Coherence Numbers of Domains

Andrej Bauer
University of Ljubljana
Slovenia

Dana S. Scott
Carnegie Mellon University
Pittsburgh, USA

Mathematical Foundations of Programming Semantics XVIII
Tulane University, New Orleans, March 2002

Motivating the Audience

A classification of PER models on domains.

An unexpected theorem about reflexive domains.

Terminology

- (Scott) domain: ω-algebraic bounded complete dcpo

Terminology

- (Scott) domain: ω-algebraic bounded complete dcpo
- Continuous retract: $D \triangleleft E$ (“E contains D”)

$$D \underset{r}{\overset{i}{\rightleftarrows}} E \qquad\qquad r \circ i = \mathrm{id}_D$$

Terminology

- (Scott) domain: ω-algebraic bounded complete dcpo
- Continuous retract: $D \triangleleft E$ (“E contains D”)

$$D \underset{r}{\overset{i}{\rightleftarrows}} E \qquad\qquad r \circ i = \mathrm{id}_D$$

- Universal domain U for a class $\mathcal{C}$ of domains:

  $U \in \mathcal{C}$ and $E \triangleleft U$ for all $E \in \mathcal{C}$.

Terminology

- (Scott) domain: ω-algebraic bounded complete dcpo
- Continuous retract: $D \triangleleft E$ ("E contains D")

$$D \underset{r}{\overset{i}{\rightleftarrows}} E \qquad\qquad r \circ i = \mathrm{id}_D$$

- Universal domain U for a class $\mathcal{C}$ of domains: $U \in \mathcal{C}$ and $E \triangleleft U$ for all $E \in \mathcal{C}$.
- Reflexive domain: $[D \to D] \triangleleft D$ and $D \neq \{\bot\}$

PER Models

A reflexive domain D is a model of untyped λ-calculus, therefore a *combinatory algebra*:

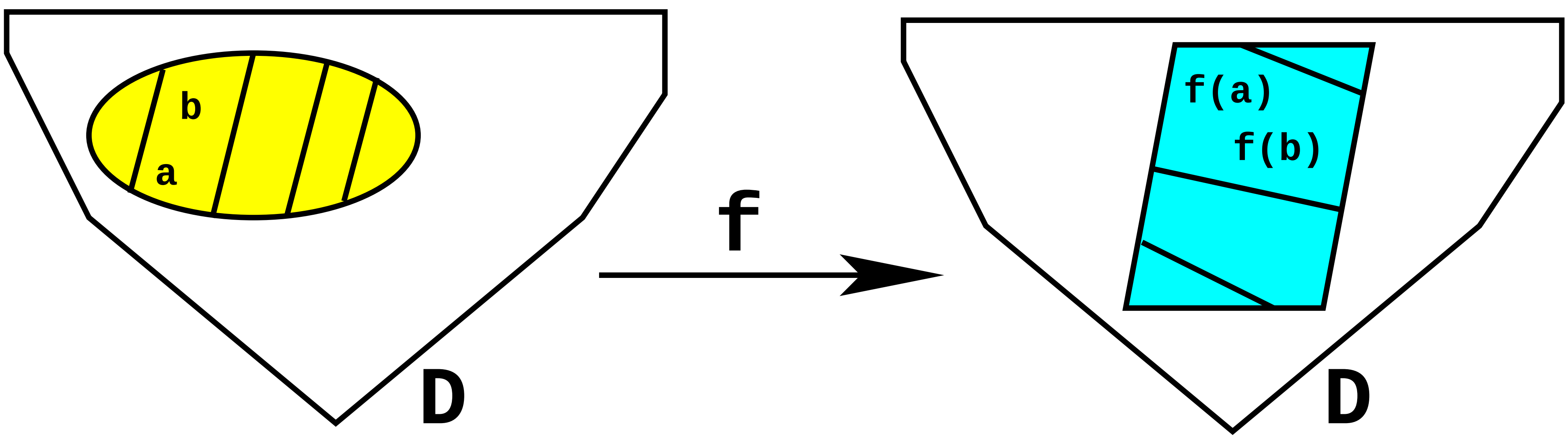
$$K = \lambda x.\, \lambda y.\, x \qquad\qquad S = \lambda x.\, \lambda y.\, \lambda z.\, (xz)(yz)$$

PER Models

A reflexive domain D is a model of untyped λ-calculus, therefore a *combinatory algebra*:

$$K = \lambda x.\, \lambda y.\, x \qquad\qquad S = \lambda x.\, \lambda y.\, \lambda z.\, (xz)(yz)$$

PER(D), category of *partial equivalence relations* on D:



How many PER models are there?

