Notes by Yan-Bin Jia

Markov Chains

Let's first recall the definition of independent trials.

A set of possible outcomes $E_1, E_2, \cdots,$ is given. With each outcome E_k there is associated a probability p_k. The prob. of a sampled sequence is defined as

$$P\{(E_{j_0}, E_{j_1}, \cdots, E_{j_k})\} = p_{j_0} p_{j_1} \cdots p_{j_k}.$$

In the theory of Markov chains.

The outcome of any trial depends on the outcome of the directly preceding trial only.

<u>Conditional prob.</u> p_{jk} : given that E_j has occurred at some trial, the prob. of E_k at the next trial.

a_k : probability of E_k at the initial trial.

For instance, here are prob. of some sample sequences

$$P\{(E_j, E_k)\} = a_j p_{jk}$$

$$P\{(E_j, E_k, E_r)\} = a_j p_{jk} p_{kr}$$

and generally

$$P\{(E_{j_0}, E_{j_1}, \cdots, E_{j_n})\} = a_{j_0} p_{j_0 j_1} p_{j_1 j_2} \cdots p_{j_{n-1} j_n}$$

Ex Random walk

-3 -2 -1 0 1 2 3

events: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$P_{jk} = 0$ if $|j-k| > 1$

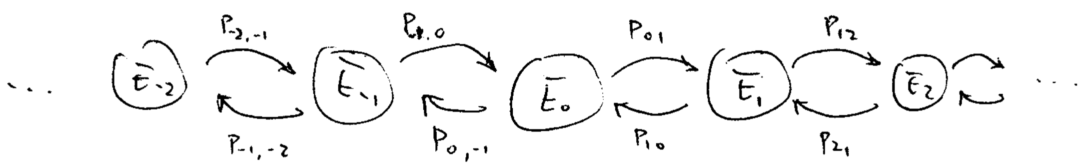
for a symmetric random walk we might say

$$P_{jj} = 0$$
$$P_{jk} = \tfrac{1}{2} \quad \text{if } |j-k| = 1$$

Graphically we would draw arrows with labelled probs. —

Indeed that is often a good way to think of a Markov chain.

Draw a graph (like a finite state automata) and label the transitions with probs:



Definition of Markov Chain

Def A sequence of trials with possible outcomes $E_1, E_2, \cdots$ is called a Markov chain if the probabilities of sample sequences are defined by

$$P\{(E_{j_0}, E_{j_1}, \cdots, E_{j_n})\} = a_{j_0} P_{j_0 j_1} \cdots P_{j_{n-1} j_n}$$

in terms of a probability distribution $\{a_k\}$ for E_k at the initial (or zero-th) trial and fixed conditional probabilities P_{jk} of E_k given that E_j has occurred at the preceding trial.

E_k : states of the system

a_k : the prob. of E_k at the initial trial

P_{jk} : the prob. of a transition from E_j to E_k.

The matrix (finite or infinite) of transition probabilities

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} & \cdots \\ P_{21} & P_{22} & P_{23} & \cdots \\ P_{31} & P_{32} & P_{33} & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

is a square matrix with non-negative elements and unit row sums.

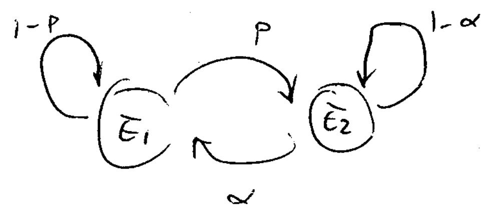
Such a matrix is called a Stochastic matrix

Any stochatic matrix with initial distribution $\{a_k\}$ completely defines a Markov chain with states $E_1, E_2, \cdots$, Ex 1

There are only two possible states E_1 and E_2. The matrix of transition probs is

$$P = \begin{pmatrix} 1-p & p \\ \alpha & 1-\alpha \end{pmatrix}$$

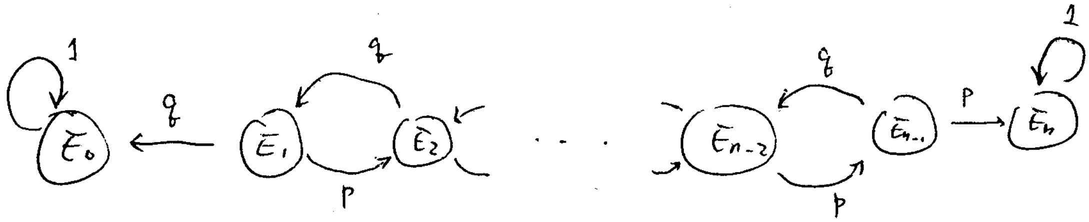
or in the graphical form



Remark This chain could be realized by the following motion of a particle moving along the x-axis. Its absolute speed remains constant but the direction of the motion can be reversed. The system is in state E_1 if the particle moves in the positive direction and in state E_2 if the motion is to the left. Then p is the prob. of a reversal when the particle moves to the right and α the prob. of a reversal when it moves to the left.

