Not not to be or not to be?

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What is wrong with these theorems?

Theorem: Three points are either collinear or not.

Theorem: A non-constant polynomial has a complex root.

Theorem: Most functions are everywhere discontinuous.

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Theorem: Most functions are everywhere discontinuous.

But such functions are irrelevant for computer science.

Branches of Math tailored for Comp. Science

- Theory of computability
- Computational complexity & algorithms
- Numerical analysis
- Domain theory
- Cryptography
- Queueing theory
- Finite model theory
- Machine learning
- Type theory
- ...

Did you ever ask yourself . . .

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One way to do this is *realizability theory*.

Overview

1. Building a Realizability World
2. Life in a Realizability World
3. Practical Considerations

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   category $=$ objects $+$ morphisms

2. Apply tools of categorical logic to study it.

   Take categorical logic course in the Philosophy Dept.

Computational Views of the World

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- The *relative* view:
  1. Which data can be *represented*?
  2. How do we *compute* with data?

Relative Computability

- Data: contents of infinite RAM

- Computation: (finite) program

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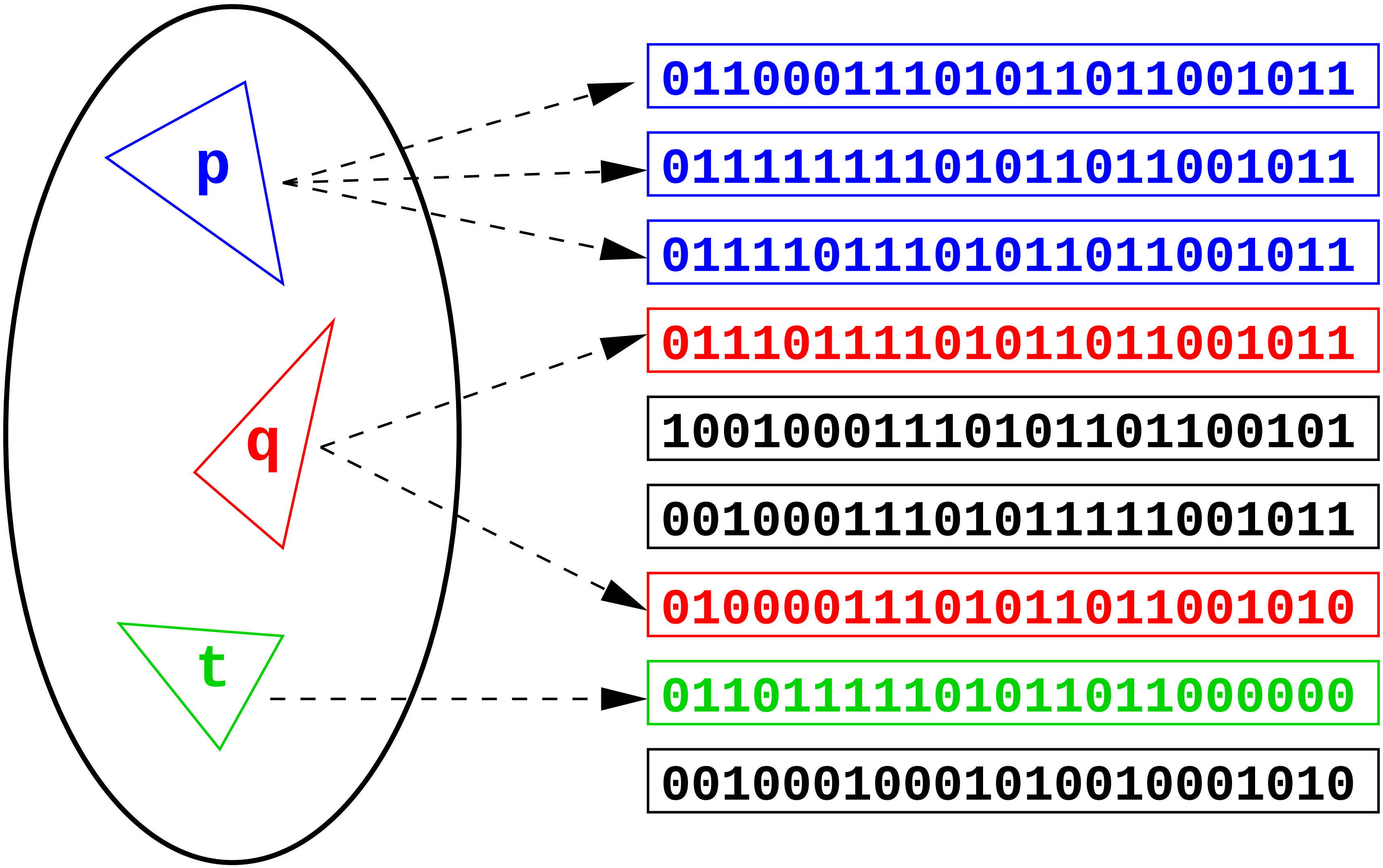
- Computation: (finite) program
  - any chosen general programming language
  - different language features give different worlds

