2 Rigid Body Concepts

The goal of this section is to develop an analogue to equation (1–3), for rigid bodies. The final differential equation we develop is given in section 2.11. In order to do this though, we need to define a lot of concepts first and relations first. Some of the longer derivations are found in appendix A. In the next section, we'll show how to write the function `dydt` needed by the numerical solver `ode` to compute the derivative $\frac{d}{dt}\mathbf{Y}(t)$ developed in this section.

2.1 Position and Orientation

The location of a particle in space at time t can be described as a vector $x(t)$, which describes the translation of the particle from the origin. Rigid bodies are more complicated, in that in addition to translating them, we can also rotate them. To locate a rigid body in world space, we'll use a vector $x(t)$, which describes the translation of the body. We must also describe the rotation of the body, which we'll do (for now) in terms of a 3×3 rotation matrix $R(t)$. We will call $x(t)$ and $R(t)$ the *spatial variables* of a rigid body.

A rigid body, unlike a particle, occupies a volume of space and has a particular shape. Because a rigid body can undergo only rotation and translation, we define the shape of a rigid body in terms of a fixed and unchanging space called *body space*. Given a geometric description of the body in body space, we use $x(t)$ and $R(t)$ to transform the body-space description into world space (figure 1). In order to simplify some equations we'll be using, we'll require that our description of the rigid body in body space be such that the *center of mass* of the body lies at the origin, $(0,0,0)$. We'll define the center of mass more precisely later, but for now, the center of mass can be thought of as a point in the rigid body that lies at the geometric center of the body. In describing the body's shape, we require that this geometric center lie at $(0,0,0)$ in body space. If we agree that $R(t)$ specifies a rotation of the body about the center of mass, then a fixed vector r in body space will be rotated to the world-space vector $R(t)r$ at time t. Likewise, if p_0 is an arbitrary point on the rigid body, in body space, then the world-space location $p(t)$ of p_0 is the result of first rotating p_0 about the origin and then translating it:

$$p(t) = R(t)p_0 + x(t). \tag{2–1}$$

Since the center of mass of the body lies at the origin, the world-space location of the center of mass is always given directly by $x(t)$. This lets us attach a very physical meaning to $x(t)$ by saying that $x(t)$ is the location of the center of mass in world space at time t. We can also attach a physical meaning to $R(t)$. Consider the x axis in body space i.e. the vector $(1,0,0)$. At time t, this vector has direction

$$R(t)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

in world space. If we write out the components of $R(t)$ as

$$R(t) = \begin{pmatrix} r_{xx} & r_{yx} & r_{zx} \\ r_{xy} & r_{yy} & r_{zy} \\ r_{xz} & r_{yz} & r_{zz} \end{pmatrix}, \tag{2–2}$$

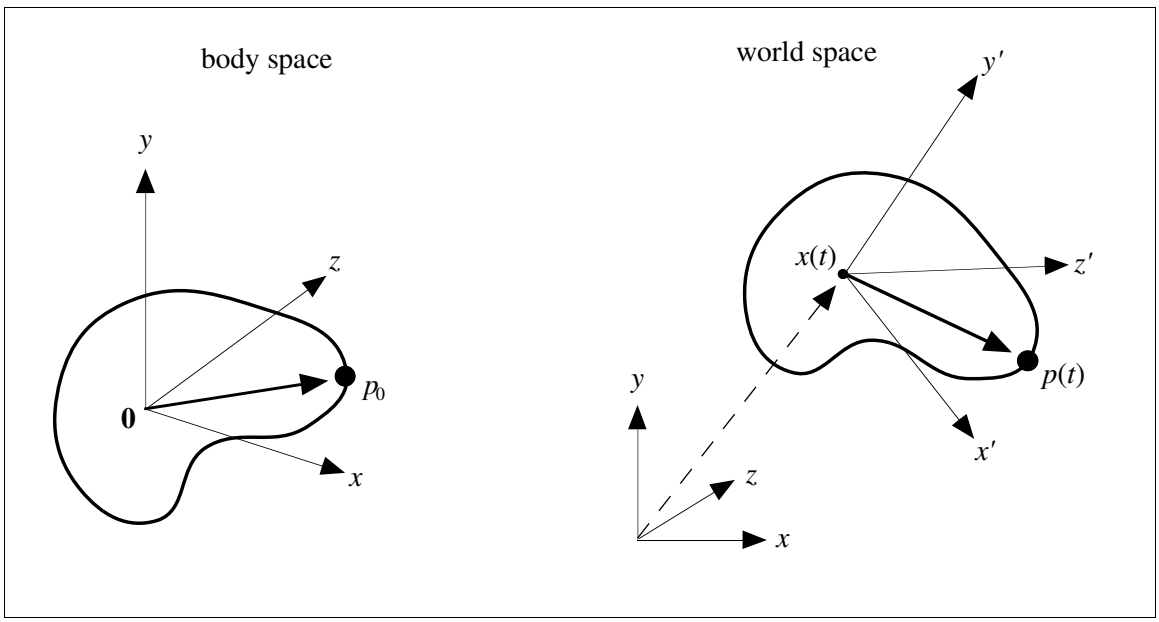

Figure 1: The center of mass is transformed to the point $x(t)$ in world space, at time t. The fixed x, y, and z axes of the body in body space transform to the vectors $x' = R(t)x$, $y' = R(t)y$ and $z' = R(t)z$. The fixed point p_0 in body space is transformed to the point $p(t) = R(t)p_0 + x(t)$.

then

$$R(t) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{pmatrix} \tag{2–3}$$

which is the first column of $R(t)$. The physical meaning of $R(t)$ is that $R(t)$'s first column gives the direction that the rigid body's x axis points in, when transformed to world space at time t. Similarly, the second and third columns of $R(t)$,

$$\begin{pmatrix} r_{yx} \\ r_{yy} \\ r_{yz} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} r_{zx} \\ r_{zy} \\ r_{zz} \end{pmatrix}$$

are the directions of the y and z axes of the rigid body in world space at time t (figure 2).

