F_c^m and see what we get….

Take a look at Figure 21…. It seems to work. Of course, this isn't proof that it will always work, but it's promising.

Location of a point in Figure 17's mountain coordinates	Location of the same point in Figure 17's car coordinates (computed by just looking at it)	Pre-multiplying the point in mountain coordinates by $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 5 \\ -4 \\ -1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ -1 \\ 1 \end{bmatrix}$
$\begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 5 \\ -1 \\ -1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ -1 \\ 1 \end{bmatrix}$
$\begin{bmatrix} 5 \\ 10 \\ 15 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 10 \\ 6 \\ 14 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \\ 15 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \\ 14 \\ 1 \end{bmatrix}$
$\begin{bmatrix} 84 \\ 84 \\ 84 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 89 \\ 80 \\ 83 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 84 \\ 84 \\ 84 \\ 1 \end{bmatrix} = \begin{bmatrix} 89 \\ 80 \\ 83 \\ 1 \end{bmatrix}$
$\begin{bmatrix} 4 \\ -4 \\ 4 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 9 \\ -8 \\ 3 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -4 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ -8 \\ 3 \\ 1 \end{bmatrix}$
$\begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix}$	$\begin{bmatrix} a+5 \\ b-4 \\ c-1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix} = \begin{bmatrix} a+5 \\ b-4 \\ c-1 \\ 1 \end{bmatrix}$

Figure 21. Converting points from the mountain coordinate frame of Figure 17 to the car coordinate frame of Figure 17.

9. The Relationship between F and T Transformations

OK, let's break the suspense. It turns out that for <u>any</u> pair of frames, a and b, it is always the case that

$$F_a^b \text{ is equal to } T_b^a$$

We know how to figure out that 4x4 matrix for frame transformations that differ only by a translation. Let's keep going and understand what we need to do if we also have some rotations.

10. Coordinate Frames that Differ by a Rotation Around the Z Axis

Consider the two frames depicted in Figure 22. By looking at Figure 22(a), we can see that $x_k = y_j$, $z_k = z_j$, and $x_j = -1 * y_k$. Let's look at a few example points:

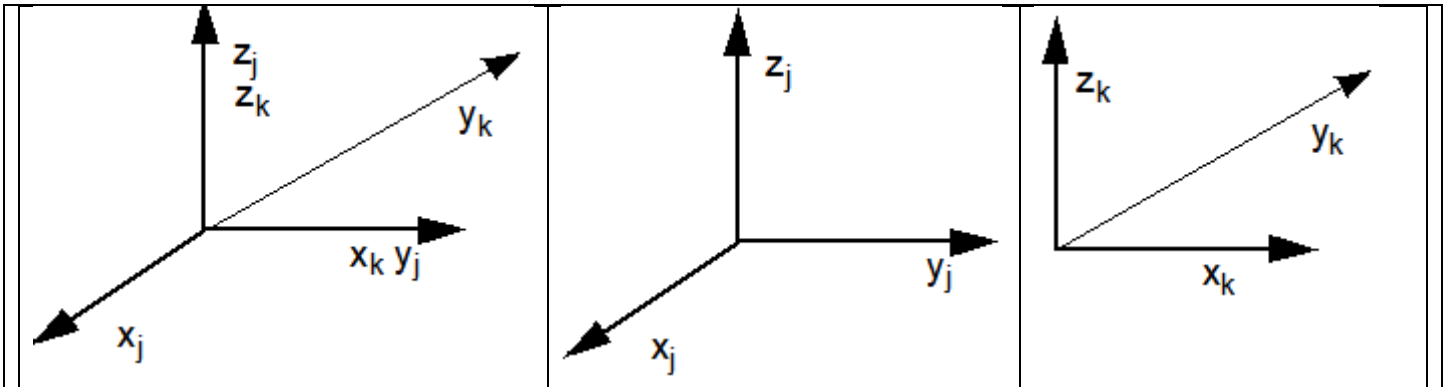
- The origin of the j axis in j coordinates: the point (0,0,0) (aka $[0\ \ 0\ \ 0\ \ 1]^T$) is the same as the origin of the j axis in k coordinates (0,0,0).
- The point (a,b,c) in j coordinates is located at (b, -a, c) in k coordinates.