Modest Sets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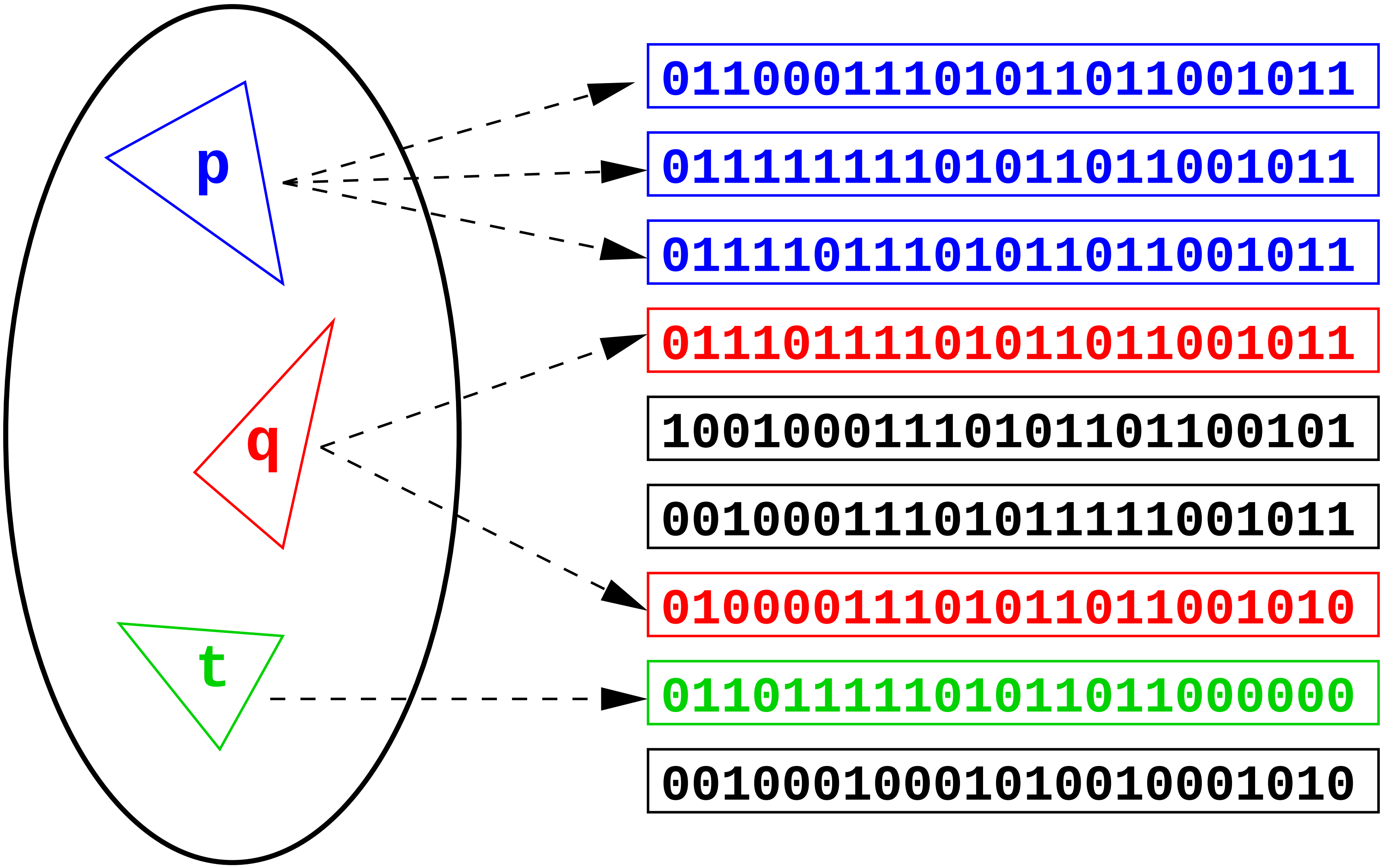
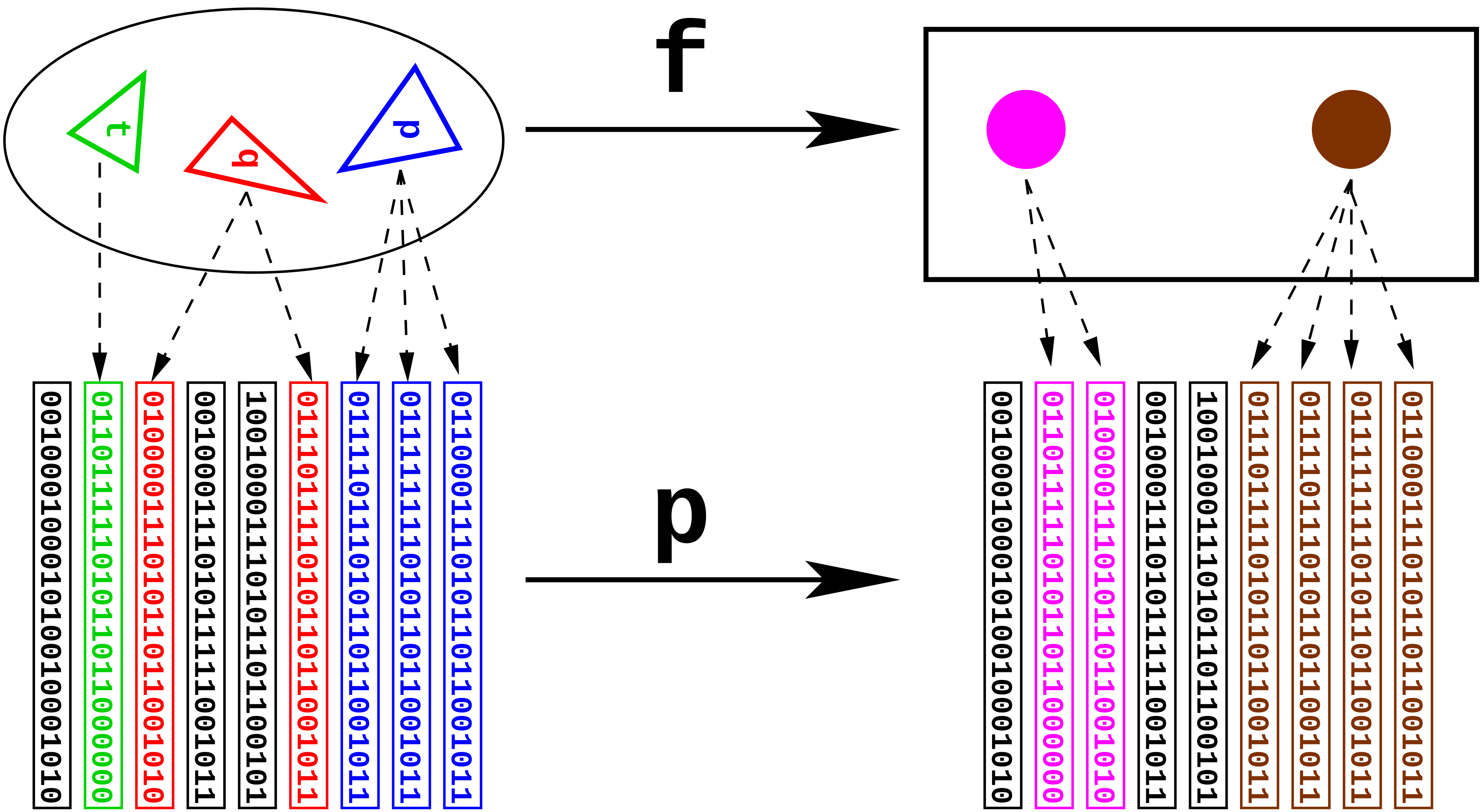


