15-780 Homework 1
Deadline: 10:30 am on February 3
There are 150 total points: each component of each question is worth 10 points.

1) Consider the following statement:

"The bouncers checked the IDs of everyone who entered the club who wasn't a VIP. Some men entered the club, and it happened that their IDs were only checked by men. No man was a VIP."

a) Translate each sentence into first-order logic. Please explain the meaning of any variables or predicates you define.

b) Convert each sentence into CNF, showing each step.

c) Can we conclude that some of the bouncers were men? Prove your answer using resolution, showing each step of the proof.

2) In 3x3 Sudoku, each number from the set $\{1, 2, 3\}$ must appear exactly once in each row and each column of a 3x3 grid. In standard Sudoku, the grid is 9x9 and each number from 1 to 9 must appear exactly once in each row, each column, and each of the nine 3x3 subgrids (see the lecture notes for an example).

a) Show how to formulate a given 3x3 Sudoku problem instance as a propositional logic formula in conjunctive normal form. Try to use as few variables and clauses as possible, and explain what the variables represent. How many variables and how many clauses are there in your formulation?

b) If you generalized your approach to represent standard 9x9 Sudoku, how many variables and how many clauses would there be? Justify your answer.

c) For the 3x3 puzzle, what is the minimal number of squares that must be revealed for there to be a unique solution? Justify your answer.

d) For the 3x3 puzzle, what is the maximal number of squares that can be revealed such that there is more than one possible solution? Justify your answer.

3) In this problem, we investigate whether any formula that can be represented using the three connectives $\wedge$, $\vee$, and $\neg$, can also be represented using a single (but more complex) connective.

a) Prove that any formula represented using connectives $\wedge$, $\vee$, and $\neg$, can be transformed into an equivalent formula using only NAND, where x NAND y is defined to be $\neg(x \wedge y)$.

b) Prove that any formula represented using connectives $\wedge$, $\vee$, and $\neg$, can be transformed into an equivalent formula using only NOR, where x NOR y is defined to be $\neg(x \vee y)$.

Define x XOR y as $(x \vee y) \wedge (\neg x \vee \neg y)$. Let S be the set of formulas that use only the connective

XOR, literals x, y and the constants T, F, defined recursively as follows:

- $T, F, x, y \in S$.
- if $\alpha, \beta \in S$, then α XOR $\beta \in S$.

c) Consider each of the four assignments to the variables x, y: TT, TF, FT, FF. Prove that every formula in S is true for an even number of these assignments.

d) Prove that there exists a formula that can be represented using the connectives $\wedge$, $\vee$, and $\neg$ that cannot be transformed into an equivalent formula using only XOR.

4) In this problem we will prove some elementary facts of number theory from the first-order Peano axioms.

Here are the first-order Peano axioms (this version due to Wikipedia), along with their translations into first-order logic (using the convention that free variables are universally quantified). Assume that we are using first-order logic with equality, as described in section 8.2 of Russel/Norvig.

- For every natural number x, $x = x$. That is, equality is reflexive.

$$number(x) \Rightarrow x = x.$$

- For all natural numbers x and y, if $x = y$, then $y = x$. That is, equality is symmetric.

$$number(x) \wedge number(y) \Rightarrow (x = y \Rightarrow y = x)$$

- For all natural numbers x, y and z, if $x = y$ and $y = z$, then $x = z$. That is, equality is transitive.

$$number(x) \wedge number(y) \wedge number(z) ==> (x = y \wedge y = z \Rightarrow x = z)$$

- For all a and b, if a is a natural number and $a = b$, then b is also a natural number. That is, the natural numbers are closed under equality.

$$number(a) \wedge a = b \Rightarrow number(b)$$

- 0 is a natural number.

$$number(0)$$

- For every natural number n, $S(n)$ is a natural number denoting the successor of n.

$$number(n) \Rightarrow number(S(n))$$

- There is no object n with $S(n) = 0$. That is, zero is never a successor.

$$\neg(S(n) = 0)$$

- For all natural numbers m and n, if $S(m) = S(n)$, then $m = n$. That is, S is an injection.

$$number(m) \wedge number(n) \wedge S(m) = S(n) \Rightarrow m = n$$

Also suppose we are given the following definition of addition:

- $plus(a, b) = plus(b, a)$
- $plus(a, 0) = a$
- $plus(a, S(b)) = S(plus(a, b))$

a) State that 2+3 = 5. Prove that this statement is entailed by our assumptions.

b) Define a predicate $even(n)$ using first-order-logic. Show that 4 is even.

c) Suppose M is a satisfying model for the knowledge base consisting of the assumptions above. Can you put a lower bound on how many distinct objects M has? Prove your answer.

d) One satisfying model for our knowledge base is the standard or intended model, in which there is one object for each natural number, and $S(n)$ and plus(a, b) have the usual meanings. Are there others? If no, argue why not; if yes, give an example of one such model.