How do we align the k coordinate frame with the j coordinate frame (i.e. what is F_k^j) ? We, if we look at the axes, we can see that we would need to perform a rotation about k's z axis by -90°. (use your right hand to confirm that it really is negative 90°!)

Is there a generic matrix that we can use for rotations about the z axis? Of course!

$$\text{Rot z } (\theta) = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So that would mean that $\text{Rotz}(-90^o) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$



(a) the j and k coordinate frames overlaid on each other.

(b) the j coordinate frame, shown on its own for clarity.

(c) the k coordinate frame, shown on its own for clarity.

Figure 22. Two coordinate frames, j and k, that differ only by a rotation about the z axis. To transform the k coordinate frame into the j coordinate frame, we rotate by -90° about k's z axis.

Let's use that matrix! When we compute the matrix product of our matrix with $[0\ 0\ 0\ 1]^T$, we get $[0\ 0\ 0\ 1]^T$ as expected. compute the matrix product of our matrix with $[a\ b\ c\ 1]^T$, we get $[b\ -a\ c\ 1]^T$ as expected.

At this point we know that we can design a matrix to convert points between two coordinate frames that only differ by a translation, or by a rotation about the z axis. It should not surprise you to learn that you can also design matrices to convert points between two coordinate frames that only differ by a rotation about the x or y axes too.

A summary of all of these matrices can be found in figure 23.

If to align the k coordinate frame with the j coordinate frame you have to: F_k^j	Then to convert a point in j coordinates into a point in k coordinates, premultiply that point by: T_j^k	The nickname for this transformation is:
Translate • along k's x axis by a • along k's y axis by b • along k's z axis by c	$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$	Trans (a, b, c)
Rotate about k's x axis by θ	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	Rot x (θ)
Rotate about k's y axis by θ	$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	Rot y (θ)
Rotate about k's z axis by θ	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	Rot z (θ)

Figure 23. Summary of transformation matrices

11. Combining multiple transformations

So far we have learned how to create the matrix that will compute the coordinates of a point in one coordinate frame given the coordinates of that point in another coordinate frame, subject to the following condition: the two frames may only differ by a translation (along the 3 axes), or by a rotation about a single axis. It is indeed possible to convert points between two coordinate frames that differ, perhaps by two translations, or by a rotation then a translation and then another rotation.

The key to understanding how to do this is to understand that there are two approaches to view any sequence of translations and rotations. Both approaches give you the same final result, but sometimes one is easier than the other. We'll discuss each one separately. The first way is known as using a "moving axis" approach and the second is called using a "fixed axis" approach.

11.1. The moving axis approach

In moving coordinate systems, each step happens relative to the steps that have come before it. For example, Figure 24 shows the world and gripper coordinate frames for a particular robotic system. Note that not only is the gripper coordinate frame translated from the world coordinate frame, but there also must be some sort of rotation that caused the gripper's x axis to point up instead of out of the page towards you as the world coordinates do.

In the "moving axes" approach, we say that to get from world coordinates to gripper coordinates (F_w^g), you need to do the following sequence of moves:

1. Rotate about x_w by -90 degrees. Call this new frame intermediate frame 1, and we'll call its axes x_1, y_1, and z_1.
2. Rotate about the new z_1 by -90 degrees. Call this new frame intermediate frame 2, and we'll call its axes x_2, y_2, and z_2.
3. Translate by (0,0,5) relative to intermediate frame 2. This results in the gripper coordinate frame.

6.7 Computing the Transformation Matrix Using Moving Axes

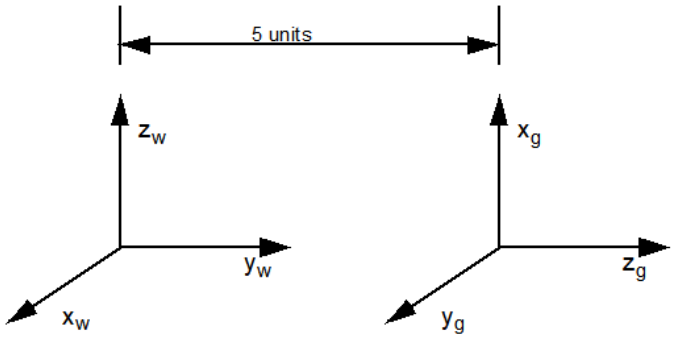


Figure 24. World and gripper coordinate frames for some robot. There are several ways we can think about transforming between the two coordinate frames.