Given reflexive domains D and E, when is

$$\mathsf{PER}(D) \simeq \mathsf{PER}(E) \quad ?$$

How many PER models are there?

Given reflexive domains D and E, when is

$$\mathsf{PER}(D) \simeq \mathsf{PER}(E) \quad ?$$

Some well-known PER models:

- $\mathsf{PER}(\mathcal{P}\omega)$ where $\mathcal{P}\omega$ is the graph model
- $\mathsf{PER}(T^{\omega})$ where T^{ω} is Plotkin's universal coherent domain
- $\mathsf{PER}(U)$ where U is universal for Scott domains

These are all different.

The Answer

Theorem:

For reflexive D and E,

$$\mathsf{PER}(\mathrm{D}) \simeq \mathsf{PER}(\mathrm{E}) \quad \Longleftrightarrow \quad \mathsf{coh}(\mathrm{D}) = \mathsf{coh}(\mathrm{E})$$

where $\mathsf{coh}(\mathrm{D})$ is the *coherence number* of D,

$1 \leq \mathsf{coh}(\mathrm{D}) \leq \omega$.

Coherent Domains

[Plotkin'78]

D is *coherent* when, for all $S \subseteq D$,

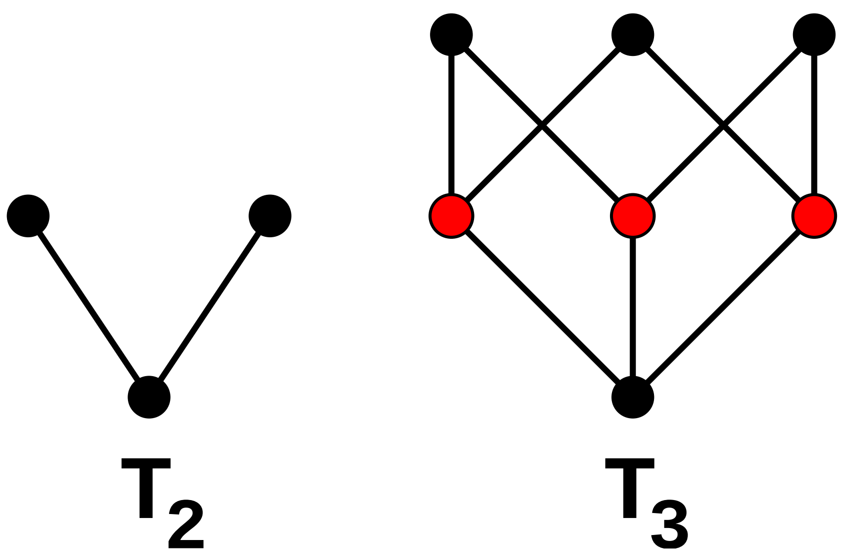
$$(\forall x_1, x_2 \in S. \{x_1, x_2\} \text{ bounded}) \implies S \text{ bounded}$$

Coherent Domains

[Plotkin'78]

D is *coherent* when, for all $S \subseteq D$,

$$(\forall x_1, x_2 \in S. \{x_1, x_2\} \text{ bounded}) \Longrightarrow S \text{ bounded}$$



Coherence Numbers

Definition:

D is n-*coherent*, when for all $S \subseteq D$,

$$(\forall x_1, \ldots, x_n \in S. \{x_1, \ldots, x_n\} \text{ bounded}) \implies S \text{ bounded}$$

Coherence Numbers

Definition:

D is *n-coherent*, when for all $S \subseteq D$,

$$(\forall x_1, \ldots, x_n \in S. \{x_1, \ldots, x_n\} \text{ bounded}) \implies S \text{ bounded}$$

Additionally, we say that every D is *ω-coherent*.

Coherence Numbers

Definition:

D is n-*coherent*, when for all $S \subseteq D$,

$$(\forall x_1, \ldots, x_n \in S. \{x_1, \ldots, x_n\} \text{ bounded}) \implies S \text{ bounded}$$

Additionally, we say that every D is ω-*coherent*.