Ex 2 (Random Walk with Absorbing Barriers)

Possible states: $E_0, E_1, \ldots, E_n$

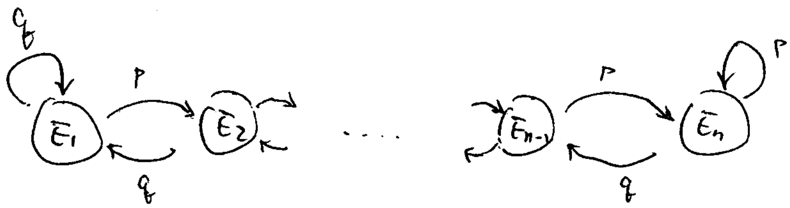


$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ q & 0 & p & 0 & \cdots & 0 & 0 & 0 \\ 0 & q & 0 & p & \cdots & 0 & 0 & 0 \\ & & & \vdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & q & 0 & p \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

- Each of the "interior" states $E_1, \ldots, E_{n-1}$ admit transitions only to its left neighbor (with prob. p) and its right neighbor (with prob. q)

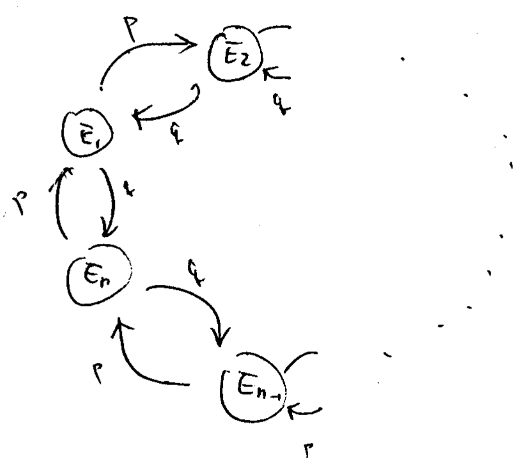
- No transition is possible from E_0 or E_n to any other state. Once E_0 or E_n is reached, the system stays there forever.

Ex 3 (Reflecting Barriers)



$$P = \begin{pmatrix} q & p & 0 & 0 & \cdots & 0 & 0 & 0 \\ q & 0 & p & 0 & \cdots & 0 & 0 & 0 \\ 0 & q & 0 & p & \cdots & 0 & 0 & 0 \\ & & & \vdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & q & 0 & p \\ 0 & 0 & 0 & 0 & \cdots & 0 & q & p \end{pmatrix}$$

Ex 4 (Cyclical Random Walks)



$$P = \begin{pmatrix} 0 & p & 0 & 0 & \cdots & 0 & 0 & q \\ q & 0 & p & 0 & \cdots & 0 & 0 & 0 \\ 0 & q & 0 & p & \cdots & 0 & 0 & 0 \\ & & & \vdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & q & 0 & p \\ p & 0 & 0 & 0 & & 0 & 0 & q \end{pmatrix}$$

More generally, we may permit the transitions between any two states. Let $q_0, q_1, \ldots, q_{n-1}$ be, resp. the prob of staying fixed or moving $1, 2, \ldots, n-1$ units to the right (where k units to the right is the same as $n-k$ units to the left) the P is shown on the left

$$P = \begin{pmatrix} q_0 & q_1 & q_2 & \cdots & q_{n-2} & q_{n-1} \\ q_{n-1} & q_0 & q_1 & \cdots & q_{n-3} & q_{n-2} \\ & & & \vdots & & \\ q_1 & q_2 & q_3 & \cdots & q_{n-1} & q_0 \end{pmatrix}$$

Higher Transition Probabilities

Denote by $p_{jk}^{(l)}$ the prob. of a transition from E_j to E_k in exact l steps. This is the sum of the probs. of all possible paths $E_j E_{j_1} \cdots E_{j_{l-1}} E_k$, i.e.

$$p_{jk}^{(l)} = \sum_{j_1, \dots, j_{l-1}} \mathbf{P}\{(E_{j_0} E_{j_1}, \dots, E_{j_{l-1}}, E_k)\}$$

In particular $p_{jk}^{(1)} = p_{jk}$ and

$$p_{jk}^{(2)} = \sum_s p_{js} p_{sk}$$

We define $p_{jk}^{(0)} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$ and recursively we have

$$p_{jk}^{(i+1)} = \sum_s p_{js} p_{sk}^{(i)}$$

In fact, if we arrange $p_{jk}^{(l)}$ into a matrix denoted by $\mathbf{P}^l$, then $\mathbf{P}^l$ is just the chain product of l identical matrices $\mathbf{P}$.