2.2 Linear Velocity

For simplicity, we'll call $x(t)$ and $R(t)$ the *position* and *orientation* of the body at time t. The next thing we need to do is define how the position and orientation change over time. This means we need expressions for $\dot{x}(t)$ and $\dot{R}(t)$. Since $x(t)$ is the position of the center of mass in world space, $\dot{x}(t)$ is the velocity of the center of mass in world space. We'll define the *linear velocity* $v(t)$ as this velocity:

$$v(t) = \dot{x}(t). \tag{2–4}$$

If we imagine that the orientation of the body is fixed, then the only movement the body can undergo is a pure translation. The quantity $v(t)$ gives the velocity of this translation.

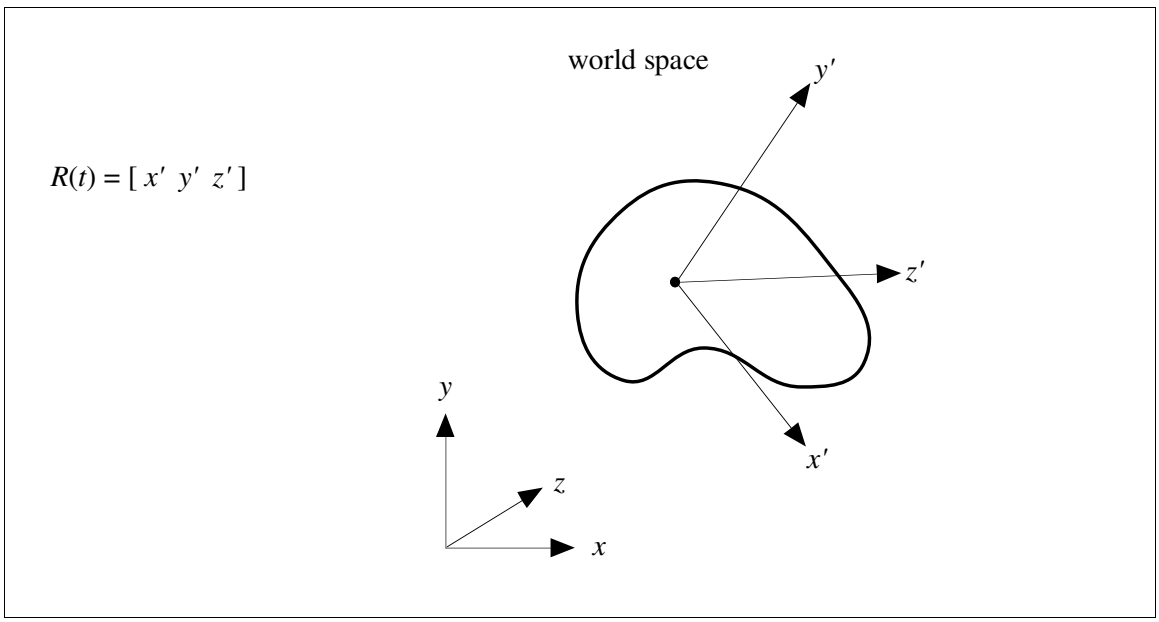

Figure 2: Physical interpretation of the orientation matrix $R(t)$. At time t, the columns of $R(t)$ are the world-space directions that the body-space x, y, and z axes transform to.

2.3 Angular Velocity

In addition to translating, a rigid body can also spin. Imagine however that we freeze the position of the center of mass in space. Any movement of the points of the body must therefore be due to the body spinning about some axis that passes through the center of mass. (Otherwise the center of mass would itself be moving). We can describe that spin as a vector $\omega(t)$. The *direction* of $\omega(t)$ gives the direction of the axis about which the body is spinning (figure 3). The magnitude of $\omega(t)$, $|\omega(t)|$, tells how fast the body is spinning. $|\omega(t)|$ has the dimensions of revolutions/time; thus, $|\omega(t)|$ relates the angle through which the body will rotate over a given period of time, if the angular velocity remains constant. The quantity $\omega(t)$ is called the *angular velocity*.

For linear velocity, $x(t)$ and $v(t)$ are related by $v(t) = \frac{d}{dt}x(t)$. How are $R(t)$ and $\omega(t)$ related? (Clearly, $\dot{R}(t)$ cannot be $\omega(t)$, since $R(t)$ is a matrix, and $\omega(t)$ is a vector.) To answer this question, let's remind ourselves of the physical meaning of $R(t)$. We know that the columns of $R(t)$ tell us the directions of the transformed x, y and z body axes at time t. That means that the columns of $\dot{R}(t)$ must describe the *velocity* with which the x, y, and z axes are being transformed. To discover the relationship between $\omega(t)$ and $R(t)$, let's examine how the change of an arbitrary vector in a rigid body is related to the angular velocity $\omega(t)$.