Coherence number:

$$\mathrm{coh}(D) = \min \left\{ 1 \leq n \leq \omega \;\middle|\; D \text{ is } n\text{-coherent} \right\}$$

Examples

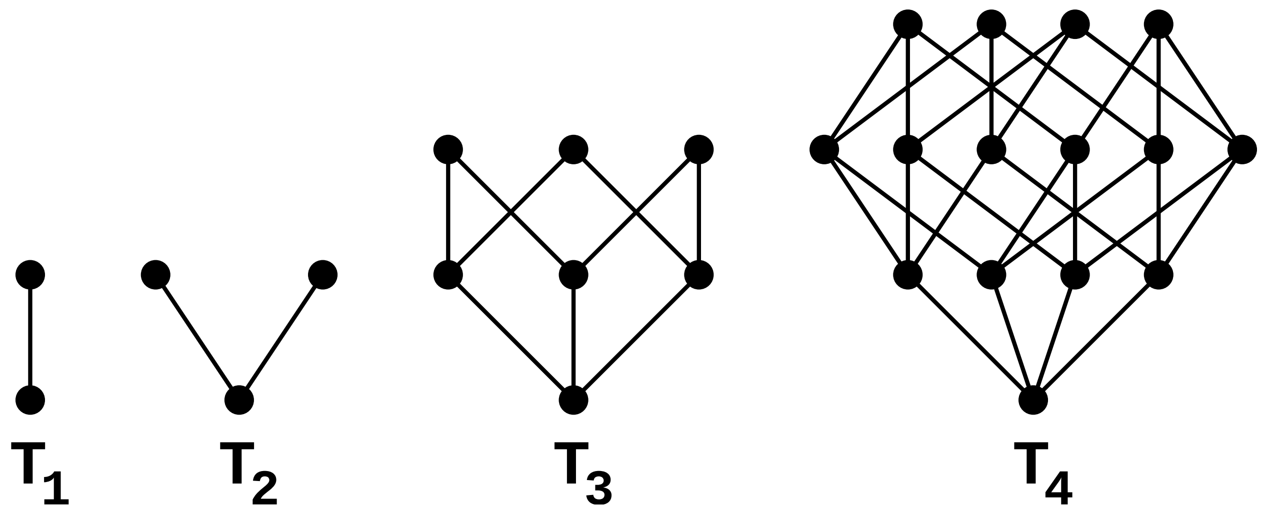
1-coherent = lattice

2-coherent = Plotkin coherent

Examples

1-coherent = lattice

2-coherent = Plotkin coherent



$$T_1 = \Sigma = \{\bot, \top\}$$

$$T_n = \mathcal{P}(\{1, \ldots, n\}) \setminus \{\{1, \ldots, n\}\} \qquad (2 \le n < \omega)$$

Observations about Coherence Numbers

Proposition: $\mathrm{coh}(T_n) = n$

Proposition:

Let $2 \leq n < \omega$. Then $T_n \triangleleft D \iff \mathrm{coh}(D) \geq n$.

Observations about Coherence Numbers

Proposition: $\mathsf{coh}(T_n) = n$

Proposition:

Let $2 \leq n < \omega$. Then $T_n \triangleleft D \iff \mathsf{coh}(D) \geq n$.

Proposition:

- $\mathsf{coh}(D \times E) = \max(\mathsf{coh}(D), \mathsf{coh}(E))$
- $\mathsf{coh}(D^\omega) = \mathsf{coh}(D)$
- $\mathsf{coh}([D \to E]) = \mathsf{coh}(E)$

n-Coherent Domains

Let Coh_n be the category of n-coherent domains.

$$\mathsf{Lat} = \mathsf{Coh}_1 \subseteq \mathsf{Coh}_2 \subseteq \mathsf{Coh}_3 \subseteq \cdots \subseteq \mathsf{Coh}_\omega = \mathsf{Dom}$$

Each Coh_n is a cartesian-closed subcategory of Dom.

n-Coherent Domains

Let Coh_n be the category of n-coherent domains.

$$\mathsf{Lat} = \mathsf{Coh}_1 \subseteq \mathsf{Coh}_2 \subseteq \mathsf{Coh}_3 \subseteq \cdots \subseteq \mathsf{Coh}_\omega = \mathsf{Dom}$$

Each Coh_n is a cartesian-closed subcategory of Dom.

