16-811: Math Fundamentals for Robotics, Fall 2024
Assignment 5

DUE: Thursday, November 14, 2024, by 11:59pm

(**Note:** There is no code submission for this assignment, only pdf.)

1. Consider a plane curve $y(x)$ over the interval $[x_0, x_1]$, with specified endpoints $y_0 = y(x_0)$ and $y_1 = y(x_1)$. Assume that $y_0 > 0$ and $y_1 > 0$ and that $y(x) \geq 0$ for $x_0 \leq x \leq x_1$. Now imagine rotating the curve about the x-axis to obtain a surface of revolution. Find the C^2 curve $y(x)$ with specified endpoints that minimizes the surface area of this surface of revolution.

   [Hint: This problem explores further some of the limitations of the Calculus of Variations. Depending on the endpoint conditions there may or may not be a C^2 solution. What does the optimal "curve" look like when it is not C^2? Can you say how the endpoint conditions matter? Be aware: There are many subtleties; don't expect to cover all, but explore what you can.]

2. Using Calculus of Variations, show that the shortest curve between two points on a sphere is an arc of a great circle. [Hints: There are different ways to solve this problem. Here is one approach: Use spherical (u, v) coordinates, where $x = R\sin v\cos u$, $y = R\sin v\sin u$, $z = R\cos v$, with R the radius of the sphere. (Do not worry about singularities in the representation.) Rephrase 3D arclength $\sqrt{dx^2 + dy^2 + dz^2}$ as a function of u, v, du, dv. View the curve as a function $v(u)$. Observe that u does not appear directly in the resulting integrand for arclength, so replace the Euler-Lagrange equation with another, as in lecture. You may find the following identity useful:

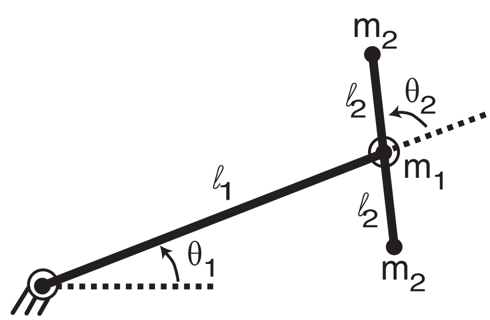
$$\int \frac{a\,dw}{\sqrt{\sin^4 w - a^2\sin^2 w}} = -\sin^{-1}\left(\frac{\cot w}{\sqrt{\frac{1}{a^2} - 1}}\right) + k,$$

   where a and k are appropriate constants.]

3. In the brachistochrone problem ("ski-slope" in lecture), suppose the right endpoint (x_1, y_1) is not specified exactly, but is merely constrained to satisfy an equation of the form $g(x, y) = 0$. Show that the optimizing curve $y(x)$ is perpendicular to the iso-contour $g(x, y) = 0$ at (x_1, y_1).

   [Hints: (i) Use an equation from lecture rather than derive everything from scratch. (ii) Observe that the curves are written differently, one explicitly as $y(x)$, the other implicitly as $g(x, y) = 0$. Think about how to compare the slopes of the two curves despite these different representations.]

4. (a) Using Lagrangian Dynamics, derive the relationship between joint torques and the angular state (angles, angular velocities, and angular accelerations) of the following balanced manipulator:



   There is no gravity (in practice, gravity is perpendicular to the sheet of the paper).

   Legend: All of link #1's mass, m_1, is concentrated at distance ℓ_1 from its rotational joint (which is attached to the ground). In turn, link #2 rotates around this distal point, with two masses, m_2, located symmetrically, each at distance ℓ_2, from the joint. In practice, these two masses might constitute one counter-balanced end-effector or two different but equally weighted end-effectors. — This is a variation of a basic SCARA-type robot arm, used in industrial assembly.

   (b) When $\ddot{\theta}_2 = 0$, explain the terms relating $\ddot{\theta}_1$ to τ_1.