Figure 4 shows a rigid body with angular velocity $\omega(t)$. Consider a vector $r(t)$ at time t specified in world space. Suppose that we consider this vector fixed to the body; that is, $r(t)$ moves along with the rigid body through world space. Since $r(t)$ is a direction, it is independent of any translational effects; in particular, $\dot{r}(t)$ is independent of $v(t)$. To study $\dot{r}(t)$, we decompose $r(t)$ into vectors a and b, where a is parallel to $\omega(t)$ and b is perpendicular to $\omega(t)$. Suppose the rigid body were to maintain a constant angular velocity, so that the tip of $r(t)$ traces out a circle centered on the $\omega(t)$ axis (figure 4). The radius of this circle is $|b|$. Since the tip of the vector $r(t)$ is instantaneously moving along this circle, the instantaneous change of $r(t)$ is perpendicular to both b and $\omega(t)$. Since the tip of $r(t)$ is moving in a circle of radius b, the instantaneous velocity of $r(t)$ has magnitude

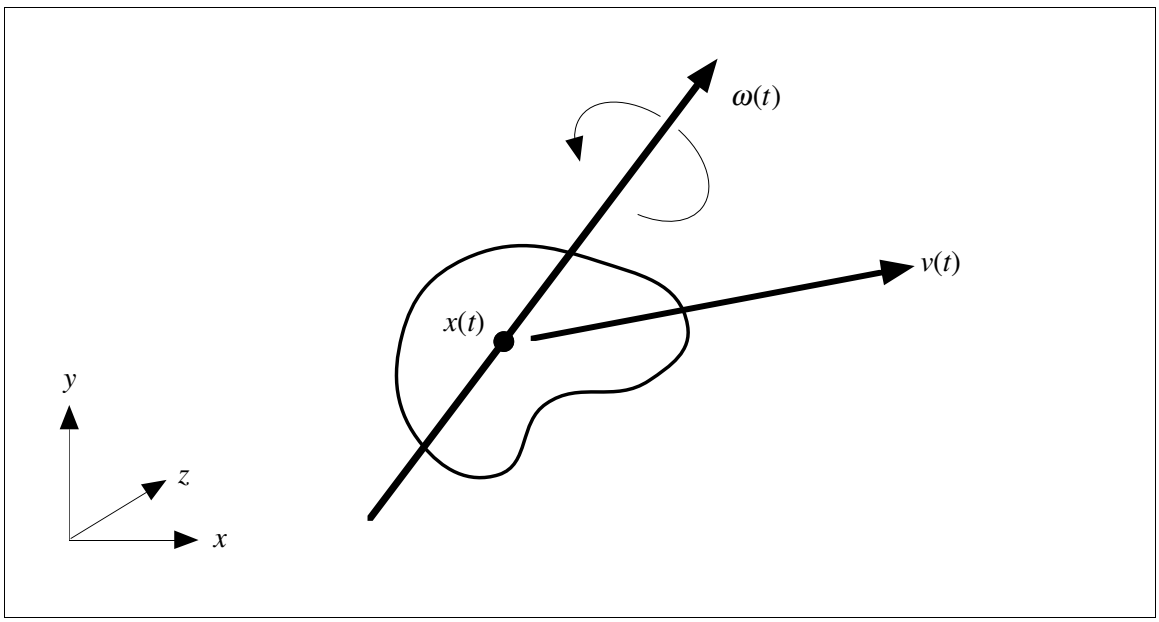

Figure 3: Linear velocity $v(t)$ and angular velocity $\omega(t)$ of a rigid body.

$|b||\omega(t)|$. Since b and $\omega(t)$ are perpendicular, their cross product has magnitude

$$|\omega(t) \times b| = |\omega(t)|\,|b|. \tag{2–5}$$

Putting this together, we can write $\dot{r}(t) = \omega(t) \times (b)$. However, since $r(t) = a + b$ and a is parallel to $\omega(t)$, we have $\omega(t) \times a = 0$ and thus

$$\dot{r}(t) = \omega(t) \times b = \omega(t) \times b + \omega(t) \times a = \omega(t) \times (b + a). \tag{2–6}$$

Thus, we can simply express the rate of change of a vector as

$$\dot{r}(t) = \omega(t) \times r(t). \tag{2–7}$$

Let's put all this together now. At time t, we know that the direction of the x axis of the rigid body in world space is the first column of $R(t)$, which is

$$\begin{pmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{pmatrix}.$$

At time t, the derivative of the first column of $R(t)$ is just the rate of change of this vector: using the cross product rule we just discovered, this change is

$$\omega(t) \times \begin{pmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{pmatrix}.$$