When we use “moving axes” we list the moves that we did from left to right, compute the individual matrices for each part, and then multiply them together. For example, in this situation, we did the following steps:

1. Rot x (-90)
2. Rot z (-90)
3. Trans (0,0,5)

So, listing our equations from left to right we have:

Rot x(-90) * Rot z(-90) * Trans(0,0,5) =

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The resulting matrix, F_w^g , will transform a point from gripper coordinates to world coordinates (i.e., it’s also T_g^w). For example, consider the point (1, 2, 3) in gripper coordinates. We compute that in world coordinates by premultiplying by our new matrix as follows:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 1 \\ 1 \end{bmatrix}$$

Which you should be able to verify is correct by looking at **Error! Reference source not found.**.

11.2. Using Fixed Axes

The alternative approach is the "fixed axes" approach. In this technique, all of your moves are relative to the original world coordinate frame. In the "fixed axes" approach, the picture is still as depicted in Figure 24, but this time the sequence of steps is:

1. Rot x(-90) (about the world's x axis)
2. Rot y (-90) (about the world's y axis)
3. Trans(0,5,0) (relative to the (x, y, and z directions specified by the world's axes)

When we using "fixed axes" computation, (i.e. each new rotation is relative to the original coordinate frame), we list our equations from right to left. So the above sequence of steps turns into

Trans (0,5,0) * Rot y (-90) * Rot x(-90)

The matrices for this product are as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Check it out! this is the same equation we got when we did the computation using moving axes! (Phew!)

11.3. Fixed Axes: IMPORTANT WARNING

Rotations can get very complicated when you are using a fixed axis approach if your current axes do not share the same origin as the original axes that you are rotating about.

For example, consider the axes of Figure 25 and suppose you want to rotate the w frame by -90 degrees about the g's y axis. The rotation is depicted in Figure 26 is not what one might expect! To visualize what is happening, you need to imagine that the two frames are locked together as you perform the rotation.

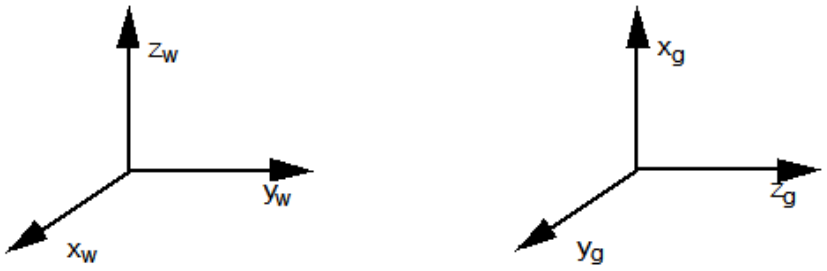


Figure 25. Two coordinate frames.