Ex For independent trials it is clear without calculation that $\mathbf{P}^l = \mathbf{P}$ for all l.

Absolute Probabilities

Recall a_j stands for the state E_j at the initial trial. The (unconditional) prob. of entering the state E_k at the lth step is

$$a_k^{(l)} = \sum_j a_j \, p_{jk}^{(l)} \qquad (1)$$

Usually we let the process start from a fixed state E_i; that is, we put $a_i = 1$. In this case $a_k^{(n)} = p_{ik}^{(n)}$.

We feel intuitively that the influence of the initial state should gradually wear off so that for large n the distribution (1) should be nearly independent of the initial distribution $\{a_j\}$. This is the case if $p_{jk}^{(n)}$ converges to a limit independent of j, that is, if P^n converges to a matrix with identical rows. We shall soon see that this is usually so.

Closures and Closed Sets

We shall say that E_k can be reached from E_j if there exists some $n \geq 0$ s.t. $P_{jk}^{(n)} > 0$.

Def 1 A set C of states is closed if no state outside C can be reached from any state E_j in C.

Def 2 For an arbitrary set C of states the smallest closed set containing C is called the closure of C.

Def 3 A single state E_k forming a closed set will be called absorbing.

Def 4 A Markov chain is irreducible if there exists no closed set other than the set of all states.

Clearly, C is closed if and only if $P_{jk} = 0$ wherever $j \in C$ and $k \notin C$

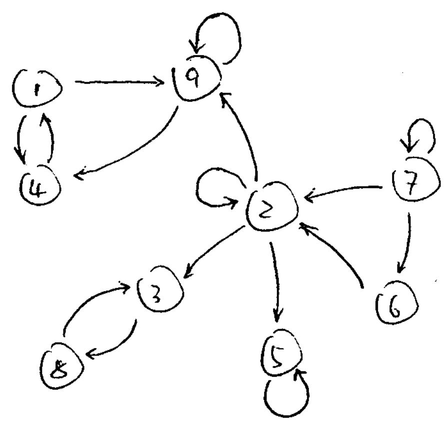
Thm The matrix formed by deleting from P^d all rows and columns corresponding to states outside a closed set C is a stochastic matrix.

This means that we have a Markov Chain defined on C and this subchain can be studied independent of all other states. For instance, the state E_k is absorbing if and only if $P_{kk} = 1$. In this case the matrix reduces to a single element.

Ex

Consider a 9 x 9 matrix of transition, where "$*$" denotes positive elements

$$P = \begin{pmatrix} 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * \\ 0 & * & * & 0 & * & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * \end{pmatrix}$$



E_5 is absorbing.

E_3 and E_8 formed a closed set.

E_1, E_4, and E_9 formed another closed set

The closure of E_2 consists of $E_1, E_2, E_3, E_4, E_5, E_8, E_9$

The closure of E_7. Consists of all nine states.

The closure of E_6 consists of all states but E_7

The above can be simplified by numbering the states in the order

$$\underline{E_5}\ \underline{E_3\, E_8}\ \underline{E_1\, E_4\, E_9}\ E_2\, E_6\, E_7$$

The closed sets then contain only adjacent states.

Consider a chain with states $E_1, \dots, E_n$, s.t. $E_1, \dots, E_r$ form a closed set C ($r < n$). Then the corresponding $r \times r$ submatrix of its transition prob. matrix P is stochastic.

$$
P = \begin{array}{cc} r & n-r \\ \end{array}
\begin{array}{c} \\ \left(\begin{array}{cc} Q & 0 \\ U & V \end{array} \right) \end{array}
\begin{array}{c} r \\ n-r \end{array}
$$

The higher-order transition probs admits a similar partitioning

$$
P^{\ell} = \begin{pmatrix} Q^{\ell} & 0 \\ U_{\ell} & V^{\ell} \end{pmatrix}
$$

note subscript
not supersoript

There are three obvious but interesting facts regarding P^{ℓ}:

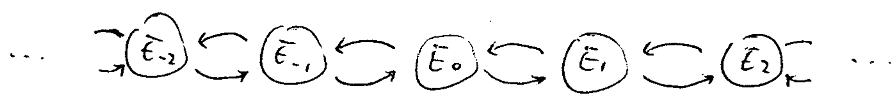
1. $p_{jk}^{(\ell)} = 0$ if $E_j \in C$ and $E_k \notin C$
2. For each pair $E_j, E_k \in C$, $p_{jk}^{(n)}$ is obtained from the recursion formula on page 7 with the summation restricted to the states of the closed set C.
3. The appearance of V^{ℓ} indicates that the statement 2 remains true when C is replaced by it complement $\overline{C}$.

As a consequence, it will be possible to simplify the further study of Markov chains by considering separately the states of the closed set C and those of the complement $\overline{C}$.