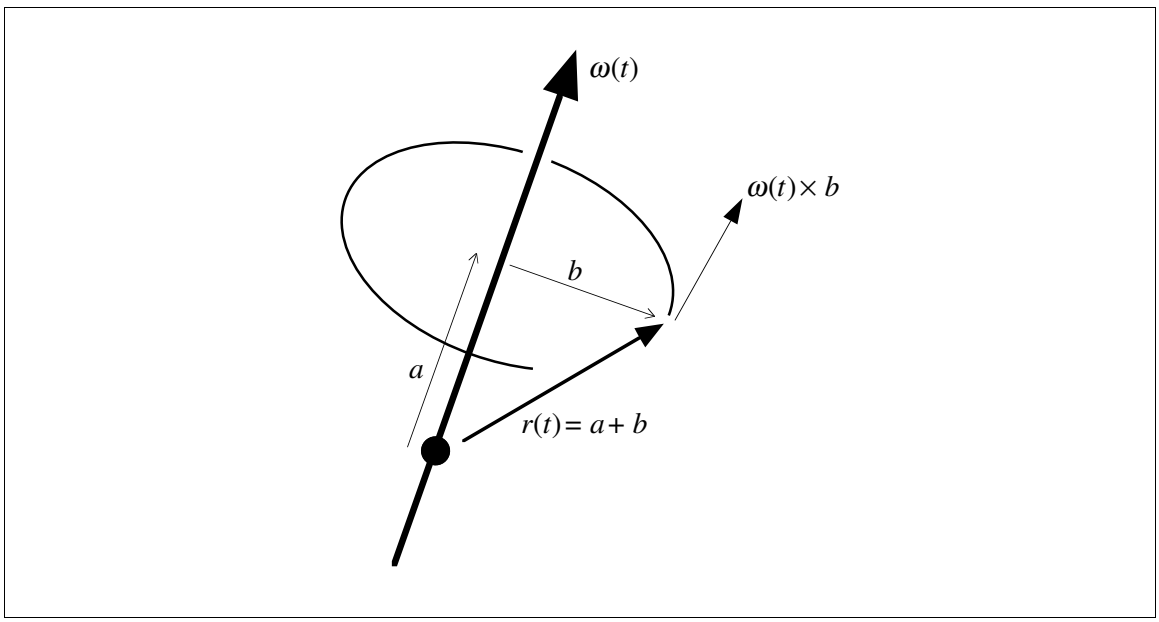

Figure 4: The rate of change of a rotating vector. As the tip of $r(t)$ spins about the $\omega(t)$ axis, it traces out a circle of diameter $|b|$. The speed of the tip of $r(t)$ is $|\omega(t)||b|$.

The same obviously holds for the other two columns of $R(t)$. This means that we can write

$$\dot{R} = \left(\omega(t) \times \begin{pmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{pmatrix} \quad \omega(t) \times \begin{pmatrix} r_{yx} \\ r_{yy} \\ r_{yz} \end{pmatrix} \quad \omega(t) \times \begin{pmatrix} r_{zx} \\ r_{zy} \\ r_{zz} \end{pmatrix} \right). \tag{2–8}$$

This is too cumbersome an expression to tote around though. To simplify things, we'll use the following trick. If a and b are 3-vectors, then $a \times b$ is the vector

$$\begin{pmatrix} a_y b_z - b_y a_z \\ -a_x b_z + b_x a_z \\ a_x b_y - b_x a_y \end{pmatrix}.$$

Given the vector a, let us define a^* to be the matrix

$$\begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix}.$$

Then[2]

$$a^* b = \begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix} \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} a_y b_z - b_y a_z \\ -a_x b_z + b_x a_z \\ a_x b_y - b_x a_y \end{pmatrix} = a \times b. \tag{2–9}$$

[2]This looks a little too "magical" at first. Did someone discover this identity accidentally? Is it a relation that just happens to work? This construct can be derived by considering what's known as *infinitesimal* rotations. The interested reader might wish to read chapter 4.8 of Goldstein[9] for a more complete derivation of the a^* matrix.

Using the "$*$" notation, we can rewrite $\dot{R}(t)$ more simply as

$$\dot{R}(t) = \left(\omega(t)^* \begin{pmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{pmatrix} \quad \omega(t)^* \begin{pmatrix} r_{yx} \\ r_{yy} \\ r_{yz} \end{pmatrix} \quad \omega(t)^* \begin{pmatrix} r_{zx} \\ r_{zy} \\ r_{zz} \end{pmatrix} \right). \tag{2–10}$$

By the rules of matrix multiplication, we can factor this into

$$\dot{R}(t) = \omega(t)^* \left(\begin{pmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{pmatrix} \quad \begin{pmatrix} r_{yx} \\ r_{yy} \\ r_{yz} \end{pmatrix} \quad \begin{pmatrix} r_{zx} \\ r_{zy} \\ r_{zz} \end{pmatrix} \right) \tag{2–11}$$

which is a matrix-matrix multiplication. But since the matrix on the right is $R(t)$ itself, we get simply that

$$\dot{R}(t) = \omega(t)^* R(t). \tag{2–12}$$

This, at last, gives us the relation we wanted between $\dot{R}(t)$ and $\omega(t)$. Note the correspondence between $\dot{r}(t) = \omega(t) \times r(t)$ for a vector, and $\dot{R}(t) = \omega(t)^* R(t)$ for the rotation matrix.

2.4 Mass of a Body

In order to work out some derivations, we'll need to (conceptually) perform some integrations over the volume of our rigid body. To make these derivations simpler, we're going to temporarily imagine that a rigid body is made up of a large number of small particles. The particles are indexed from 1 to N. The mass of the ith particle is m_i, and each particle has a (constant) location r_{0i} in body space. The location of the ith particle in world space at time t, denoted $r_i(t)$, is therefore given by the formula

$$r_i(t) = R(t) r_{0i} + x(t). \tag{2–13}$$

The total mass of the body, M, is the sum

$$M = \sum_{i=1}^{N} m_i. \tag{2–14}$$

(Henceforth, summations are assumed to be summed from 1 to N with index variable i.)