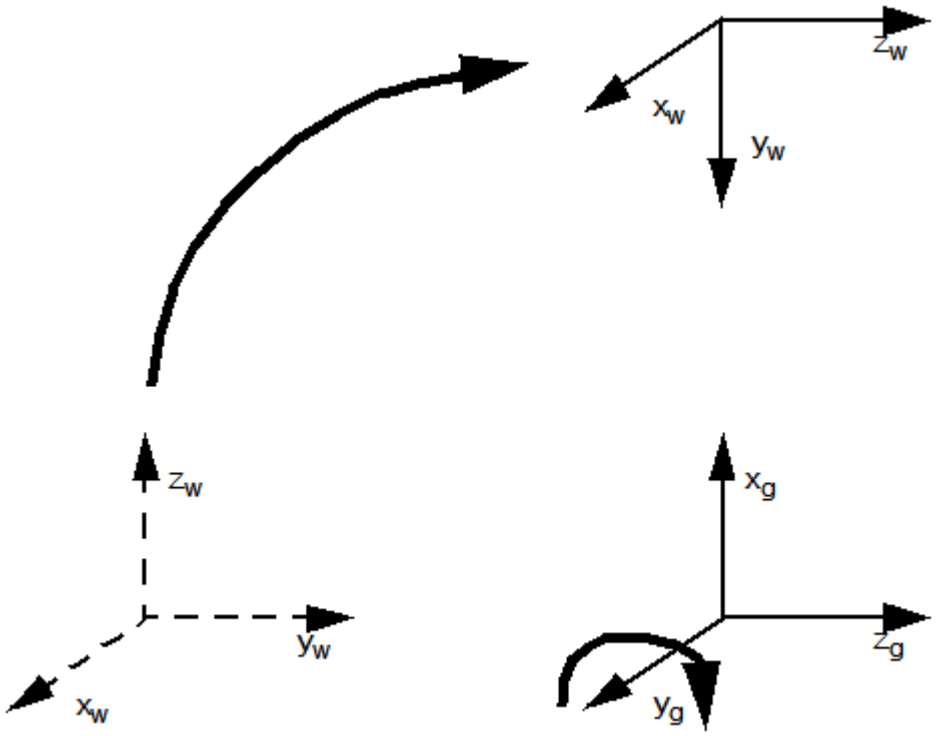


Figure 26. A rotation of the original w coordinate frame from Figure 25 (shown as dotted arrows) about the y_g coordinate frame from Figure 25.

12. Forward Kinematics

One task that we often wish to do is the following: given the joint angles of a robot arm, compute the transformation between world and gripper coordinates. Of course, at this point given any fixed joint angles, we already have the tools to compute this transformation. However, what we really would like to do is to come up with a transformation matrix that is a function of the joint angles of the robot.

12.1. A First Example

Consider the robot arm given in Figure 27. This arm has two links (of length L1 & L2) and one joint which can rotate about its Z axis. There are 3 coordinate frames, world coordinates, joint coordinates, and gripper coordinates. Figure 28 contains a picture of the same arm, but this time, the joint has been rotated by 30 degrees (about its z axis - the only axis about which it can rotate).

Now look at the * in both figures. It should be clear that the world coordinates of the * do not change between Figure 27 and Figure 28, but the location of * in link coordinates does change, as does its location in gripper coordinates.

There is one point in this image whose location does not move in any frame as the joint moves. The point located at the very center of the joint (i.e. with Joint coordinates (0,0,0)) is always at world coordinates location (L1, 0, 0), and always at (-L2, 0, 0) in gripper coordinates, no matter how you move the joint.

Suppose that we want to be able to convert between gripper coordinates and world coordinates, as a function of the angle of the joint.