Classification of States

Def The state E_j has period $t > 1$ if $p_{jj}^{(m)} = 0$ unless $m = \nu t$ is a multiple of t, and t is the largest integer with this property. The state E_j is aperiodic if no such $t > 1$ exists.

Ex 1 In an unrestricted random walk, all states have period 2.



Ex 2. In a random walk with absorbing barriers, the interior states have period 2, but the absorbing states E_0 and E_n are aperiodic.

Let $f_{jk}^{(m)}$ stand for the probability that in a process starting from E_j the first entry to E_k occurs at the mth step; We put $f_{jk}^{(0)}=0$ and

$$f_{jk} = \sum_{m=1}^{\infty} f_{jk}^{(m)}$$

$$\mu_j = \sum_{m=1}^{\infty} m\, f_{jj}^{(m)}$$

Obviously, f_{jk} is the probability that starting from E_j, the system will ever reach E_k; thus $f_{jk} \le 1$.

When $f_{jk}=1$, that is, a pass through E_k is certain, $\{f_{jk}^{(m)}\}$ is a proper probability distribution referred to as the <u>first-passage distribution</u> for E_k, and $\mu_j < \infty$ is the <u>mean recurrence time</u> for E_j.

To calculate $f_{jk}^{(m)}$, we note that if the first passage through E_k occurs at the νth trial $(1 \le \nu \le m-1)$, the (conditional) probability of E_k at the mth trial equals $P_{kk}^{(m-\nu)}$. So we have

$$P_{jk}^{(m)} = \sum_{\nu=1}^{m} f_{jk}^{(\nu)} P_{kk}^{(m-\nu)} .$$

Letting successively $m=1,2,\cdots$, we get recursively $f_{jk}^{(1)}, f_{jk}^{(2)}, \cdots$, Conversely, if $f_{jk}^{(n)}$ are known for $m=1,2,\cdots$, then the same equation determines $P_{jk}^{(m)}$.

Def 1 The state E_j is <u>persistent</u> if $f_{jj}=1$ and <u>transient</u> if $f_{jj}<1$. A persistent state E_j is called <u>null state</u> if its mean recurrence time $\mu_j=\infty$.

The above theorem applies to periodic states also. It classifies all persistent states into null states and non-null states, the latter of which are of special interest.

Def 2 An aperiodic persistent state E_j with $\mu_j<\infty$ is called <u>ergodic</u>.

The theorem on the next page expresses the conditions for the different types in terms of the transition probabilities $P_{jj}^{(m)}$

Th^m (i) E_j is transient if and only if

$$\sum_{m=0}^{\infty} p_{jj}^{(m)} < \infty .$$

In this case

$$\sum_{m=1}^{\infty} p_{ij}^{(m)} < \infty \qquad \text{for all } i.$$

(ii) E_j is a (persistent) null state if and only if

$$\sum_{m=0}^{\infty} p_{jj}^{(m)} = \infty , \quad \text{but } p_{jj}^{(m)} \longrightarrow 0$$

as $m \to \infty$. In this case

$$p_{ij}^{(m)} = 0 \qquad \text{for all } i.$$

(iii) An aperiodic (persistent) state E_j is ergodic if and only if $\mu_j < \infty$. In this case as $m \to \infty$

$$p_{ij}^{(m)} \longrightarrow f_{ij} \mu_j^{-1} .$$

<u>Corollary</u> If E_j is aperiodic, then as $m \to \infty$ either

$$p_{ij}^{(m)} \to 0$$

or

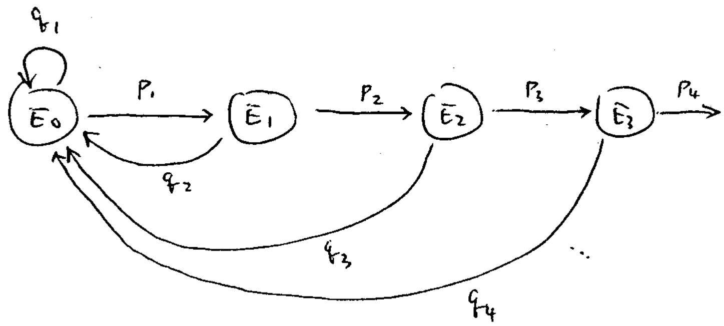
$$p_{ij}^{(m)} \to f_{ij} \mu_j^{-1} .$$

Ex (Markov chain connected with recurrent events)

Possible states $E_0, E_1, \cdots,$

Transition probabilities

$$P = \begin{pmatrix} q_1 & p_1 & 0 & 0 & 0 & \cdots \\ q_2 & 0 & p_2 & 0 & 0 & \cdots \\ q_3 & 0 & 0 & p_3 & 0 & \cdots \\ q_4 & 0 & 0 & 0 & p_4 & \cdots \\ \cdots & & & & & \end{pmatrix} \qquad \text{where } p_k + q_k = 1$$



We may interpret the state E_k as representing the "age" of the system. When the system reaches age k, the aging process continues with probability p_{k+1}, but with probability q_{k+1}, it rejuvenates and starts afresh with age zero.