2.5 Velocity of a Particle

The *velocity* $\dot{r}_i(t)$ of the ith particle is obtained by differentiating equation (2–13): using the relation $\dot{R}(t) = \omega^* R(t)$, we obtain

$$\dot{r}_i(t) = \omega^* R(t) r_{0i} + v(t). \tag{2–15}$$

We can rewrite this as

$$\begin{aligned} \dot{r}_i(t) &= \omega(t)^* R(t) r_{0i} + v(t) \\ &= \omega(t)^* \left(R(t) r_{0i} + x(t) - x(t) \right) + v(t) \\ &= \omega(t)^* \left(r_i(t) - x(t) \right) + v(t) \end{aligned} \tag{2–16}$$

using the definition of $r_i(t)$ from equation (2–13). Recall from the definition of the "$*$" operator that $\omega(t)^* a = \omega(t) \times a$ for any vector a. Using this, we can simply write

$$\dot{r}_i(t) = \omega(t) \times \left(r_i(t) - x(t) \right) + v(t). \tag{2–17}$$

Note that this separates the velocity of a point on a rigid body into two components (figure 5): a linear component $v(t)$, and an angular component $\omega \times (r_i(t) - x(t))$.

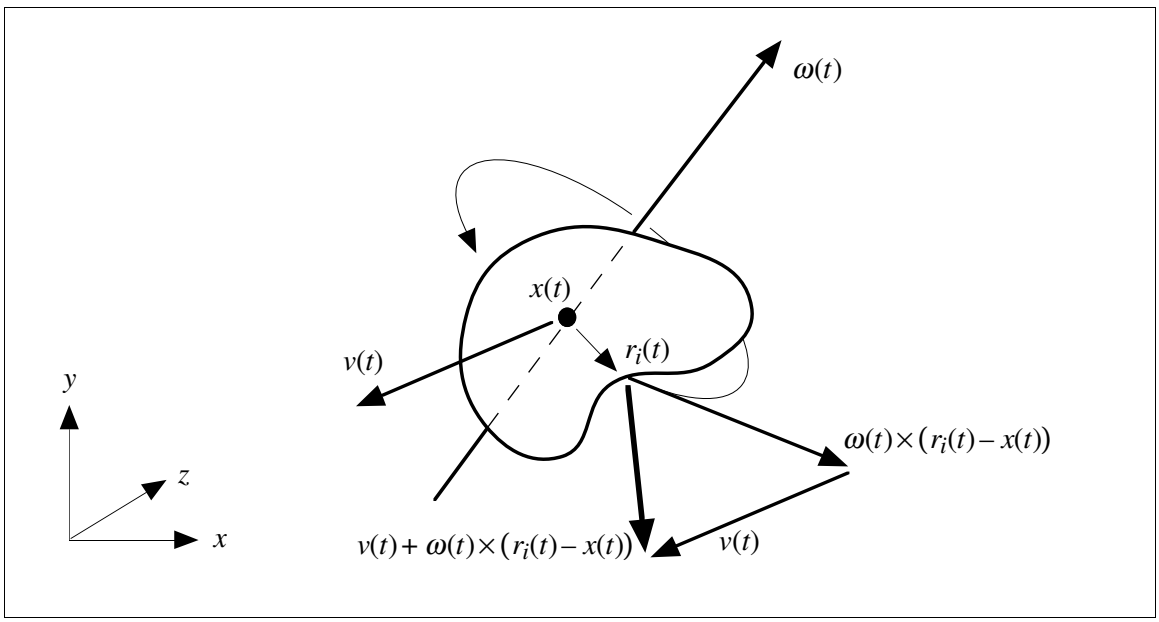

Figure 5: The velocity of the ith point of a rigid body in world space. The velocity of $r_i(t)$ can be decomposed into a linear term $v(t)$ and an angular term $\omega(t) \times (r_i(t) - x(t))$.

2.6 Center of Mass

Our definition of the center of mass is going to enable us to likewise separate the dynamics of bodies into linear and angular components. The center of mass of a body in world space is defined to be

$$\frac{\sum m_i r_i(t)}{M} \tag{2–18}$$

where M is the mass of the body (i.e. the sum of the individual masses m_i). When we say that we are using a center of mass coordinate system, we mean that in body space,

$$\frac{\sum m_i r_{0i}}{M} = \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2–19}$$

Note that this implies that $\sum m_i r_{0i} = \mathbf{0}$ as well.

We have spoken of $x(t)$ as being the location of the center of mass at time t. Is this true? Yes: since the ith particle has position $r_i(t) = R(t)r_{0i} + x(t)$ at time t, the center of mass at time t is

$$\frac{\sum m_i r_i(t)}{M} = \frac{\sum m_i (R(t)r_{0i} + x(t))}{M} = \frac{R(t)\sum m_i r_{0i} + \sum m_i x(t)}{M} = x(t)\frac{\sum m_i}{M} = x(t).$$

Additionally, the relation

$$\sum m_i (r_i(t) - x(t)) = \sum m_i (R(t)r_{0i} + x(t) - x(t)) = R(t)\sum m_i r_{0i} = \mathbf{0} \tag{2–20}$$

is also very useful.

