1 A few notes on Lecture 14

1.1 The distance bound

Recall that we want to bound

$$\begin{aligned}|Z_i - Z_{i-1}| = |E[f(X_1, \ldots, X_{i-1}, &X_i, X_{i+1}, \ldots, X_n) \\ -f(X_1, \ldots, X_{i-1}, &\widehat{X}_i, X_{i+1}, \ldots, X_n) \mid X_1, X_2, \ldots, X_i]|\end{aligned} \tag{1}$$

Note that we do not want to just use the $2\sqrt{2}$-Lipschitz property, since that will be too weak, and will only give us

$$\Pr[|f - Ef| \leq \lambda] \leq \exp\{-\lambda^2/O(n)\}.$$

We want something much better!

Claim 1.1

$$\begin{aligned}&f(X_1, \ldots, X_{i-1}, X_i, X_{i+1}, \ldots, X_n) - f(X_1, \ldots, X_{i-1}, \widehat{X}_i, X_{i+1}, \ldots, X_n) \\ &\leq 2\big(\min_{j \neq i} d(X_i, X_j) + \min_{j \neq i} d(\widehat{X}_i, X_j)\big).\end{aligned}$$

Proof: Let $A = X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n$ be all the points except X_i and $\widehat{X}_i$, and let $T(A)$ be the optimal TSP tour on A. Note that $f(A) = \text{length}(T(A))$. For any point x and set S, define $d(x, S) = \min_{y \in S} d(x, y)$.

Note that if take $T(A)$, and to it we add two edges from X_i to its closest point in A, and from $\widehat{X}_i$ to *its* closest point in A, then we have an Eulerian graph on the $n+1$ points $A \cup \{X_i, \widehat{X}_i\}$ of total length at most

$$f(A) + 2\big(d(X_i, A) + d(\widehat{X}_i, A)\big). \tag{2}$$

Using the triangle inequality to shortcut repeated vertices gives us TSP tour of length at most (2), and hence the length of the optimal tour on $A \cup \{X_i, \widehat{X}_i\}$ has length

$$f(A \cup \{X_i, \widehat{X}_i\}) \leq f(A) + 2\big(d(X_i, A) + d(\widehat{X}_i, A)\big). \tag{3}$$

Finally, using the fact that

$$\begin{aligned}f(A) \leq f(A \cup \{X_i\}) \leq f(A \cup \{X_i, \widehat{X}_i\}) \\ f(A) \leq f(A \cup \{\widehat{X}_i\}) \leq f(A \cup \{X_i, \widehat{X}_i\})\end{aligned}$$

implies that

$$\begin{aligned}|f(A \cup \{X_i\}) - f(A \cup \{\widehat{X}_i\})| &\leq f(A \cup \{X_i, \widehat{X}_i\}) - f(A) \\ &\leq 2\big(d(X_i, A) + d(\widehat{X}_i, A)\big),\end{aligned} \tag{4}$$

the last inequality using (3). This is just a rephrasing of the claim that we want to prove. ■

Corollary 1.2 *Define* $B = \{X_j \mid j > i\}$. *Then*

$$f(X_1, \ldots, X_{i-1}, X_i, X_{i+1}, \ldots, X_n) - f(X_1, \ldots, X_{i-1}, \widehat{X}_i, X_{i+1}, \ldots, X_n) \leq 2\big(d(X_i, B) + d(\widehat{X}_i, B)\big). \tag{5}$$

Proof: The quantity on the right of (5) is larger than the quantity on the right of (4), since $B \subseteq A$. ■

Lemma 1.3

$$|Z_i - Z_{i-1}| \leq 2\big(E[d(X_i, B) \mid X_i] + E[d(\widehat{X}_i, B)]\big). \tag{6}$$

Proof: Plug in the result of Claim 1.1 into (1), and note that $d(X_i, B)$ is independent of $X_1, X_2, \ldots, X_{i-1}$, whereas $d(\widehat{X}_i, B)$ is independent of all $X_1, \ldots X_i$. Simplifying gives us the lemma. ■

1.2 The Rest of the Argument

Suppose we throw down $n-i$ points randomly in $\mathcal{U}$, and define the random variable $Q_i(x)$ to be the distance of x to the closest point amongst these $n-i$. Let R be the set of random points, and hence $Q_i(x) = d(x, R)$. We proved that

Claim 1.4 *For any $x \in \mathcal{U}$,*

$$E[Q_i(x)] \leq \frac{O(1)}{\sqrt{n-i}}. \tag{7}$$

Proof: This was the geometric proof, and I am going to omit it. ■

Hence we can upper bound both $E[d(X_i, B) \mid X_i]$ and $E[d(\widehat{X}_i, B)]$ by $\frac{O(1)}{\sqrt{n-i}}$. Finally, using (6), we get

$$|Z_i - Z_{i-1}| \leq 2\left(\frac{O(1)}{\sqrt{n-i}} + \frac{O(1)}{\sqrt{n-i}}\right) \tag{8}$$

This implies that we can set $c_i = \frac{O(1)}{\sqrt{n-i}}$ in Azuma's inequality, which is much better than the bound that we get just plugging in the $2\sqrt{2}$-Lipschitz-ness of f. Now $\sum_i c_i^2 = O(\log n)$, and hence we get

$$\Pr[|f - Ef| \leq \lambda] \leq \exp\{-\lambda^2 / O(\log n)\}, \tag{9}$$

as claimed.