3.1. From Quadratic to Generalized Problems

The most common way of dealing with the above problem is to transform it into a (linear) generalized eigenvalue problem. For example, defining

$$v = \begin{pmatrix} \lambda u \\ u \end{pmatrix}$$

we can rewrite (9.27) as

$$\begin{pmatrix} -C & -K \\ I & 0 \end{pmatrix} v = \lambda \begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix} v \ . \tag{9.28}$$

It is clear that there is a large number of different ways of rewriting (9.27), the one above being one of the simplest. One advantage of (9.27) is that when M is Hermitian positive definite, as is often the case, then so also is the second matrix of the resulting generalized problem (9.28). If all matrices involved, namely K, C, and M, are Hermitian it might be desirable to obtain a generalized problem with Hermitian matrices, even though this does not in any way guarantee that the eigenvalues will be real. We can write instead of (9.28)

$$\begin{pmatrix} C & K \\ K & 0 \end{pmatrix} v = \lambda \begin{pmatrix} -M & O \\ O & K \end{pmatrix} v \quad . \tag{9.29}$$

An alternative to the above equation is

$$\begin{pmatrix} C & M \\ M & 0 \end{pmatrix} v = \mu \begin{pmatrix} -K & O \\ O & M \end{pmatrix} v \tag{9.30}$$

where we have set $\mu = 1/\lambda$. By comparing (9.29) and (9.30) we note the interesting fact that M and K have simply been interchanged. This could also have been observed directly from the original equation (9.27) by making the change of variable $\mu = 1/\lambda$. For practical purposes, we may therefore select between (9.30) and (9.29) the formulation that leads to the more economical computations. We will select (9.29) in the rest of this chapter.

While the difference between (9.30) and (9.29) may be insignificant, there are important practical implications in chosing between (9.28) and (9.29). Basically, the decision comes down to choosing an intrinsically non-Hermitian generalized eigen-problem with *a Hermitian positive definite* B matrix, versus a generalized eigen-problem where *both matrices in the pair are Hermitian indefinite.* In the case where M is a (positive) diagonal matrix, then the first approach is not only perfectly acceptable, but may even be the method of choice in case Arnoldi's method using a polynomial preconditioning is to be attempted. In case all matrices involved are Hermitian positive definite, there are strong reasons why the second approach is to be preferred. These are explained by Parlett and Chen [120]. Essentially, one can use a Lanczos type algorithm, similar to one of versions described in subsection 2.6, in spite of the fact that the B matrix that defines the inner products is indefinite.

PROBLEMS

P-9.1 Examine how the eigenvalues and eigenvectors of a pair of matrices (A, B) change when both A and B are multiplied by the same nonsingular matrix to the left or to the right.

P-9.2 In section 2.4 and 2.3 the shifts σ_1, σ_2 were assumed to be such that $1 - \sigma_1\sigma_2 \neq 0$. What happens if this were not to be the case? Consider both the linear shifts, Section 2.4 and Wielandt deflation 2.3.

P-9.3 Given the right and left eigenvectors u_1, and w_1 associated with an eigenvalue λ_1 of the pair A, B and such that $(Bu_1, Bw_1) = 1$, show that the matrix pair

$$A_1 = A - \sigma_1 B u_1 w_1^H B^H \ , \quad B_1 = B - \sigma_2 A u_1 w_1^H B^H$$

has the same left and right eigenvectors as A, B. The shifts σ_1, σ_2 are assumed to satisfy the condition $1 - \sigma_1\sigma_2 \neq 0$.

P-9.4 Show that when (A, B) are Hermitian and B is positive definite then $C = B^{-1}A$ is self-adjoint with respect to the B-inner product, i.e., that (9.22) holds.

P-9.5 Redo the proof of Proposition 9.1 with the usual definitions of eigenvalues ($Au = \lambda Bu$). What is gained? What is lost?

P-9.6 Show that algorithm 9.3 is a reformulation of Algorithm 9.2, applied to the pair (A', B') where $A' = B$ and $B' = (A - \sigma B)$.