Does Coh_n have a universal domain?

n-Coherent Domains

Let Coh_n be the category of n-coherent domains.

$$\mathsf{Lat} = \mathsf{Coh}_1 \subseteq \mathsf{Coh}_2 \subseteq \mathsf{Coh}_3 \subseteq \cdots \subseteq \mathsf{Coh}_\omega = \mathsf{Dom}$$

Each Coh_n is a cartesian-closed subcategory of Dom.

Does Coh_n have a universal domain?

T_1^ω is universal for Coh_1.

T_2^ω is universal for Coh_2.

Therefore by induction. . .

Universal domain for Coh_n

Let $\mathbb{F}_n = \left\{x \subseteq \mathbb{N} \;\middle|\; |x| < n\right\}$, and define:

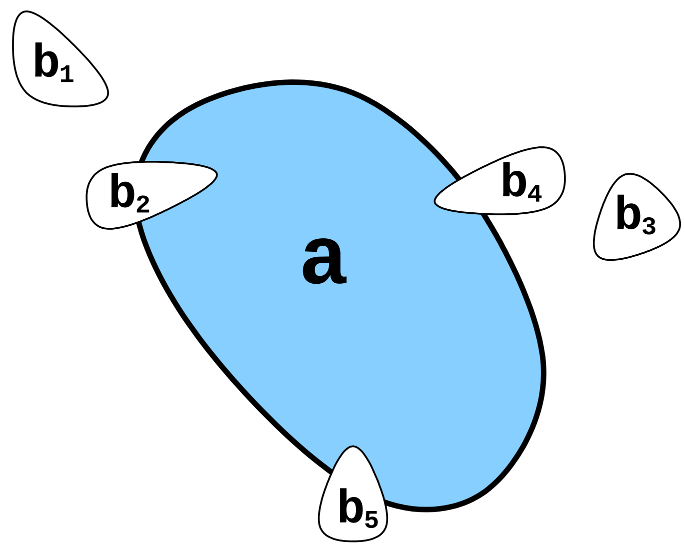
$$\mathbb{C}_n = \left\{\langle a, B\rangle \in \mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{F}_n \;\middle|\; \forall b \in B.\, b \not\subseteq a\right\}$$

Universal domain for Coh_n

Let $\mathbb{F}_n = \{x \subseteq \mathbb{N} \mid |x| < n\}$, and define:

$$\mathbb{C}_n = \{\langle a, B\rangle \in \mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{F}_n \mid \forall b \in B.\, b \not\subseteq a\}$$

A typical element $\langle a, \{b_1, b_2, \ldots\}\rangle$ of $\mathbb{C}_n$:

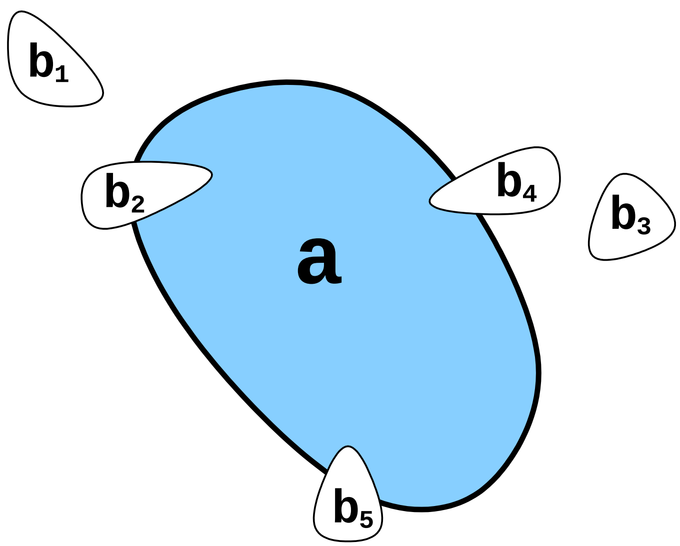


Universal domain for Coh_n

Let $\mathbb{F}_n = \{x \subseteq \mathbb{N} \mid |x| < n\}$, and define:

$$\mathbb{C}_n = \{\langle a, B\rangle \in \mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{F}_n \mid \forall b \in B.\, b \not\subseteq a\}$$

A typical element $\langle a, \{b_1, b_2, \ldots\}\rangle$ of $\mathbb{C}_n$:



$$\mathbb{C}_1 \cong \mathcal{P}\mathbb{N} \cong \mathsf{T}_1^\omega \qquad \mathbb{C}_2 \cong \mathsf{T}_2^\omega$$

Universal domain for Coh_n

Theorem: $\mathbb{C}_n$ is universal for Coh_n.