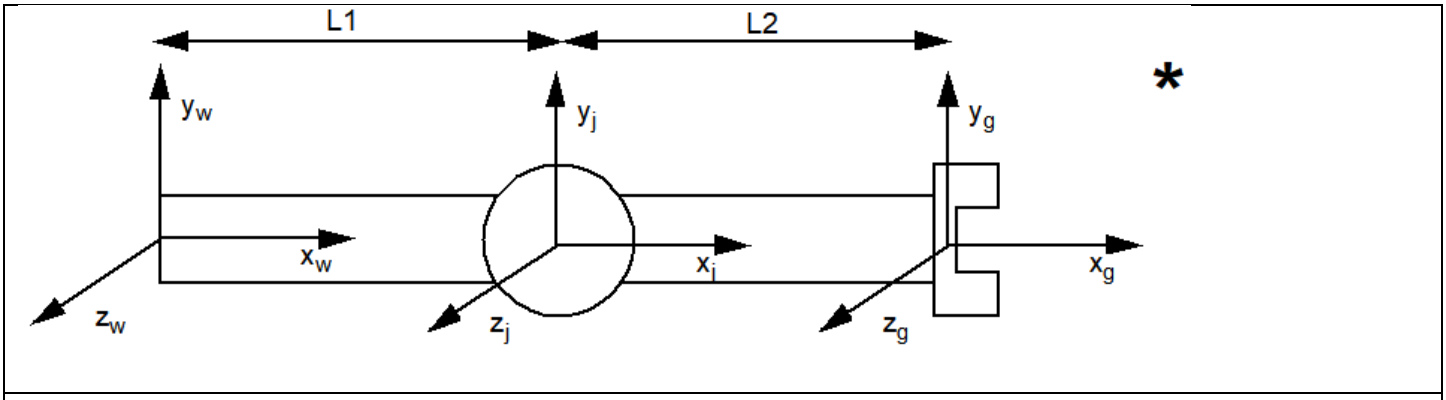


Figure 27. A simple robot

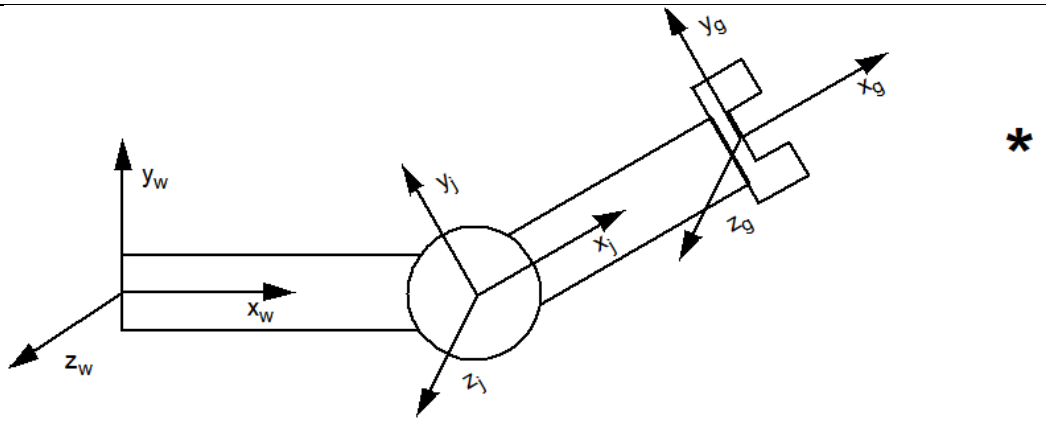


Figure 28. The arm from Figure 27 with the joint rotated by 30 degrees.

Let's begin by just looking at Figure 27 again and considering the case where the joint is not rotated at all. In this case, to convert a point from gripper coordinates to world coordinates all we do is add L1+L2 to whatever the x value is. e.g., suppose the * is located at (4, 3, 0) in gripper coordinates. Then it's obviously located at (4+L1+L2, 3, 0) in world coordinates.

But wait! To convert from gripper to world coordinates isn't as simple as that. Because if the joint is rotated as shown in Figure 28, then it's no longer a simple addition of L1+L2.

What we want is a way to easily convert between gripper and world coordinates *as a function of joint angle*. Let's call the angle that the joint is rotated ψ. We want one matrix that has the variable ψ built into it, and if we plug in the value for ψ, that matrix will be our matrix that we multiply a point in gripper coordinates by to get the point in world coordinates.

Now let's do the math. To move our frame from world coordinates to gripper coordinates, we need to translate a distance of L1 along the x axis, and then rotate by whatever angle our joint is twisted to (e.g. 0 degrees in Figure 27 or 30 degrees in Figure 28). We're going to do it as relative motion, so we end up multiplying the matrices Trans(L1, 0, 0) by Rot z (ψ) from left to right. So we have the following:

$$\begin{bmatrix} 1 & 0 & 0 & L1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\psi & -\sin\Psi & 0 & 0 \\ \sin\Psi & \cos\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\Psi & 0 & L1 \\ \sin\Psi & \cos\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 29. Performing a frame transformation from world coordinates to joint coordinates

But of course, this only moves us from world coordinates to joint coordinates. We wanted to move our frame from world coordinates to gripper coordinates. Once we have a frame in joint coordinates, moving it to gripper coordinates is simply a translation of [L2, 0, 0]. So we need to multiply the two matrices above by Trans[L2, 0, 0].

$$\begin{bmatrix} 1 & 0 & 0 & L1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\psi & -\sin\Psi & 0 & 0 \\ \sin\Psi & \cos\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & L2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\Psi & 0 & L2\cos\psi + L1 \\ \sin\Psi & \cos\psi & 0 & L2\sin\Psi \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 30. Performing a frame transformation from world coordinates to gripper coordinates.