NOTES AND REFERENCES. The reader is referred to Stewart and Sun [172] for more details and references on the theory of generalized eigenproblems. There does not seem to be any exhaustive coverage of the generalized eigenvalue problems, theory and algorithms, in one book. In addition, there seems to be a dichotomy between the need of users, mostly in finite elements modeling, and the numerical methods that numerical analysts develop. One of the first papers on the numerical solution of quadratic eigenvalue problems is Borri and Mantegazza [9]. Quadratic eigenvalue problems are rarely solved in structural engineering. The models are simplified first by neglecting damping and the leading eigenvalues of the resulting generalized eigenproblem are computed. Then the eigenvalues of the whole problem are approximated by performing a projection process onto the computed invariant subspace of the approximate problem [76]. This may very well change in the future, as models are improving and computer power is making rapid gains. ♠

Chapter X

Origins of Matrix Eigenvalue Problems

This chapter gives a brief overview of some applications that give rise to matrix eigenvalue problems. There are two broad classes of such applications. The first, and by far the largest currently, consists of problems related to the analysis of vibrations. These typically generate symmetric generalized eigenvalue problems. The second is the class of problems related to stability analysis, such as for example the stability analysis of an electrical network. In general, this second class of problems generates nonsymmetric matrices. The list of applications discussed in this chapter is by no means exhaustive. In fact the number of such applications is constantly growing as the software to solve large eigenvalue problems improves.

1. Introduction

The numerical computation of eigenvalues of large matrices is a problem of major importance in many scientific and engineering applications. We list below just a few of the applications areas where eigenvalue calculations arise:

- Structural dynamics
- Electrical Networks
- Combustion processes
- Macro-economics
- Normal mode techniques
- Quantum chemistry
- Markov chain techniques
- Chemical reactions
- Magnetohydrodynamics
- Control theory

This list is certainly not exhaustive. The most commonly solved eigenvalue problems today are those issued from the first item in the list, namely those problems associated with the vibration analysis of large structures. Complex structures such as those of an aircraft or a turbine are represented by finite element models involving a large number of degrees of freedom. To compute the natural frequencies of the structure one usually solves a generalized eigenvalue problem of the form $Ku = \lambda Mu$ where typically, but not always, the stiffness and mass matrices K and M respectively, are both symmetric positive definite.

In the past decade tremendous advances have been achieved in the solution methods for symmetric eigenvalue problems especially those related to problems of structures. The well-known structural analysis package, NASTRAN, which was developed by engineers in the sixties and seventies now incorporates the state of the art in numerical methods for eigenproblems such as block Lanczos techniques.

Similar software for the nonsymmetric eigenvalue problem on the other hand remains lacking. There seems to be two main causes for this. First, in structural engineering where such problems occur in models that include damping, and gyroscopic effects, it is a common practice to replace the resulting quadratic problem by a small dense problem much less difficult to solve using heuristic arguments. A second and more general reason is due to a prevailing view among applied scientists that the large nonsymmetric eigenvalue problems arising from their more accurate models are just intractable or difficult to solve numerically. This often results in simplified models to yield smaller matrices that can be handled by standard methods. For example, one-dimensional models may be used instead of two-dimensional or three-dimensional models. This line of reasoning is not totally unjustified since nonsymmetric eigenvalue problems can be hopelessly difficult to solve in some situations due for example, to poor conditioning. Good numerical algorithms for non-Hermitian eigenvalue problems tend also to be far more complex that their Hermitian counterparts. Finally, as was reflected in earlier chapters, the theoretical results that justify their use are scarcer.

The goal of this chapter is mainly to provide motivation and it is independent of the rest of the book. We will illustrate the main ideas that lead to the various eigenvalue problems in some of the applications mentioned above. The presentation is simplified in order to convey the overall principles.

2. Mechanical Vibrations

Consider a small object of mass m attached to an elastic spring suspended from the lid of a rigid box, see Figure 10.1. When stretched by a distance Δl the spring will exert a force of magnitude $k\Delta l$ whose direction is opposite to the direction of the displacement. Moreover, if there is a fluid in the box, such as oil, a displacement will cause a damping, or drag force to the movement, which is usually proportional to the velocity of the movement. Let us call l the distance of the center of the object from the top of the box when the mass is at equilibrium and denote by y the position of the mass at time t, with the initial position $y = 0$ being that of equilibrium. Then at any given time there are four forces acting on m:

1. The gravity force mg pulling downward;

2. The spring force $-k(l+y)$;
3. The damping force $-c\dfrac{dy}{dt}$;
4. The external force $F(t)$.

By Newton's law of motion,

$$m\frac{d^2y}{dt^2} = mg - k(l+y) - c\frac{dy}{dt} + F(t)\ .$$

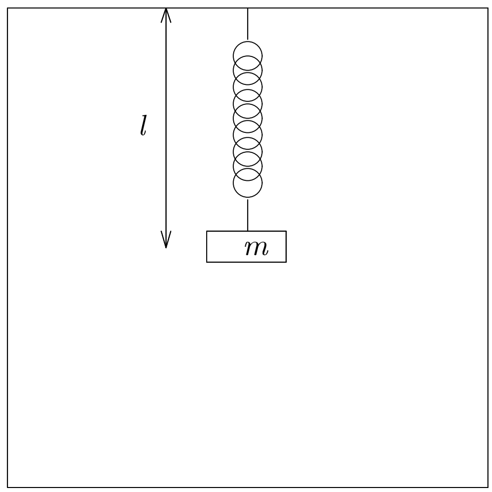

Figure 10.1 Model problem in mechanical vibrations

If we write the equation at steady state, i.e., setting $y \equiv 0$ and $F(t) \equiv 0$, we get $mg = kl$. As a result the equation simplifies into

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F(t)\ . \tag{10.1}$$

Free vibrations occur when there are no external forces and when the damping effects are negligible. Then (10.1) becomes

$$m\frac{d^2y}{dt^2} + ky = 0 \tag{10.2}$$

the general solution of which is of the form

$$y(t) = R\cos\left(\frac{k}{m}t - \phi\right)$$

which means that the mass will oscillate about its equilibrium position with a period of $2\pi/\omega_0$, with $\omega_0 \equiv k/m$.

Damped free vibrations include the effect of damping but exclude any effects from external forces. They lead to the homogeneous equation:

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0$$

whose characteristic equation is $mr^2 + cr + k = 0$.

When $c^2 - 4km > 0$ then both solutions r_1, r_2 of the characteristic equation are negative and the general solution is of the form

$$y(t) = ae^{r_1 t} + be^{r_2 t}$$

which means that the object will return very rapidly to its equilibrium position. A system with this characteristic is said to be *overdamped*.

When $c^2 - 4km = 0$ then the general solution is of the form

$$y(t) = (a + bt)e^{-ct/2m}$$

which corresponds to critical damping. Again the solution will return to its equilibrium but in a different type of movement from the previous case. The system is said to be *critically damped*.

Finally, the case of *underdamping* corresponds to the situation when $c^2 - 4km < 0$ and the solution is of the form

$$y(t) = e^{-ct/2m}\,[a\cos\mu t + b\sin\mu t]$$

with

$$\mu = \frac{\sqrt{4km - c^2}}{2m}\ .$$

This time the object will oscillate around its equilibrium but the movement will die out quickly.

In practice the most interesting case is that of *forced vibrations*, in which the exterior force F has the form $F(t) = F_0 \cos \omega t$. The corresponding equation is no longer a homogeneous equation, so we need to seek a particular solution to the equation (10.1) in the form of a multiple of $\cos(\omega t - \delta)$. Doing so, we arrive after some calculation at the solution

$$\eta(t) = \frac{F_0 \cos(\omega t - \delta)}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \tag{10.3}$$

where

$$\tan \delta = \frac{c\omega}{k - m\omega^2} \ .$$

See Exercise P-10.3 for a derivation. The general solution to the equations with forcing is obtained by adding this particular solution to the general solution of the homogeneous equation seen earlier.

The above solution is only valid when $c \neq 0$. When $c = 0$, i.e., when there are no damping effects, we have what is referred to as *free forced vibrations* . In this case, letting $\omega_0^2 = \frac{k}{m}$, a particular solution of the nonhomogeneous equation is

$$\frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

when $\omega \neq \omega_0$ and

$$\frac{F_0 t}{2m\omega_0} \sin \omega_0 t \tag{10.4}$$

otherwise. Now every solution is of the form

$$y(t) = a \cos \omega t + b \ \sin \omega t + \frac{F_0}{2m\omega_0} \ t \ \sin \omega_0 t.$$

The first two terms in the above solution constitute a periodic function but the last term represents an oscillation with a dangerously increasing amplitude.

This is referred to as a resonance phenomenon and has been the cause of several famous disasters in the past, one of the most recent ones being the Tacoma bridge disaster (Nov. 7, 1940). Another famous such catastrophe, is that of the Broughton suspension bridge near Manchester England. In 1831 a column of soldiers marched on it in step causing the bridge to enter into resonance and collapse. It has since become customary for soldiers to break step when entering a bridge.

