15-859N — Spectral Graph Theory and Scientific Computing — Fall 2007

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Assignment 2 Due date: November 22

1 Chebyshev Polynomials

[10 points]

In this problem we will develop some important identities for Chebyshev Polynomials.

1. Consider the following matrix:

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

   Give a one sentence explanation why $A(\theta)$ is rigid counter clockwise rotation by θ degrees and A_θ^n is a rotation by $n\theta$ degrees.

2. We can abstract A_θ to a matrix $A = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ where c and s are variables in some polynomial ring such that $c^2 + s^2 = 1$. Show that

$$A^n = \begin{pmatrix} T_n(c) & -sQ_n(c) \\ sQ_n(c) & T_n(c) \end{pmatrix}$$

   where T_n and Q_n are polynomials in c satisfying:

$$\begin{aligned} T_0(c) &= 1 \\ T_1(c) &= c \\ T_{n+1}(c) &= cT_n(c) - (1 - c^2)Q_n(c) \end{aligned}$$

   and

$$\begin{aligned} Q_0(c) &= 0 \\ Q_1(c) &= 1 \\ Q_{n+1}(c) &= cQ_n(c) + T_n(c) \end{aligned}$$

3. Use these identities to show that:

$$\begin{aligned} T_{n+1}(c) &= 2cT_n(c) - T_{n-1}(c) \\ Q_{n+1}(c) &= 2cQ_n(c) - Q_{n-1}(c). \end{aligned}$$

   Thus T and Q are Chebyshev Polynomials of the first and second kind respectively. Explain why all the roots of T_n and Q_n lie in the interval $[-1, +1]$ and in this interval T and Q return values in this interval.

4. Show how to diagonalize A for $|c| \geq 1$.

5. Use this diagonal form to show that

$$T_n(c) = \frac{(c + \sqrt{c^2 - 1})^n + (c - \sqrt{c^2 - 1})^n}{2} = \frac{(c + \sqrt{c^2 - 1})^n + (c + \sqrt{c^2 - 1})^{-n}}{2}$$