$c \Vdash x$ “c realizes (represents) x”

Realized Functions



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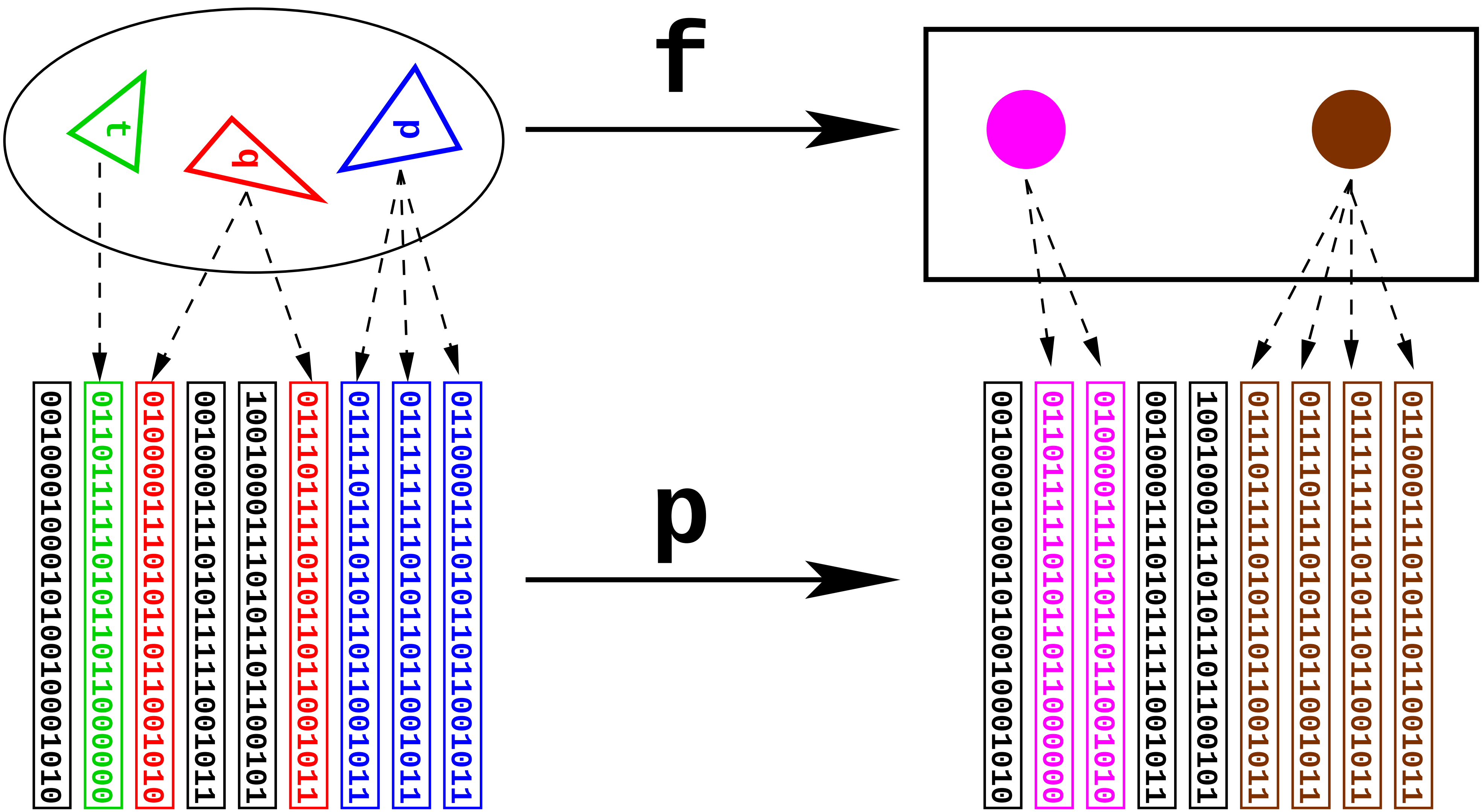