Note that in reality the case $c = 0$ is fallacious since some damping effects always exist. However, in practice when c is very small the particular solution (10.3) can become very large when $\omega^2 = k/m$. Thus, whether c is zero or simply very small, *dangerous oscillations can occur whenever the forcing function F has a period equal to that of the free vibration case.*

We can complicate matters a little in order to introduce matrix eigenvalue problems by taking the same example as before and add another mass suspended to the first one, as is shown in Figure 10.2.

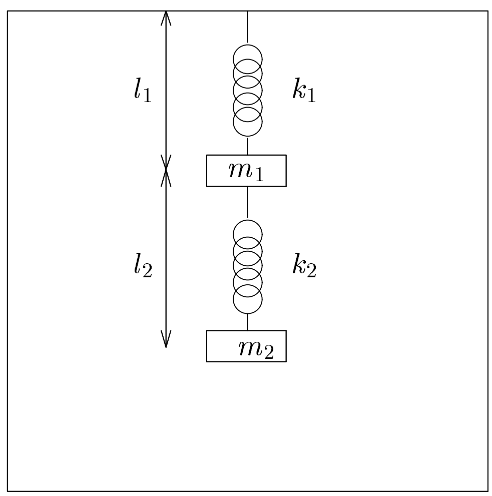

Figure 10.2 A spring system with two masses.

Assume that at equilibrium, the center of gravity of the first mass is at distance l_1 from the top and that of the second is at distance l_2 from the first one. There are now two unknowns, the displacement y_1 from the equilibrium of the first mass and the displacement y_2 from its equilibrium position of the second mass. In addition to the same forces as those for the single mass case, we must now include the effect of the spring force pulling from the other spring. For the first mass this is equal to

$$k_2[l_2 - y_1 + y_2],$$

which clearly corresponds to a displacement of the second mass relative to the first one. A force equal to this one in magnitude but opposite in sign acts on the second mass in addition to the other forces. Newton's law now yields

$$m_1 \frac{d^2 y_1}{dt^2} = m_1 g - k_1(l_1 + y_1) - c_1 \frac{dy_1}{dt} + k_2(l_2 + y_2 - y_1) + F_1(t) \ ,$$

$$m_2 \frac{d^2 y_2}{dt^2} = m_2 g - k_2(l_2 + y_1) - c \frac{dy_2}{dt} - k_2(l_2 + y_2 - y_1) + F_2(t) \ .$$

At equilibrium the displacements as well as their derivatives, and the external forces are zero. As a result we must have $0 = m_1 g - k_1 l_1 + k_2 l_2$, and $0 = m_2 g - 2k_2 l_2$. Hence the simplification

$$m_1 \frac{d^2 y_1}{dt^2} + c_1 \frac{dy_1}{dt} + (k_1 + k_2) y_1 - k_2 y_2 = F_1(t) \ , \tag{10.5}$$

$$m_2 \frac{d^2 y_2}{dt^2} + c_2 \frac{dy_2}{dt} - k_2 y_1 + 2k_2 y_2 = F_2(t) \ . \tag{10.6}$$

Using the usual notation of mechanics for derivatives, equations (10.5) and (10.6) can be written in condensed form as

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{pmatrix} + \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & 2k_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \tag{10.7}$$

or,

$$M\ddot{y} + C\dot{y} + Ky = F \tag{10.8}$$

in which M, C and K are 2×2 matrices. More generally, one can think of a very large structure, for example a high rise building, as a big collection of masses and springs that are interacting with each other just as in the previous example. In fact equation (10.8) is the typical equation considered in structural dynamics but the matrices M, K, and C can be very large. One of the major problems in structural engineering it to attempt to avoid vibrations, i.e., the resonance regime explained earlier for the simple one mass case. According to our previous discussion this involves avoiding the eigenfrequencies, ω_0 in the previous example, of the system. More exactly, an analysis is made before the structure is build and the proper frequencies are computed. There is usually a band of frequencies that must be avoided. For example, an earthquake history of the area may suggest avoiding specific frequencies. Here, the proper modes of the system are determined by simply computing oscillatory solutions of the form $y(t) = y_0 e^{i\omega t}$ that satisfies the free undamped vibration equation

$$M\ddot{y} + Ky = 0$$

or

$$-\omega^2 M y_0 + K y_0 = 0 \ .$$

3. Electrical Networks.

Consider a simple electrical circuit consisting of a resistance or R Ohms, an inductance of L Henrys and a capacitor of C Farads connected in series with a generator of E volts. In a closed circuit, the sum of the voltage drops is equal to the input voltage $E(t)$. The voltage drop across the resistance is RI where I is the intensity while it is $L\dot{I}$ across the inductance and Q/C across the capacitor where Q is the electric charge whose derivative is I. Therefore the governing equations can be written in terms of Q as follows,

$$L\ddot{Q} + R\dot{Q} + Q/C = E(t) \ ,$$

which resembles that of mechanical vibrations.