The probability that a recurrence time equals k is given by the product $p_1 p_2 \cdots p_{k-1} q_k$. Given a prescribed distribution $\{f_k\}$ for the recurrence times, we can choose the p_ks. It suffices to put $q_1 = 1 - p_1 = f_1$, $q_2 = 1 - p_2 = f_2/p_1$, and so on. Generally

$$p_k = \frac{1 - f_1 - \cdots - f_k}{1 - f_1 - \cdots - f_{k-1}}$$

Ex (continued)

The nature of the transition probability matrix P shows that a first return at the mth trial can only occur through the sequence

$$E_0 \rightarrow E_1 \rightarrow E_2 \rightarrow \cdots \rightarrow E_{m-1} \rightarrow E_0$$

and so

$$f_{00}^{(1)} = q_1$$

$$f_{00}^{(m)} = p_1 p_2 \cdots p_{m-1} q_m , \qquad \text{for } m \geq 1.$$

In the case where p_k are defined by

$$p_k = \frac{1 - f_1 - \cdots - f_k}{1 - f_1 - \cdots - f_{k-1}},$$

this reduces to $f_{00}^{(m)} = \{f_m\}$.

- Thus E_0 is transient if $\Sigma f_m < 1$.
- For a persistent E_0 the mean recurrence time μ_0 of E_0 coincides with the expectation of the distribution $\{f_m\}$.
- If E_0 has period t, then $f_m = 0$ except m is a multiple of t.

Irreducible Chains

Two states of the same type have the same period or they are both aperiodic; both are transient or else both are persistent; in the latter case both mean recurrence times are infinite or else both are finite.

Th^m 1 All states of an irreducible chain are of the same type.

Th^m 2 For a persistent E_j there exists unique irreducible closed set C containing E_j and such that for every E_i, E_k of states in C

$$f_{ik} = 1 \quad \text{and} \quad f_{ki} = 1.$$

Th^m 2 implies that the closure of a persistent state is irreducible.

Ex Suppose $p_{jk} = 0$ whenever $k \leq j$ but $p_{j,j+1} > 0$.

Transitions take place only to higher states, and so no return to any state is possible. Every state E_j is transient. The closure of E_j consists of the states $E_j, E_{j+1}, E_{j+2}, \cdots$, which contains the closed set of $E_{j+1}, E_{j+2}, \ldots$, So there exists no irreducible sets.

Decompositions

Th^{m} 2 on page 18 implies that no transient state can ever be reached from a persistent state. If the chain contains both types of states, this means that the matrix P can be partitioned symbolically in the form

$$P = \begin{pmatrix} Q & 0 \\ U & V \end{pmatrix}$$

where Q corresponds to the persistent states. (However, Q may be further decomposable.)

Th^{m} The states of a Markov chain can be divided in a unique manner into non-overlapping sets $T, C_1, C_2, \cdots$, such that

(i) T consists of all transient states.

(ii) If state E_j is in C_ν then $f_{jk} = 1$ for all $E_k \in C_\nu$, while $f_{jk} = 0$ for all E_k outside C_ν.

So C_ν is irreducible and contains persistent states of the same type.

Th^{m} In a finite chain there exist no null states, and it is impossible that are states are transient.

Invariant Distributions

We shall now concentrate on irreducible chains since every persistent state belongs to an irreducible set whose asymptotic behavior can be studied independently of the remaining states. All states of such a chain are of the same type. So we begin with the simplest case, namely, chains with finite mean recurrence time μ_j. We consider now chains composed of ergodic states and postpone the discussion of periodic chains.

Th$^{\underline{m}}$ In an irreducible chain with only ergodic elements the limits

$$u_k = \lim_{n\to\infty} P_{jk}^{(n)} \qquad (1)$$

exist and are independent of the initial state j. Furthermore $u_k > 0$,

$$\sum_k u_k = 1 \qquad (2)$$

and

$$u_j = \sum_i u_i P_{ij}. \qquad (3)$$

Conversely, suppose that the chain is irreducible and aperiodic, and that there exist numbers $u_k > 0$ satisfying (2) and (3), then all states are ergodic, the u_ks are given by (1) and

$$u_k = 1/\mu_k \qquad (4)$$

where μ_k is the mean recurrence time of E_k.

