Appendix A Motion Equation Derivations

In this appendix, we'll fill in some of the missing details from section 2, with regards to the equations $\dot{P}(t) = F(t)$, $\dot{L}(t) = \tau(t)$, and $L(t) = I(t)\omega(t)$. The derivation method used here is somewhat nonstandard, and was proposed by Andy Witkin. The derivation in this appendix is (we feel) much shorter and considerably more elegant than the one found in traditional sources such as Goldstein[9].

We've described the external force acting on a rigid body in terms of forces $F_i(t)$, where $F_i(t)$ is the external force acting on the ith particle. However, for a rigid body to maintain its shape, there must be some "internal" constraint forces that act between particles in the same body. We will make the assumption that these constraint forces act passively on the system and do not perform any net work. Let $F_{ci}(t)$ denote the net internal constraint force acting on the ith particle. The work performed by F_{ci} on the ith particle from time t_0 to t_1 is

$$\int_{t_0}^{t_1} F_{ci}(t) \cdot \dot{r}_i(t)\, dt$$

where $\dot{r}_i(t)$ is the velocity of the ith particle. The net work over all the particles is the sum

$$\sum_i \int_{t_0}^{t_1} F_{ci}(t) \cdot \dot{r}_i(t)\, dt = \int_{t_0}^{t_1} \sum_i F_{ci}(t) \cdot \dot{r}_i(t)\, dt,$$

which must be zero for any interval t_0 to t_1. This means that the integrand

$$\sum_i F_{ci}(t) \cdot \dot{r}_i(t) \tag{A–10}$$

is itself always zero for any time t. (Henceforth we'll just write these expressions as $\sum F_{ci} \cdot \dot{r}_i = 0$.)

We can use this fact to eliminate any mention of the constraint forces F_{ci} from our derivations. First, some quick notes about the "$*$" operator defined in section 2.3: since $a^*b = a \times b$, and $a \times b = -b \times a$, we get

$$-a^*b = b \times a = b^*a. \tag{A–11}$$

Since a^* is an antisymmetric matrix,

$$(a^*)^T = -a^*. \tag{A–12}$$

Finally, since the "$*$" operator is a linear operator,

$$(\dot{a})^* = (\dot{a^*}) = \frac{d}{dt}(a^*) \tag{A–13}$$

and for a set of vectors a_i

$$\sum a_i^* = \left(\sum a_i\right)^*. \tag{A–14}$$

Recall that we can write the velocity $\dot{r}_i$ as $\dot{r}_i = v + \omega \times (r_i - x)$ where r_i is the particle's location, x is the position of the center of mass, and v and ω are linear and angular velocity. Letting $r_i' = r_i - x$ and using the "$*$" notation,

$$\dot{r}_i = v + \omega^* r_i' = v - r_i'^* \omega. \tag{A–15}$$

Substituting this into $\sum F_{ci} \cdot \dot{r}_i$, which is always zero, yields

$$\sum F_{ci} \cdot (v - r_i'^* \omega) = 0. \tag{A–16}$$

Note that this equation must hold for arbitrary values of v and ω. Since v and ω are completely independent, if we choose ω to be zero, then $\sum F_{ci} \cdot v = 0$ for any choice of v, from which we conclude that in fact $\sum F_{ci} = \mathbf{0}$ is always true. This means that the constraint forces produce no net force. Similarly, choosing v to be zero we see that $\sum -F_{ci} \cdot (r_i'^* \omega) = 0$ for any ω. Rewriting $F_{ci} \cdot (r_i'^* \omega)$ as ${F_{ci}}^T (r_i'^* \omega)$ we get that

$$\sum -{F_{ci}}^T r_i'^* \omega = \left(\sum -{F_{ci}}^T r_i'^* \right) \omega = 0 \tag{A–17}$$

for any ω, so $\sum -{F_{ci}}^T r_i'^* = \mathbf{0}^T$. Transposing, we have

$$\sum -(r_i'^*)^T F_{ci} = \sum (r_i')^* F_{ci} = \sum r_i' \times F_{ci} = \mathbf{0} \tag{A–18}$$

which means that the internal forces produce no net torque.

We can use the above to derive the rigid body equations of motion. The net force on each particle is the sum of the internal constraint force F_{ci} and the external force F_i. The acceleration $\ddot{r}_i$ of the ith particle is

$$\ddot{r}_i = \frac{d}{dt}\dot{r}_i = \frac{d}{dt}(v - r_i'^* \omega) = \dot{v} - \dot{r}_i'^* \omega - r_i'^* \dot{\omega}. \tag{A–19}$$

Since each individual particle must obey Newton's law $f = ma$, or equivalently $ma - f = \mathbf{0}$, we have

$$m_i \ddot{r}_i - F_i - F_{ci} = m_i(\dot{v} - \dot{r}_i'^* \omega - r_i'^* \dot{\omega}) - F_i - F_{ci} = \mathbf{0} \tag{A–20}$$

for each particle.