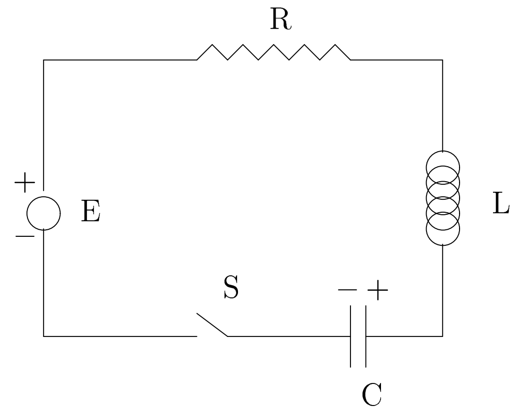

Figure 10.3 A simple series electric circuit.

Realistic electric networks can be modeled by a large number of circuits interconnected to each other. Resonance here might be sought rather than avoided, as occurs when tuning a radio to a given electromagnetic wave which is achieved by varying the capacity C.

The problem of power system networks is different in that there are instabilities of exponential type that occur in these systems under small disturbances. The problem there is to control these instabilities. Although very complex in nature, the problem of power systems instability can be pictured from the above simple circuit in which the resistance R is made negative, i.e., we assume that the resistance is an active device rather than a passive one. Then it can be seen that the circuit may become unstable because the solution takes the form $ae^{s_1 t} + be^{s_2 t}$ in which s_1, s_2 may have positive real parts, which leads to unstable solutions.

4. Quantum Chemistry

In quantum theory the properties of elementary particles such as electrons, are described by their wave function Ψ which is solution of the Schrödinger equation

$$\hat{H}\Psi = E\Psi \tag{10.9}$$

in which $\hat{H}$ is the energy operator, and E is the energy of the particle. The operator $\hat{H}$ is called the Hamiltonian and is defined by

$$\hat{H} = -\frac{h^2}{2m}\Delta + q \tag{10.10}$$

where h is the Plank constant, m is the mass of the particle and q is the potential energy. The equation (10.9) is an eigenvalue problem involving an unbounded operator. The way in which it is typically handled is by starting from an initial configuration

$$\Psi = \sum_{i=1}^{N} c_i \chi_i$$

and then solve the problem in the subspace spanned by $(\chi_i)_{i=1,\ldots,N}$. This amounts to solving the generalized matrix eigenvalue problem $Hc = ESc$ where the matrices H and S are defined by $H = (\hat{H}\chi_j, \chi_i)_{i,j=1\ldots N}$, $S = (\chi_j, \chi_i)_{i,j=1\ldots N}$. A better approximation to the sought eigenfunctions are then obtained and used as new χ_i's. This is referred to as the configuration interaction method a variation of which is Davidson's method.

5. Stability of Dynamical Systems

Consider a dynamical system governed by the differential equation

$$\frac{dy}{dt} = F(y) \tag{10.11}$$

where $y \in \mathbb{R}^n$ is some vector-valued function of t and F is a function from $\mathbb{R}^n$ to itself. We will assume that the system is time autonomous in that the variable t does not appear in the right hand side of (10.11). Note that F can be a complicated partial differential operator and is usually nonlinear.

The stability of a nonlinear system that satisfies the equation $\dot{y} = F(y)$ is usually studied in terms of its steady state solution. The steady state solution $\bar{y}$ is, by definition, the limit of $y(t)$ as t tends to infinity. This limit, when it exists, will clearly depend on the initial conditions of the differential equation. The solution $\bar{y}$ can be found by solving the steady-state equation $F(y) = 0$ because the variation of y with respect to time will tend to zero at infinity. A system governed by equation (10.11) is said to be locally stable if there exists an ϵ such that

$$\|y(t) - \bar{y}\| \rightarrow 0 \ , \ \text{as } t \rightarrow \infty$$

whenever $\|y(0) - \bar{y}\| \leq \epsilon$. For obvious reasons, it is said that the steady state solution is attracting. The important result on the stability of dynamical systems, is that in most cases the stability of the dynamical system can be determined by its linear stability, i.e., by the stability of the linear approximation of F at $\bar{y}$. In other words the system is stable if all the eigenvalues of the Jacobian matrix

$$J = \left\{ \frac{\partial f_i(\bar{y})}{\partial x_j} \right\}_{i,j=1,\ldots,n}$$

have negative real parts and unstable if at least one eigenvalue has a positive real part. If some eigenvalues of J lie on the imaginary axis, then the stability of the system cannot be determined by its linear stability, see [66]. In this case the system may or may not be stable depending on the initial condition among other things.