Universal domain for Coh_n

Theorem: $\mathbb{C}_n$ is universal for Coh_n.

Proof: a generalization of [Plotkin'78].

Universal domain for Coh_n

Theorem: $\mathbb{C}_n$ is universal for Coh_n.

Proof: a generalization of [Plotkin'78].

Theorem:

- $\mathbb{C}_n \lhd \mathsf{T}_n^\omega \quad$ for $n < \omega$,
- $\mathbb{C}_\omega \lhd \prod_{n<\omega} \mathsf{T}_n$.

Hence T_n^ω is universal for Coh_n,

and $\prod_{n<\omega} \mathsf{T}_n$ is universal for $\mathsf{Coh}_\omega = \mathsf{Dom}$.

A Theorem about Reflexive Domains

Theorem:

Let D be a domain with $n = \mathrm{coh}(D)$.

The following are equivalent:

1. D is reflexive

A Theorem about Reflexive Domains

Theorem:

Let D be a domain with $n = \mathsf{coh}(D)$.

The following are equivalent:

1. D is reflexive
2. $D \times D \lhd D$

A Theorem about Reflexive Domains

Theorem:

Let D be a domain with $n = \mathsf{coh}(D)$.

The following are equivalent:

1. D is reflexive
2. $D \times D \lhd D$
3. $E \times D \lhd D$ for some E with $\mathsf{coh}(E) = \mathsf{coh}(D)$

A Theorem about Reflexive Domains

Theorem:

Let D be a domain with $n = \mathsf{coh}(D)$.

The following are equivalent:

1. D is reflexive
2. $D \times D \lhd D$
3. $E \times D \lhd D$ for some E with $\mathsf{coh}(E) = \mathsf{coh}(D)$
4. $T_k \times D \lhd D$ for all $k < 1 + n$

A Theorem about Reflexive Domains

Theorem:

Let D be a domain with $n = \mathsf{coh}(D)$.

The following are equivalent:

1. D is reflexive
2. $D \times D \lhd D$
3. $E \times D \lhd D$ for some E with $\mathsf{coh}(E) = \mathsf{coh}(D)$
4. $T_k \times D \lhd D$ for all $k < 1 + n$
5. D is universal for Coh_n

Reflexive domains are easy to come by

Corollary:

D^ω is reflexive for any non-trivial domain D.

Reflexive domains are easy to come by

Corollary:

D^ω is reflexive for any non-trivial domain D.

Proof: $D^\omega \times D^\omega \cong D^\omega$

Reflexive domains are easy to come by

Corollary:

D^ω is reflexive for any non-trivial domain D.

Proof: $D^\omega \times D^\omega \cong D^\omega$

Corollary:

$[E \to D]$ is reflexive if $\max(E)$ is infinite and D non-trivial.

Reflexive domains are easy to come by

Corollary:

D^ω is reflexive for any non-trivial domain D.

Proof: $D^\omega \times D^\omega \cong D^\omega$

Corollary:

$[E \to D]$ is reflexive if $\max(E)$ is infinite and D non-trivial.

Proof: $\mathbb{N}_\perp \lhd E$, therefore $D^\omega \lhd [\mathbb{N}_\perp \to D] \lhd [E \to D]$

Classification of PER models

Theorem:

For reflexive D and E,

$$\mathsf{coh}(D) = \mathsf{coh}(E) \quad \Longrightarrow \quad \mathsf{PER}(D) \simeq \mathsf{PER}(E)\ .$$

Proof:

Sufficient to prove $D \lhd E$ and $E \lhd D$.