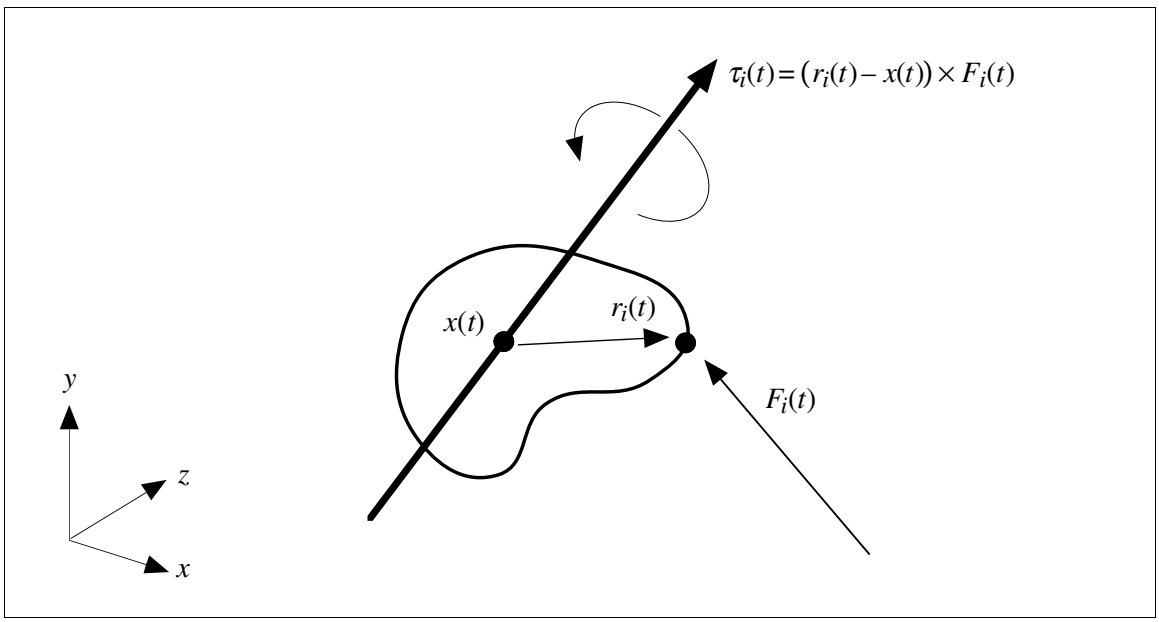

Figure 6: The torque $\tau_i(t)$ due to a force $F_i(t)$ acting at $r_i(t)$ on a rigid body.

2.7 Force and Torque

When we imagine a force acting on a rigid body due to some external influence (e.g. gravity, wind, contact forces), we imagine that the force acts on a particular particle of the body. (Remember that our particle model is conceptual only. We can have a force act at any geometrical location on or inside the body, because we can always imagine that there happens to be a particle at that exact location.) The location of the particle the force acts on defines the location at which the force acts. We will let $F_i(t)$ denote the total force from external forces acting on the ith particle at time t. Also, we define the external *torque* $\tau_i(t)$ acting on the ith particle as

$$\tau_i(t) = (r_i(t) - x(t)) \times F_i(t). \tag{2–21}$$

Torque differs from force in that the torque on a particle depends on the location $r_i(t)$ of the particle, relative to the center of mass $x(t)$. We can intuitively think of the direction of $\tau_i(t)$ as being the axis the body would spin about due to $F_i(t)$, if the center of mass were held firmly in place (figure 6).

The total external force $F(t)$ acting on the body is the sum of the $F_i(t)$:

$$F(t) = \sum F_i(t) \tag{2–22}$$

while the total external torque is defined similarly as

$$\tau(t) = \sum \tau_i(t) = \sum (r_i(t) - x(t)) \times F_i(t). \tag{2–23}$$

Note that $F(t)$ conveys no information about where the various forces acted on the body; however, $\tau(t)$ does tell us something about the distribution of the forces $F_i(t)$ over the body.

2.8 Linear Momentum

The linear momentum p of a particle with mass m and velocity v is defined as

$$p = mv. \tag{2–24}$$

The total linear momentum $P(t)$ of a rigid body is the sum of the products of the mass and velocity of each particle:

$$P(t) = \sum m_i \dot{r}_i(t). \tag{2–25}$$

From equation (2–17), the velocity $\dot{r}_i(t)$ of the ith particle is $\dot{r}_i(t) = v(t) + \omega(t) \times (r_i(t) - x(t))$. Thus, the total linear momentum of the body is

$$\begin{aligned} P(t) &= \sum m_i \dot{r}_i(t) \\ &= \sum \Big(m_i v(t) + \omega(t) \times (r_i(t) - x(t)) \Big) \\ &= \sum m_i v(t) + \omega(t) \times \sum m_i \left(r_i(t) - x(t) \right). \end{aligned} \tag{2–26}$$

Because we are using a center of mass coordinate system, we can apply equation (2–20) and obtain

$$P(t) = \sum m_i v(t) = \Big(\sum m_i \Big) \, v(t) = M v(t). \tag{2–27}$$

This gives us the nice result that the total linear momentum of our rigid body is the same as if the body was simply a particle with mass M and velocity $v(t)$. Because of this, we have a simple transformation between $P(t)$ and $v(t)$: $P(t) = Mv(t)$ and $v(t) = P(t)/M$. Since M is a constant,

$$\dot{v}(t) = \frac{\dot{P}(t)}{M}. \tag{2–28}$$

The concept of linear momentum lets us express the effect of the total force $F(t)$ on a rigid body quite simply. Appendix A derives the relation

$$\dot{P}(t) = F(t) \tag{2–29}$$

which says that the change in linear momentum is equivalent to the total force acting on a body. Note that $P(t)$ tells us nothing about the rotational velocity of a body, which is good, because $F(t)$ also conveys nothing about the change of rotational velocity of a body!