It is often the case that Jacobian matrices are very large nonsymmetric and sparse such as for example when F originates from the discretization of a partial differential operator. This is also the case when simulating electrical power systems, since the dimension of the Jacobian matrices will be equal to the number of nodes in the network multiplied by the number of unknowns at each node, which is usually four.

6. Bifurcation Analysis

The behavior of phenomena arising in many applications can be modeled by a parameter dependent differential equation of the form

$$\frac{dy}{dt} = F(y, \alpha) \tag{10.12}$$

where y is a vector valued function and α is typically a real parameter. There are several problems of interest when dealing with an equation of the form (10.12). A primary concern in some applications is to determine how stability properties of the system will change as the parameter α varies. For example α might represent a mass that is put on top of a structure to study its resistance to stress. When this mass increases to reach a critical value the structure will collapse. Another important application is when controlling the so-called panel flutter that causes wings of airplanes to disrupt after strong vibrations. Here the bifurcation parameter is the magnitude of the velocity of air. Christodoulou and Scriven have recently solved a rather challenging problem involving bifurcation and stability analysis in fluid flow [17]. In what is referred to as bifurcation theory a set of analytical and numerical tools that are used to analyze the change of solution behavior as α varies and part of the spectrum of the Jacobian moves from the left half plane (stable plane) to the right half (unstable) plane.

A typical situation is when one *real* eigenvalue passes from the left plane to the right half plane. Thus, the Jacobian becomes singular in between. This could correspond to either a 'turning 'point or a 'real bifurcation 'point. The change of behavior of the solution can happen in several different ways as is illustrated in Figure 4. Often bifurcation analysis amounts to the detection of all such points. This is done by a marching procedure along one branch until crossing the primary bifurcation point and taking all possible paths from there to detect the secondary bifurcation points etc..

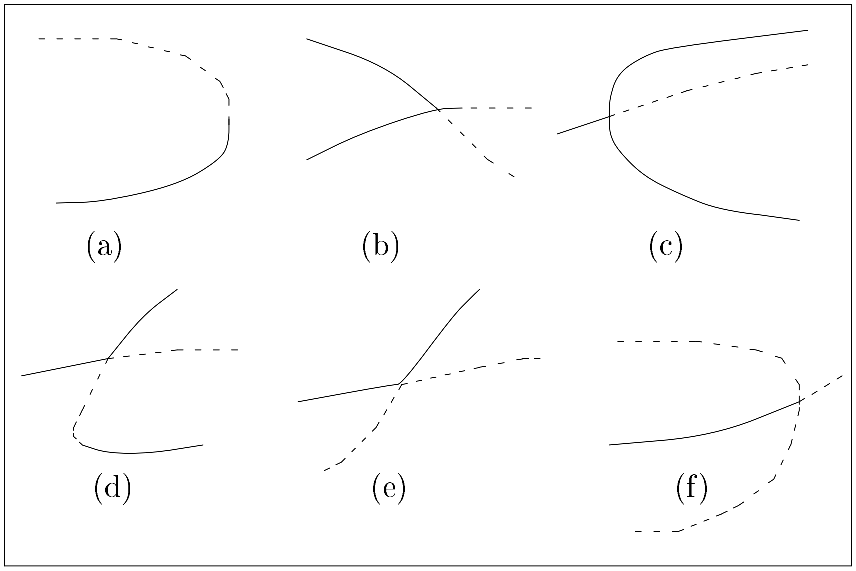

Figure 10.4 Bifurcation patterns. Stable branches solid lines, unstable branches dashed lines.

An interesting case is when a pair of complex imaginary eigenvalues cross the imaginary axis. This is referred to as Hopf bifurcation. Then at the critical value of α where the crossing occurs, the system admits a periodic solution. Also, the trajectory of y, sometimes referred to as the phase curve in mechanics, forms a closed curve in the y plane referred to as the phase plane (this can be easily seen for the case $n = 2$ by using the parameter t to represent the curve).