If $\quad c \Vdash x \quad$ then $\quad p(c) \Vdash f(x)$.

Computation-aware Sets and Functions

Foundation of classical mathematics:

sets & functions

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Foundation of *computation-aware* mathematics:

modest sets & realized functions

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4. Cartesian product $A \times B$

5. Function space $A \to B$

Example: real numbers $\mathbb{R}$

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- exponent $e \in \mathbb{Z}$
- *signed* binary digit representation:

$$x = 2^e \cdot \sum_{k=0}^{\infty} \frac{d_i}{2^{k+1}}$$

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The Particle Physics of Realizability

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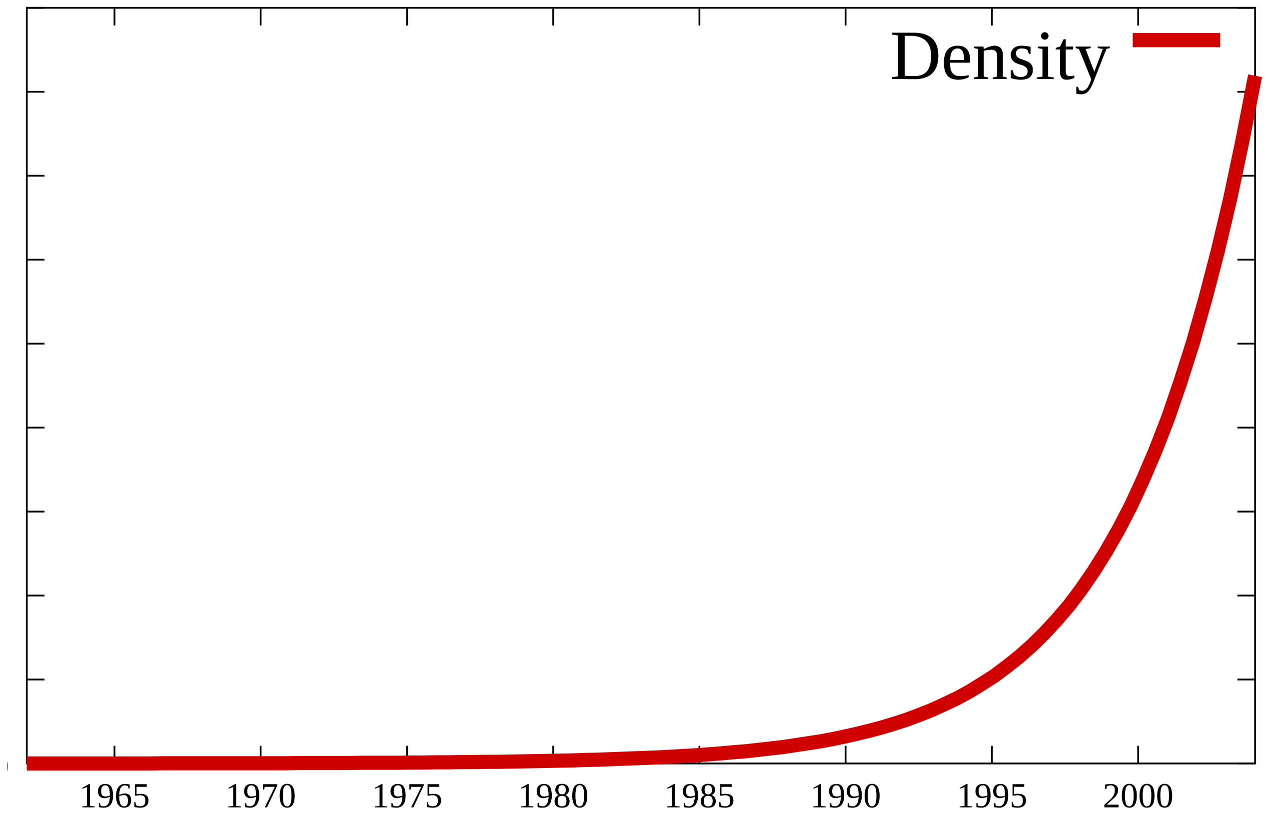
"The two basic realizers are S and K."