Since the relationship between $P(t)$ and $v(t)$ is simple, we will be using $P(t)$ as a state variable for our rigid body, instead of $v(t)$. We could of course let $v(t)$ be a state variable, and use the relation

$$\dot{v}(t) = \frac{F(t)}{M}. \tag{2–30}$$

However, using $P(t)$ instead of $v(t)$ as a state variable will be more consistent with the way we will be dealing with angular velocity and acceleration.

2.9 Angular Momentum

While the concept of linear momentum is pretty intuitive ($P(t) = Mv(t)$), the concept of angular momentum (for a rigid body) is not. The only reason that one even bothers with the angular momentum of a rigid body is that it lets you write simpler equations than you would get if you stuck with angular velocity. With that in mind, its probably best not to worry about attaching an intuitive physical explanation to angular momentum—all in all, its a most unintuitive concept. Angular momentum ends up simplifying equations because it is conserved in nature, while angular *velocity* is not: if you have a body floating through space with no torque acting on it, the body's angular momentum is constant. This is not true for a body's angular velocity though: even if the angular momentum of a body is constant, the body's angular *velocity* may not be! Consequently, a body's angular velocity can vary even when no force acts on the body. Because of this, it ends up being simpler to choose angular momentum as a state variable over angular velocity.

For linear momentum, we have the relation $P(t) = Mv(t)$. Similarly, we define the total angular momentum $L(t)$ of a rigid body by the equation $L(t) = I(t)\omega(t)$, where $I(t)$ is a 3×3 matrix (technically a rank-two tensor) called the *inertia tensor*, which we will describe momentarily. The inertia tensor $I(t)$ describes how the mass in a body is distributed relative to the body's center of mass. The tensor $I(t)$ depends on the orientation of a body, but does not depend on the body's translation. Note that for both the angular and the linear case, momentum is a linear function of velocity—its just that in the angular case the scaling factor is a matrix, while its simply a scalar in the linear case. Note also that $L(t)$ is independent of any translational effects, while $P(t)$ is independent of any rotational effects.

The relationship between $L(t)$ and the total torque $\tau(t)$ is very simple: appendix A derives

$$\dot{L}(t) = \tau(t), \tag{2–31}$$

analogous to the relation $\dot{P}(t) = F(t)$.

2.10 The Inertia Tensor

The inertia tensor $I(t)$ is the scaling factor between angular momentum $L(t)$ and angular velocity $\omega(t)$. At a given time t, let r_i' be the displacement of the ith particle from $x(t)$ by defining $r_i' = r_i(t) - x(t)$. The tensor $I(t)$ is expressed in terms of r_i' as the symmetric matrix

$$I(t) = \sum \begin{pmatrix} m_i(r_{iy}'^2 + r_{iz}'^2) & -m_i r_{ix}' r_{iy}' & -m_i r_{ix}' r_{iz}' \\ -m_i r_{iy}' r_{ix}' & m_i(r_{ix}'^2 + r_{iz}'^2) & -m_i r_{iy}' r_{iz}' \\ -m_i r_{iz}' r_{ix}' & -m_i r_{iz}' r_{iy}' & m_i(r_{ix}'^2 + r_{iy}'^2) \end{pmatrix} \tag{2–32}$$

For an actual implementation, we replace the finite sums with integrals over a body's volume in world space. The mass terms m_i are replaced by a density function. At first glance, it seems that we would need to evaluate these integrals to find $I(t)$ whenever the orientation $R(t)$ changes. This would be prohibitively expensive to do during a simulation unless the body's shape was so simple (for example, a sphere or cube) that that the integrals could be evaluated symbolically.

Fortunately, by using body-space coordinates we can cheaply compute the inertia tensor for any orientation $R(t)$ in terms of a precomputed integral in body-space coordinates. (This integral is typically computed before the simulation begins and should be regarded as one of the input

parameters describing a physical property of the body.) Using the fact that $r_i'^T r_i' = r_{ix}'^2 + r_{iy}'^2 + r_{iz}'^2$, we can rewrite $I(t)$ as the difference

$$I(t) = \sum m_i r_i'^T r_i' \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} m_i r_{ix}'^2 & m_i r_{ix}' r_{iy}' & m_i r_{ix}' r_{iz}' \\ m_i r_{iy}' r_{ix}' & m_i r_{iy}'^2 & m_i r_{iy}' r_{iz}' \\ m_i r_{iz}' r_{ix}' & m_i r_{iz}' r_{iy}' & m_i r_{iz}'^2 \end{pmatrix} \tag{2–33}$$

Taking the outer product multiplication of r_i' with itself, that is

$$r_i' r_i'^T = \begin{pmatrix} r_{ix}' \\ r_{iy}' \\ r_{iz}' \end{pmatrix} \begin{pmatrix} r_{ix}' & r_{iy}' & r_{iz}' \end{pmatrix} = \begin{pmatrix} r_{ix}'^2 & r_{ix}' r_{iy}' & r_{ix}' r_{iz}' \\ r_{iy}' r_{ix}' & r_{iy}'^2 & r_{ix}' r_{iz}' \\ r_{iz}' r_{ix}' & r_{iz}' r_{iy}' & r_{iz}'^2 \end{pmatrix} \tag{2–34}$$

and letting $\mathbf{1}$ denote the 3×3 identity matrix, we can express $I(t)$ simply as

$$I(t) = \sum m_i ((r_i'^T r_i')\mathbf{1} - r_i' r_i'^T) \tag{2–35}$$

How does this help?