Now, our final equation looks pretty messy, but it's not too bad. Suppose that ψ is 0 (i.e. we've got Figure 27). Then we have:

$$\begin{bmatrix} \cos\psi & -\sin\Psi & 0 & L2\cos\psi + L1 \\ \sin\Psi & \cos\psi & 0 & L2\sin\Psi \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & L2+L1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 31. Performing a frame transformation from world coordinates to gripper coordinates when the joint angle is zero

Remember that we computed how to transform a frame from world coordinates to gripper coordinates. Which turns points in gripper coordinates into points in world coordinates.

Wow! Figure 31 is actually exactly what we predicted it would be. Look at it. It's the matrix Trans(L1+L2, 0, 0).

12.2. A Second Example

Just for fun, let's compute the transformation when the joint is rotated by 90 degrees, so the gripper is pointing straight up in the air. Well, we plug in 90 degrees for ψ and we get:

$$\begin{bmatrix} \cos\psi & -\sin\Psi & 0 & L2\cos\psi + L1 \\ \sin\Psi & \cos\psi & 0 & L2\sin\Psi \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & L1 \\ 1 & 0 & 0 & L2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 32. Performing a frame transformation from world coordinates to gripper coordinates when the joint angle is 90 degrees

Does this make sense? Think about it. Suppose that you have a point that is located right at the origin of the gripper. So in gripper coordinates, it's location is (0,0,0). Where is that in world coordinates? Let's multiply:

$$\begin{bmatrix} 0 & -1 & 0 & L1 \\ 1 & 0 & 0 & L2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} L1 \\ L2 \\ 0 \\ 1 \end{bmatrix}$$

Figure 33. Transforming a point from gripper coordinates to world coordinates when the joint angle is 90 degrees

Hey, it works!

13. Some Tips on Using Mathematica

Clearly it's a major pain to do anything with more than one or two manipulations by hand. Mathematica can really help. Here are a couple of hints on how I computed the information for this document. I'm assuming some familiarity with the very basics of Mathematica.

For starters, I wrote Mathematica functions to represent each of the rotation and translation matrices. Just in case you haven't seen a Mathematica function before, here's how you write a function that takes to variables, a and b, and returns the mean (average) of a and b (note: the underscore defines what's a variable. Don't use underscores for anything else or you'll have problems with your code:

```
avg[a_, b_] := ((a+b)/2)
```

Matrices are represented as lists of lists. So, here is my function for Rotx:

```
Rotx[theta_] := (
                    {{1, 0, 0, 0},
                     {0, Cos[theta], -1*Sin[theta], 0},
                     {0, Sin[theta], Cos[theta], 0},
                     {0, 0, 0, 1}}
                )
```

Remember that Mathematica uses radians. So to Rotx by 90 degrees I say

```
Rotx[Pi/2]
```

Now, let's use this function. If I want, I could run my function to see what the matrix is to Rotate x by 0 (this should be the identity matrix, right?!) (Note: From this point onwards, I'll show you the "in" and "out" messages from Mathematica so you can distinguish my input from its output)

```
In[2]:= Rotx[0]

Out[2]= {{1,0,0,0},{0,1,0,0},{0,0,1,0},{0,0,0,1}}
```

Hmmm, correct, but not really very pleasing to the eye. We can use the built-in MatrixForm function to display matrices with a little more beauty:

In[3]:=

```
MatrixForm[Rotx[0]]
```

Out[3]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So what do we get when we rotate about x by Pi/2?:

In[4]:=

```
MatrixForm[Rotx[Pi / 2]]
```

Out[4]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Looking good!

8. References

Much of this tutorial is derived from "Essential Kinematics for Autonomous Vehicles" by Alonzo Kelly, Carnegie Mellon University Robotics Institute technical report number CMU-RI-TR-94-14, May 1994, available on the web at https://www.ri.cmu.edu/publications/essential-kinematics-for-autonomous-vehicles/

14. Acknowledgements

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