$$\mathsf{K}xy = x \qquad\qquad Sxyz = (xz)(yz)$$

The Cosmology of Realizability

"The universe is becoming denser at an exponential rate."

(Gordon Moore, 1965)



The Language of Realizability

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is a program p such that, for $\langle d, e \rangle \Vdash_{\mathbb{R}} x$,

$$p(d, e) = 0 \qquad \text{if } x < 0$$

$$p(d, e) = 1 \qquad \text{if } x \geq 0$$

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This is a good thing!

Markov's Principle

If elements of a set A can be enumerated and $\varphi(x)$ is a semi-decidable predicate then

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Witnessed by a program which iterates through all elements $x_1, x_2, \ldots$ of A and tests $\varphi(x_i)$ until one is found to hold.

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Such a witness exists only if we can use a `throw`–`catch` programming construct, or a similar control mechanism.

The “not not” translation

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Classical	Intuitionistic
$\varphi \implies \psi$	$\neg\neg(\varphi^* \implies \psi^*)$
$\varphi \wedge \psi$	$\neg\neg(\varphi^* \wedge \psi^*)$
$\varphi \vee \psi$	$\neg\neg(\varphi^* \vee \psi^*)$
$\exists x.\, \varphi(x)$	$\neg\neg\exists x.\, \varphi(x)^*$
$\forall x.\, \varphi(x)$	$\neg\neg\forall x.\, \varphi(x)^*$

Not not a Classic Masterpiece

Not not, not to not be, or not to be:
not that is not the question:
Not not, whether not 'tis not nobler in the mind to suffer
The slings and arrows of outrageous fortune,
Or not to not take arms against a sea of troubles,
And not by not opposing not not end them?

(not not Hamlet)

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Finite combinatorics is pretty much the same.

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We prove correctness of an implementation
by showing it realizes the desired specification.

A Question for Hardware Designers

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Would negative digits be useful in hardware implementation of floating point arithmetic?

A Challenge for Computational Geometers

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Prove theorems without using the dichotomy

$$\forall x \in \mathbb{R}. \, (x < 0 \vee x \geq 0)$$

Use instead

$$\forall \epsilon > 0. \, \forall x \in \mathbb{R}. \, (x < \epsilon \vee x > -\epsilon)$$

A Task for Programming Language Designers

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Ordinary `if–then–else` control mechanism is inappropriate for exact arithmetic.

Design practical data structures for real numbers *and* invent new control mechanisms for programming with them.

In Conclusion

The world of realizability is *your* world.