Since $r_i(t) = R(t) r_{0i} + x(t)$ where r_{0i} is a constant, $r_i' = R(t) r_{0i}$. Then, since $R(t)R(t)^T = \mathbf{1}$,

$$\begin{aligned} I(t) &= \sum m_i ((r_i'^T r_i')\mathbf{1} - r_i' r_i'^T) \\ &= \sum m_i ((R(t)r_{0i})^T (R(t)r_{0i})\mathbf{1} - (R(t)r_{0i})(R(t)r_{0i})^T) \\ &= \sum m_i (r_{0i}^T R(t)^T R(t) r_{0i}\mathbf{1} - R(t) r_{0i} r_{0i}^T R(t)^T) \\ &= \sum m_i ((r_{0i}^T r_{0i})\mathbf{1} - R(t) r_{0i} r_{0i}^T R(t)^T). \end{aligned} \tag{2–36}$$

Since $r_{0i}^T r_{0i}$ is a scalar, we can rearrange things by writing

$$\begin{aligned} I(t) &= \sum m_i ((r_{0i}^T r_{0i})\mathbf{1} - R(t) r_{0i} r_{0i}^T R(t)^T) \\ &= \sum m_i (R(t)(r_{0i}^T r_{0i}) R(t)^T \mathbf{1} - R(t) r_{0i} r_{0i}^T R(t)^T) \\ &= R(t) \left(\sum m_i ((r_{0i}^T r_{0i})\mathbf{1} - r_{0i} r_{0i}^T) \right) R(t)^T. \end{aligned} \tag{2–37}$$

If we define I_{body} as the matrix

$$I_{body} = \sum m_i ((r_{0i}^T r_{0i})\mathbf{1} - r_{0i} r_{0i}^T) \tag{2–38}$$

then from the previous equation we have

$$I(t) = R(t) I_{body} R(t)^T. \tag{2–39}$$

Since I_{body} is specified in body-space, it is constant over the simulation. Thus, by precomputing I_{body} for a body before the simulation begins, we can easily compute $I(t)$ from I_{body} and the orientation matrix $R(t)$. Section 5.1 derives the body-space inertia tensor for a rectangular object in terms of an integral over the body's volume in body space.

Also, the inverse of $I(t)$ is given by the formula

$$\begin{aligned} I^{-1}(t) &= \left(R(t)I_{body}R(t)^T\right)^{-1} \\ &= \left(R(t)^T\right)^{-1} I_{body}^{-1}R(t)^{-1} \\ &= R(t)I_{body}^{-1}R(t)^T \end{aligned} \tag{2–40}$$

since, for rotation matrices, $R(t)^T = R(t)^{-1}$ and $\left(R(t)^T\right)^T = R(t)$. Clearly, I_{body}^{-1} is also a constant during the simulation.

2.11 Rigid Body Equations of Motion

Finally, we have covered all the concepts we need to define the state vector $\mathbf{Y}(t)$! For a rigid body, we will define $\mathbf{Y}(t)$ as

$$\mathbf{Y}(t) = \begin{pmatrix} x(t) \\ R(t) \\ P(t) \\ L(t) \end{pmatrix}. \tag{2–41}$$

Thus, the state of a rigid body is its position and orientation (describing spatial information), and its linear and angular momentum (describing velocity information). The mass M of the body and body-space inertia tensor I_{body} are constants, which we assume we know when the simulation begins. At any given time, the auxiliary quantities $I(t)$, $\omega(t)$ and $v(t)$ are computed by

$$v(t) = \frac{P(t)}{M}, \quad I(t) = R(t)I_{body}R(t)^T \qquad \text{and} \qquad \omega(t) = I(t)L(t). \tag{2–42}$$

The derivative $\frac{d}{dt}\mathbf{Y}(t)$ is

$$\frac{d}{dt}\mathbf{Y}(t) = \frac{d}{dt}\begin{pmatrix} x(t) \\ R(t) \\ P(t) \\ L(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ \omega(t)^* R(t) \\ F(t) \\ \tau(t) \end{pmatrix}. \tag{2–43}$$

The next section gives an implementation for the function `dydt` that computes $\frac{d}{dt}\mathbf{Y}(t)$.

One final note: rather than represent the orientation of the body as a matrix $R(t)$ in $\mathbf{Y}(t)$, it is better to use *quaternions*. Section 4 discusses using quaternions in place of rotation matrices. Briefly, a quaternion is a type of four element vector that can be used to represent a rotation. If we replace $R(t)$ in $\mathbf{Y}(t)$ with a quaternion $q(t)$, we can treat $R(t)$ as an auxiliary variable that is computed directly from $q(t)$, just as $\omega(t)$ is computed from $L(t)$. Section 4 derives a formula analogous to $\dot{R}(t) = \omega(t)^* R(t)$, that expresses $\dot{q}(t)$ in terms of $q(t)$ and $\omega(t)$.

3 Computing $\frac{d}{dt}\mathbf{Y}(t)$

Lets consider an implementation of the function `dydt` for rigid bodies. The code is written in C++, and we'll assume that we have datatypes (classes) called `matrix` and `triple` which implement, respectively, 3×3 matrices and points in 3-space. Using these datatypes, we'll represent a rigid