To better understand the last theorem, consider a process from an initial distribution $\{a_j\}$. The probability of the state E_k at the mth step is given by

$$a_k^{(m)} = \sum_j a_j \, p_{jk}^{(m)}$$

Therefore

$$\lim_{m\to\infty} a_k^{(m)} = \lim_{m\to\infty} \sum_j a_j \, p_{jk}^{(m)}$$

$$= \sum_j a_j \lim_{m\to\infty} p_{jk}^{(m)}$$

$$= u_k \sum_j a_j$$

$$= u_k .$$

In other words, whatever the initial distribution, the probability of E_k tends to u_k. On the other hand, when $\{u_k\}$ is the initial distribution i.e. $a_k = u_k$, then $a_k^{(1)} = \sum_j a_j p_{jk} = \sum_j u_j p_{jk} = u_k$ (by (3) on p. 20) and by induction $a_k^{(m)} = u_k$ for all m.

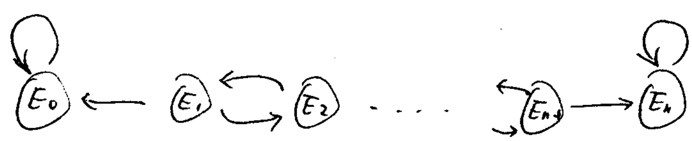
Def A probability distribution $\{u_k\}$ satisfying $u_k = \sum_i u_i p_{ik}$ is called <u>invariant</u> or <u>stationary</u>

Then the theorem on page 20 can be reformulated as

Th<u>m</u> An irreducible aperiodic chain possesses an invariant probability distribution $\{u_k\}$ if and only if it is ergodic. In this case $u_k > 0$ for all k, and the absolute probabilities $a_k^{(n)}$ tend to u_k irrespective of the initial distribution.

<u>Criterion</u> If a chain possesses an invariant probability distribution $\{u_k\}$, then $u_k = 0$ for each E_k that is either transient or a persistent null state.

<u>Ex 1</u> Chains with several irreducible components may admit of several stationary solutions. An example is presented by the random walk with two absorbing states E_0 and E_n.



Every probability distribution $(\alpha, 0, 0, \cdots, 0, 1-\alpha)$ is stationary.

<u>Ex 2</u> Given a matrix of transition probabilities P_{jk}, it is not always easy to decide whether an invariant distribution $\{u_k\}$ exists. A notable exception occurs when

$$P_{jk} = 0 \quad \text{for } |k-j| > 1, \quad \text{i.e. } P = \begin{pmatrix} P_{00} & P_{01} & & \bigcirc \\ P_{10} & P_{11} & P_{12} & \\ \bigcirc & P_{21} & P_{22} & P_{23} \\ & & \cdots & \end{pmatrix}$$

The defining relations $u_k = \sum_i u_i P_{ik}$ induces

$$u_0 = P_{00} u_0 + P_{10} u_1$$

$$u_i = P_{i-1,i}\, u_{i-1} + P_{ii} u_i + P_{i+1,i}\, u_{i+1} \qquad \text{for } i \geq 1$$

Ex 2 (continued) To avoid trivialities we assume $P_{j,j+1} > 0$ and $P_{j,j-1} > 0$ for all j. The above equations can be solved successively using that the row sums of the matrix P add to unity. We get

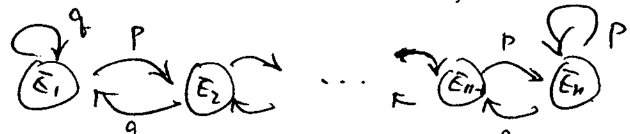
$$u_1 = \frac{P_{01}}{P_{10}} u_0, \quad u_2 = \frac{P_{01} P_{12}}{P_{10} P_{21}} u_0, \quad \cdots, \quad u_k = \frac{P_{01} P_{12} \cdots P_{k-1,k}}{P_{10} P_{21} \cdots P_{k,k-1}} u_0, \quad \cdots$$

To make the resulting sequence $u_0, u_1, \cdots,$ a probability distribution the norming factor u_0 must be chosen such that $\sum u_k = 1$. Such a choice is possible if and only if

$$\sum \frac{P_{01} P_{12} \cdots P_{k-1,k}}{P_{10} P_{21} \cdots P_{k,k-1}} < \infty \qquad (*)$$

This is the necessary and sufficient condition for the existence of an invariant probability distribution.

Ex 3 (Reflecting barriers) This (with $n \le \infty$) represents the special case



of the preceding example with $P_{j,j+1} = p$ for all $j < n$ and $P_{j,j-1} = q$ for all $j > 1$. When n is finite, there exists an invariant distribution with u_k proportional to $(p/q)^k$. With infinitely many states the convergence of $(*)$ requires that $p < q$, and in this case, $u_k = (1 - p/q)(p/q)^k$. From the general theory of random walks, it is clear that the states are transient when $p<q$, and persistent null states when $p=q$.