7. Chemical Reactions

An increasing number of matrix eigenvalue problems arise from the numerical simulation of chemical reactions. An interesting class of such reactions are those where periodic reactions occur 'spontaneously 'and trigger a wave like regime. A well-known such example is the Belousov-Zhabotinski reaction which is modeled by what is referred to as the Brusselator model. The model assumes that the reaction takes place in a tube of length one. The space variable is denoted by r, and the time variable by t. There are two chemical components reacting with one another. Their concentrations which are denoted by $x(t,r)$ and $y(t,r)$ satisfy the coupled partial differential equations

$$\begin{aligned} \frac{\partial x}{\partial t} &= \frac{D_1}{L}\frac{\partial^2 x}{\partial r^2} + A - B - (B+1)x + x^2 y \\ \frac{\partial y}{\partial t} &= \frac{D_2}{L}\frac{\partial^2 y}{\partial r^2} + Bx - x^2 y \end{aligned}$$

with the initial conditions,

$$x(0,r) = x_0(r), \quad y(0,r) = y_0(r), \quad 0 \le r \le 1$$

and the boundary conditions

$$x(t,0) = x(t,1) = A, \qquad y(t,0) = y(t,1) = \frac{B}{A} \ .$$

A trivial stationary solution to the above system is $\bar{x} = A, \bar{y} = B/A$. The linear stability of the above system at the stationary solution can be studied by examining the eigenvalues of the Jacobian of the transformation on the right-hand-side of the above equations. This Jacobian can be represented in the form

$$J = \begin{pmatrix} \frac{D_1}{L}\frac{\partial^2}{\partial_r^2} - (B+1) + 2xy & x^2 \\ B - 2xy & \frac{D_2}{L}\frac{\partial^2}{\partial_r^2} - x^2 \end{pmatrix} \ .$$

This leads to a sparse eigenvalue problem after discretization. In fact the problem addressed by chemists is a bifurcation problem, in that they are interested in the critical value of L at which the onset of periodic behavior is triggered. This corresponds to a pair of purely imaginary eigenvalues of the Jacobian crossing the imaginary axis.

8. Macro-economics

We consider an economy which consists of n different sectors each producing one good and each good produced by one sector. We denote by a_{ij} the quantity of good number i that is necessary to produce one unit of good number j. This defines the coefficient matrix A known as the matrix of technical coefficients. For a given production $(x)_{i=1,\ldots,n}$, the vector Ax will represent the quantities needed for this production , and therefore $x - Ax$ will be the net production. This is roughly Leontiev's linear model of production.

Next, we would like to take into account labor and salary in the model. In order to produce a unit quantity of good j, the sector j employs w_j workers and we define the vector of workers $w = [w_1, w_2, \ldots, w_n]^T$. Let us assume that the salaries are the same in all sectors and that they are entirely used for consumption, each worker consuming the quantity d_i of good number i. We define again the vector $d = [d_1, d_2, \ldots, d_n]^T$. The total consumption of item i needed to produce one unit of item j becomes

$$a_{ij} + w_j d_i \ .$$

This defines the so-called *socio-technical* matrix $B = A + w^T d$.

The additional assumptions on the model are that the needs of the workers are independent of their sector, and that there exists a pricing system that makes every sector profitable. By pricing system or strategy, we mean a vector $p = (p_i)_{i=1,\ldots,n}$ of the prices p_i of all the goods. The questions are

1) Does there exist a pricing strategy that will ensure a profit rate equal for all sectors? (balanced profitability)

2) Does there exist a production structure x that ensures the same growth rate τ to each sector? (balanced growth).

The answer is provided by the following theorem.

Theorem 10.1 *If the matrix B is irreducible there esists a pricing strategy p, a production structure x and a growth rate $r = \tau$ that ensure balanced profitability and balanced growth and such that*

$$B^T p = \frac{1}{1+r} p \ , \ \ Bx = \frac{1}{1+\tau} x.$$

In other words the desired pricing system and production structure are left and right eigenvectors of the matrix B respectively. The proof is a simple exercise that uses the Perron-Frobenius theorem. Notice that the profit rate r is equal to the growth rate τ; this is referred to as the golden rule of growth.