Classification of PER models

Theorem:

For reflexive D and E,

$$\mathsf{coh}(D) = \mathsf{coh}(E) \quad \Longrightarrow \quad \mathsf{PER}(D) \simeq \mathsf{PER}(E) \, .$$

Proof:

Sufficient to prove $D \triangleleft E$ and $E \triangleleft D$.

Let $n = \mathsf{coh}(D) = \mathsf{coh}(E)$.

D reflexive $\Longrightarrow$ D universal for Coh_n $\Longrightarrow$ $E \triangleleft D$.

Similarly for $D \triangleleft E$. QED.

Classification of PER models

Every $\mathsf{PER}(D)$ is equivalent to precisely one of

$$\mathsf{PER}(\mathsf{T}_1^\omega)$$

$$\mathsf{PER}(\mathsf{T}_2^\omega)$$

$$\mathsf{PER}(\mathsf{T}_3^\omega)$$

$$\vdots$$

$$\mathsf{PER}(\textstyle\prod_{n<\omega} \mathsf{T}_n)$$

Classification of PER models

Every PER(D) is equivalent to precisely one of

$$\mathsf{PER}(\mathsf{T}_1^{\omega})$$

$$\mathsf{PER}(\mathsf{T}_2^{\omega})$$

$$\mathsf{PER}(\mathsf{T}_3^{\omega})$$

$$\vdots$$

$$\mathsf{PER}(\textstyle\prod_{n<\omega} \mathsf{T}_n)$$

An analogous statement holds for realizability toposes.

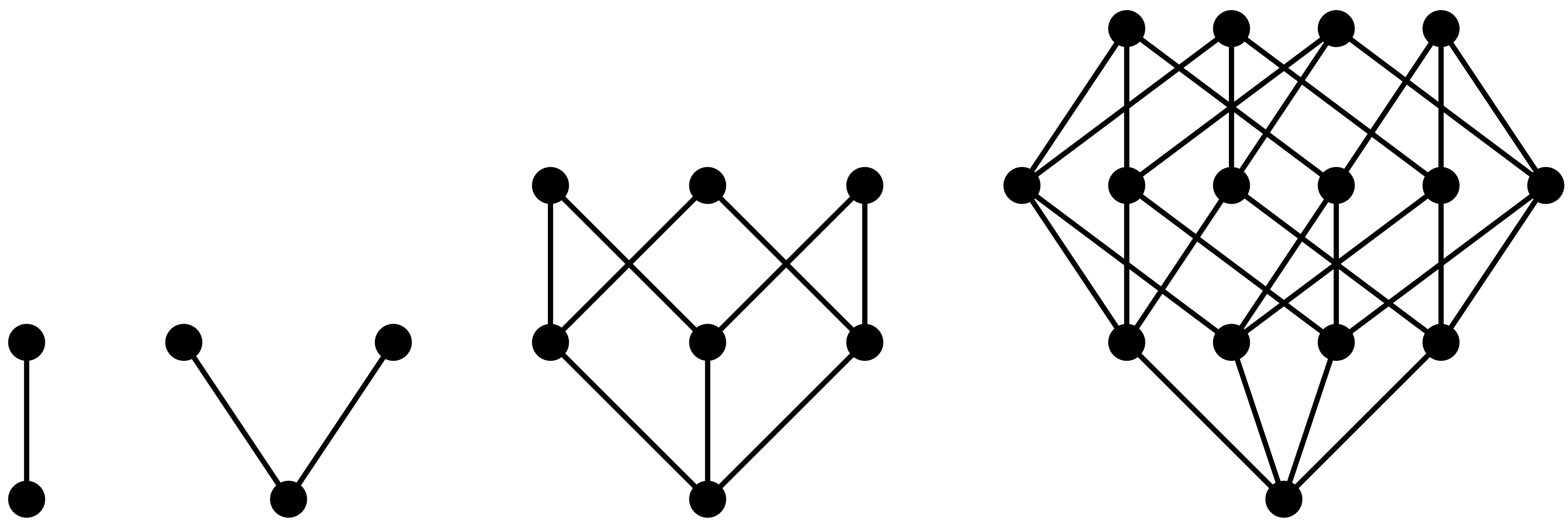
Questions

What is this good for?

Questions

What is this good for?

What is the computational understanding of T_k?



T_1 T_2 T_3 T_4

The ‘whodoneit’ interpretation