Ex 4 In the example of recurrent events on page 16, the define relations for an invariant probability distribution are

$$u_k = p_k u_{k-1}, \quad k=1,2,\dots$$
$$u_0 = q_1 u_0 + q_2 u_1 + q_3 u_2 + \cdots \qquad (1)$$

So we get

$$u_k = p_1 \dots p_k u_0$$

And it is now easily seen that

$$q_1 u_0 + q_k u_{k-1} = (1-p_1)u_0 + \dots + (1-p_k)u_{k-1}$$
$$= u_0 - u_1 + u_1 - u_2 + \dots - u_{k-1} + u_{k-1} - u_k$$
$$= u_0 - u_k.$$

Thus (1) is automatically satisfied whenever $u_k \to 0$ and an invariant probability distribution exists if and only if

$$\sum_k p_1 p_2 \cdots p_k < \infty$$

Transient States

We saw that the persistent states of any Markov chain can be divided into non-overlapping closed sets consisting of irreducible set $C_1, C_2, \dots$, and a set T of transient states. When the system starts from a transient state, two contingencies arise

1. The system ultimately passes into one of the closed sets C_ν and stays there forever.
2. The system remains forever in the transient set T.

Our problem consists in determining the corresponding probabilities.

Ex 1 (Martingales). In such a chain for every j the expectation of the probability distribution $\{p_{jk}\}$ equals j, that is, if

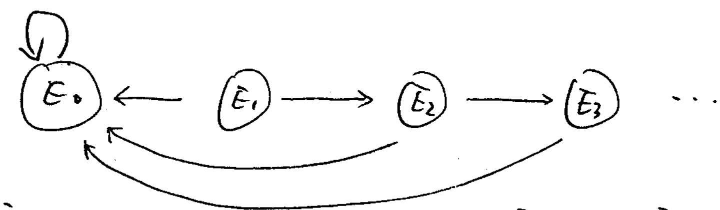
$$\sum_k p_{jk}\, k = j. \tag{1}$$

Consider a finite chain $E_0, \dots, E_n$. Letting $j=0$ and $j=n$ we see that $p_{00} = p_{nn} = 1$. So E_0 and E_n are absorbing. To avoid trivialities we assume that the chain contains no further closed sets. Then the interior states $E_1, \dots, E_{n-1}$ are transient, and so the process will ultimately terminate at E_0 or at E_n. By induction we infer from (1) that

$$\sum_{k=0}^{n} p_{jk}^{(m)}\, k = j \qquad \text{for all } m.$$

Ex 1 (continued) But $P_{jk}^{(m)} \to 0$ for every transient E_k. So for all $i > 0$, $P_{in}^{(m)} \to i/n$. In other words, the probabilities of starting with E_j and being absorbed ultimately at E_0 and E_n are $1 - \frac{i}{n}$ and $\frac{i}{n}$, resp.

Ex 2 In the following chain, E_0 is absorbing while from other states



E_j transitions are possible to the right neighbor E_{j+1} and to E_0, but to no other state. We put

$$P_{j0} = \varepsilon_j, \quad P_{j,j+1} = 1 - \varepsilon_j \qquad \text{for } j \geq 1$$

where $\varepsilon_j > 0$. With the initial state E_j, the probability of no absorption at E_0 in m trials equals

$$(1-\varepsilon_j)(1-\varepsilon_{j+1}) \cdots (1+\varepsilon_{j+m-1}).$$

This product decreases as m increases and hence it approaches a limit λ_j. So the probability of ultimate absorption equals $1 - \lambda_j$ while with probability λ_j the system remains forever at transient states. The necessary and sufficient condition for $\lambda_j > 0$ is $\sum \varepsilon_k < \infty$

Substochastic Matrix

The study of transient state depends on the submatrix of P obtained by deleting all rows and columns corresponding to persistent states. The row sums of the resulting submatrix are no longer unity

Def A square matrix Q with elements q_{ik} is <u>substochastic</u> if $q_{ik} \geq 0$ and all row sums are ≤ 1

- Every stochastic matrix is substochastic
- Every substochastic matrix can be augmented to become stochastic by adding an absorbing state E_0.

What we said about stochastic matrices applies without essential change also to substochastic matrix. Let q_{ij} denote the conditional probabilities in the substochastic matrix Q. The elements of Q^n satisfies the recurrence relation

$$q_{ik}^{(m+1)} = \sum_{v} q_{iv} q_{vk}^{(m)}$$

Denote by $\sigma_i^{(m)}$ the sum of the elements in the ith row of Q^m. Then

$$\sigma_i^{(m+1)} = \sum_\nu q_{i\nu} \sigma_\nu^{(m)} \qquad \text{for } m \geq 1 \qquad (1)$$

which remains valid also for $m=0$ provided we let $\sigma_\nu^{(0)} = 1$ for all ν.