9. Markov Chain Models

A discrete state, discrete time Markov chain is a random process with a finite (or countable) number of possible states taking place at countable times t_1, $t_2, \ldots, t_k \ldots$, and such that the probability of an event depends only on the state of the system at the previous time. In what follows, both times and states will be numbered by natural integers. Thus, the conditional probability that the system be in state j at time k, knowing that it was under state j_1 at time 1, state j_2, at state 2 etc.., state $j_k - 1$ at time $k - 1$ only depends on its state $j_k - 1$ at the time $k - 1$, or

$$\begin{aligned} P(X_k = j \mid X_1 = j_1, X_2 = j_2, \ldots, X_{k-1} = j_{k-1}) \\ = P(X_k = j \mid X_{k-1} = j_{k-1}) \end{aligned}$$

where $P(E)$ is the probability of the event E and X is a random variable.

A system can evolve from a state to another by passing through different transitions. For example, if we record at every minute the number of people waiting for the $7am$ bus at a given bus-stop, this number will pass from 0 at, say, instant 0 corresponding to

$6:45$ am to say 10 at instant 15 corresponding to 7 am. Moreover, at any given time between instant 0 and 15, the probability of another passenger coming, i.e., of the number of passengers increasing by one at that instant, only depends on the number of persons already waiting at the bus-stop.

If we assume that there are N possible states, we can define at each instant k, an $N \times N$ matrix $P^{(k)}$, called transition probability matrix, whose entries $p_{ij}^{(k)}$ are the probabilities that a system passes from state i to state j at time k, i.e.,

$$p_{ij}^{(k)} = P(X_k = j | X_{k-1} = i)$$

The matrix $P^{(k)}$ is such that its entries are nonnegative, and the row sums are equal to one. Such matrices are called *stochastic*. One of the main problems associated with Markov chains is to determine the probabilities of every possible state of the system after a very long period of time.

The most elementary question that one faces when studying such models is: how is the system likely to evolve given that it has an initial probability distribution $q^{(0)} = (q_1^{(0)}, q_2^{(0)}, \ldots, q_N^{(0)})$? It is easy to see that at the first time $q^{(1)} = q^{(0)} P^{(0)}$, and more generally

$$q^{(k)} = q^{(k-1)} P^{(k-1)}.$$

Therefore,

$$q^{(k)} = q^{(0)} P^{(0)} P^{(1)} \ldots P^{(k-1)} P^{(k)}.$$

A homogeneous systems is one whose transition probability matrix $P^{(k)}$ is independent of time. If we assume that the system is homogeneous then we have

$$q^{(k)} = q^{(k-1)} P \tag{10.13}$$

and as a result if there is a stationary distribution $\pi = \lim q^{(k)}$ it must satisfy the equality $\pi = \pi P$. In other words π is a left eigenvector of P associated with the eigenvalue unity. Conversely, one might ask what are the conditions under which there is a stationary distribution.

All the eigenvalues of P do not exceed its 1-norm which is one because P is nonnegative. Therefore if we assume that P is irreducible then by the Perron-Frobenius theorem, one is the eigenvalue of largest modulus, and there is a corresponding left eigenvector π with positive entries. If we scale this eigenvector so that $\|\pi\|_1 = 1$ then this eigenvector will be a stationary probability distribution. Unless there is only one eigenvalue with modulus one, it is not true that a limit of q_k defined by (10.13) always exists. In case there is only eigenvalue of P of modulus one, then q_k will converge to π under mild conditions on the initial probability distributions q_0.

Markov chain techniques are very often used to analyze queuing networks and to study the performance of computer systems.

PROBLEMS

P-10.1 Generalize the model problems of Section 2 involving masses and springs to an arbitrary number of masses.

P-10.2 Compute the exact eigenvalues (analytically) of the matrix obtained from discretizing the Chemical reaction model problem in Section 7. Use the parameters listed in Chapter II for the example.

P-10.3 Show that when $F(t) = F_0 \cos \omega t$ then a particular solution to (10.1) is given by

$$\frac{F_0}{(k - m\omega^2)^2 + c^2\omega^2} \left[(k - m\omega^2) \cos \omega t + c\omega \sin \omega t \right].$$

Show that (10.3) is an alternative expression of this solution.

NOTES AND REFERENCES. Many of the emerging applications of eigenvalue techniques are related to fluid dynamics and bifurcation theory [12, 70, 79, 101, 103, 80, 157, 176] aero-elasticity [33, 34, 51, 102, 67, 68, 156], chemical engineering [18, 17, 130, 71, 131] and economics [28]. An interesting account of the Tocoma bridge disaster mentioned in Section 1, and other similar phenomena can be found in Brauns's book [10]. ♠

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