To derive $\dot{P} = F = \sum F_i$, lets sum equation (A–20) over all the particles. We obtain

$$\sum m_i(\dot{v} - \dot{r}_i'^* \omega - r_i'^* \dot{\omega}) - F_i - F_{ci} = \mathbf{0}. \tag{A–21}$$

Breaking the large sum into smaller ones, we obtain

$$\begin{aligned}
\sum m_i(\dot{v} - \dot{r}_i'^* \omega - r_i'^* \dot{\omega}) - F_i - F_{ci} &= \\
\sum m_i \dot{v} - \sum m_i \dot{r}_i'^* \omega - \sum m_i r_i'^* \dot{\omega} - \sum F_i - \sum F_{ci} &= \\
\sum m_i \dot{v} - \left(\sum m_i \dot{r}_i' \right)^* \omega - \left(\sum m_i r_i' \right)^* \dot{\omega} - \sum F_i - \sum F_{ci} &= \\
\sum m_i \dot{v} - \left(\frac{d}{dt} \sum m_i r_i' \right)^* \omega - \left(\sum m_i r_i' \right)^* \dot{\omega} - \sum F_i - \sum F_{ci} &= \mathbf{0}.
\end{aligned} \tag{A–22}$$

Since we are in a center-of-mass coordinate system, equation (2–20) from section 2.6 tells us that $\sum m_i r_i' = \mathbf{0}$, which also means that $\frac{d}{dt} \sum m_i r_i' = \mathbf{0}$. Removing terms with $\sum m_i r_i'$, and the term $\sum F_{ci}$ from the above equation yields

$$\sum m_i \dot{v} - \sum F_i = \mathbf{0} \tag{A–23}$$

or simply $M\dot{v} = \dot{P} = \sum F_i$ as advertised.

To obtain $\dot{L} = \tau = \sum r_i' \times F_i$, we again start with equation (A–20). Multiplying both sides by $r_i'^*$ yields

$$r_i'^* m_i(\dot{v} - \dot{r}_i'^* \omega - r_i'^* \dot{\omega}) - r_i'^* F_i - r_i'^* F_{ci} = r_i'^* \mathbf{0} = \mathbf{0}. \tag{A–24}$$

Summing over all the particles, we obtain

$$\sum r_i'^* m_i \dot{v} - \sum r_i'^* m_i \dot{r}_i'^* \omega - \sum r_i'^* m_i r_i'^* \dot{\omega} - \sum r_i'^* F_i - \sum r_i'^* F_{ci} = \mathbf{0}. \tag{A–25}$$

Since $\sum r_i'^* F_{ci} = \mathbf{0}$, we can rearrange this to obtain

$$\left(\sum m_i r_i'\right)^* \dot{v} - \left(\sum m_i r_i'^* \dot{r}_i'^*\right) \omega - \left(\sum m_i r_i'^* r_i'^*\right) \dot{\omega} - \sum r_i'^* F_i = \mathbf{0}. \tag{A–26}$$

Using $\sum m_i r_i' = \mathbf{0}$, we are left with

$$-\left(\sum m_i r_i'^* \dot{r}_i'^*\right) \omega - \left(\sum m_i r_i'^* r_i'^*\right) \dot{\omega} - \sum r_i'^* F_i - = \mathbf{0} \tag{A–27}$$

or, recognizing that $\sum r_i'^* F_i = \sum r_i' \times F_i = \tau$,

$$-\left(\sum m_i r_i'^* \dot{r}_i'^*\right) \omega - \left(\sum m_i r_i'^* r_i'^*\right) \dot{\omega} = \tau. \tag{A–28}$$

We're almost done now: if we refer back to the matrix defined by the "$*$" notation, one can easily verify the relation that the matrix $-a^* a^*$ is equivalent to the matrix $(a^T a)\mathbf{1} - aa^T$ where $\mathbf{1}$ is the 3×3 identity matrix. (This relation is equivalent to the vector rule $a \times (b \times c) = ba^T c - ca^T b$.) Thus

$$\sum -m_i r_i'^* r_i'^* = \sum m_i ((r_i'^T r_i')\mathbf{1} - r_i' r_i'^T) = I(t). \tag{A–29}$$

Substituting into equation (A–28), this yields

$$\left(\sum -m_i r_i'^* \dot{r}_i'^*\right) \omega + I(t)\dot{\omega} = \tau. \tag{A–30}$$

The above expression is almost acceptable, as it gives an expression for $\dot{\omega}$ in terms of τ, except that it requires us to evaluate the matrix $\sum m_i r_i'^* \dot{r}_i^*$, which is as expensive computing the inertia tensor from scratch. We'll use one last trick here to clean things up. Since $\dot{r}_i' = \omega \times r_i'$ and $r_i'^* \omega = -\omega \times r_i'$, we can write

$$\sum m_i \dot{r}_i'^* r_i'^* \omega = \sum m_i (\omega \times r_i')^* (-\omega \times r_i') = \sum -m_i (\omega \times r_i') \times (\omega \times r_i') = \mathbf{0}.$$

Thus, we can add $-\sum m_i \dot{r}_i'^* r_i'^* \omega = \mathbf{0}$ to equation (A–30) to obtain

$$\left(\sum -m_i r_i'^* \dot{r}_i'^* - m_i \dot{r}_i'^* r_i'^*\right) \omega + I(t)\dot{\omega} = \tau. \tag{A–31}$$

Finally, since

$$\dot{I}(t) = \frac{d}{dt} \sum -m_i r_i'^* r_i'^* = \sum -m_i r_i'^* \dot{r}_i'^* - m_i \dot{r}_i'^* r_i'^* \tag{A–32}$$

we have

$$\dot{I}(t)\omega + I(t)\dot{\omega} = \frac{d}{dt}(I(t)\omega) = \tau. \tag{A–33}$$

Since $L(t) = I(t)\omega(t)$, this leaves us with the final result that

$$\dot{L}(t) = \tau. \tag{A–34}$$