That Q is substochastic means $\sigma_i^{(1)} \leq \sigma_i^{(0)}$ and by induction that $\sigma_i^{(m+1)} \leq \sigma_i^{(m)}$.

Fixing i, the sequence $\{\sigma_i^{(m)}\}$ decreases monotonically to a limit $\sigma_i \geq 0$ and clearly

$$\sigma_i = \sum_\nu q_{i\nu} \sigma_i \qquad (*)$$

The whole theory of the transient states depends on the solutions of $(*)$. In some cases there exists no non-zero solution. In others there may exist infinitely many linearly independent solutions, that is, different sequences of numbers satisfying

$$x_i = \sum_\nu q_{i\nu} x_\nu . \qquad (2)$$

Characterizing $\{\sigma_i\}$

Our first problem is to characterize the particular solution $\{\sigma_i\}$. Solutions $\{x_i\}$ to our interest such that $0 \le x_i \le 1$ for all i satisfy $0 \le x_i \le \sigma_i^{(0)}$; Comparing recurrent relations (1) and (2) on page 28, we see inductively that

$$x_i \le \sigma_i^{(m)} \qquad \text{for all } m$$

and so

$$0 \le x_i \le 1 \quad \text{implies} \quad x_i \le \sigma_i \le 1.$$

The solution $\{\sigma_i\}$ will be called <u>maximal</u>.

<u>Lemma</u> For a substochastic matrix Q, the linear system (2) on page 28 possesses a maximal solution $\{\sigma_i\}$ such that for any solution $\{x_i\}$ $0 \le x_i \le 1$, the condition $x_i \le \sigma_i \le 1$ holds for all i. These σ_i represent the limits of the row sums of Q^m.

Now identify Q with the submatrix P of a Markov chain obtained by retaining only the element P_{jk} for which P_j and P_k are transient. The linear system (2) on page 28 may be written in the form

$$x_i = \sum_{E_\nu \in T} P_{i\nu} x_\nu \qquad E_i \in T \qquad (1)$$

Denote by $\sigma_i^{(m)}$ the probability that no transition from a state E_i to a persistent state occur during the first m trials. The limit σ_i equals the probability that no such transition ever occurs.

Thm The Probability x_i, that, starting from E_i that the system stays forever among the transient states are given by the maximal solution of (1).

<u>Criterion</u> In an irreducible Markov chain with states $E_0, E_1, \dots$, the state E_0 is persistent if and only if the linear system

$$x_i = \sum_{\nu=1}^{\infty} P_{i\nu} x_\nu \qquad i \geq 1$$

admits of no solution with $0 \leq x_i \leq 1$ except $x_i = 0$ for all i.

Ex Consider Ex. 2 on page 22 again. We still assume that $P_{j,j+1} \neq 0$ and $P_{j,j-1} \neq 0$. The chain is irreducible because every state can be reached from every other state. Thus all states are of the same type and it suffices to look at E_0 only. Equation (1) on page 30 reduces to the recursive system

$$x_1 = P_{11} x_1 + P_{12} x_2$$

$$P_{j,j-1}(x_j - x_{j-1}) = P_{j,j+1}(x_{j+1} - x_j)$$

/* from
$x_j = P_{j,j-1} x_{j-1} + P_{jj} x_j + P_{j,j+1} x$
and
$x_j = (P_{j,j-1} + P_{jj} + P_{j,j+1}) x_j$
*/

Thus
$$x_j - x_{j+1} = \frac{P_{21} P_{32} \cdots P_{j,j-1}}{P_{23} P_{34} \cdots P_{j,j+1}} (x_1 - x_2)$$

Since $(P_{10} + P_{11} + P_{12}) x_1 = (P_{11} + P_{12}) x_2$, $P_{10} > 0$. we have $x_2 - x_1 > 0$

So a bounded non-negative solution $\{x_j\}$ exists if and only if

$$\sum_j \frac{P_{21} P_{32} \cdots P_{j,j-1}}{P_{23} P_{34} \cdots P_{j,j+1}} < \infty .$$

The chain is persistent if and only if the series diverges. In the special case of random walks we have $P_{j,j+1} = p$ and $P_{j,j-1} = q$ for all $j > 1$. So we see again that the states are persistent if and only if $p \leq q$.

Absorption in a Closed Persistent Set

Denote again by T the class of transient states and C any closed set of persistent states. Denote by y_i the probability of ultimate absorption in C, given the initial state E_i.

Th^m The probabilities y_i of ultimate absorption in the closed persistent set C are given by the minimal non-negative solution of

$$y_i = \sum_T p_{i\nu} y_\nu + \sum_C p_{i\nu}, \qquad E_i \in T$$

For more on Markov chains, please refer to

William Feller: "An Introduction to Probability Theory and Its Applications". vol 1. John Wiley & Sons. 1968.

Q.E.D.