

# Calculus of Variations

The calculus of variations arose in tight complicity with the development of mechanics. Much of the axiomatic grounding of physics in general and mechanics in particular consists of variational principles. Indeed, the rudiments of quantum theory may be derived from Newton's laws and variational principles. We will in these notes trace some of the history of the calculus of variations. One caution: while much of the treatment will seem mathematical, it is in fact very handwavy. A true and correct treatment would require us to pull out such differential geometry tools as covariant derivatives, affine connections, and  $n$ -forms.

## Applications:

Path optimization

Optimal control (min cost trajectories)

## Engineering

Vibrating membranes

Theory of elasticity

Electrostatics

## Machine Vision

Surface reconstruction

Image flow (Motion and Structure from Optical Flow)

Edge detection

## Introduction

We have seen the basic principle "To minimize  $P$  is to solve  $P'=0$ ."

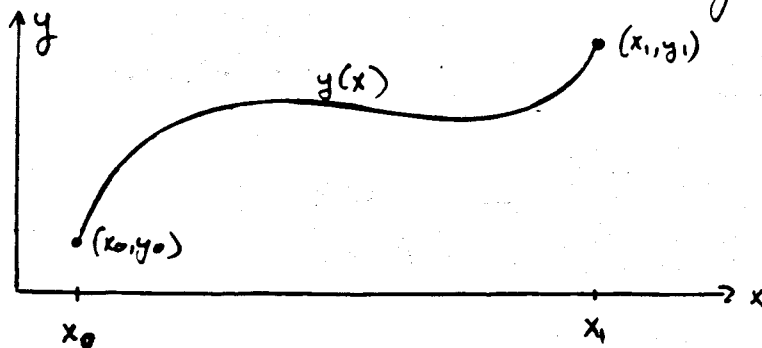
So far we have only looked at finite-dimensional problems, that is, minimizations of some function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$ . In such a problem we seek the value of  $n$  numbers that minimize  $f$ .

What about infinite-dimensional problems, that is, problems in which  $P$  depends on an infinity of numbers?

In particular, what about functionals (functions of functions)?

### Example

Suppose we connect two points in the plane,  $(x_0, y_0)$  and  $(x_1, y_1)$ , by a rectifiable curve of the form  $y = y(x)$ .



The length of the curve can be written

$$L(y) = \int_{x_0}^{x_1} \sqrt{1 + (y')^2} dx$$

$L$  is a functional.

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Now consider the problem:

"Find the shortest curve between the points  $(x_0, y_0)$  and  $(x_1, y_1)$ ."

This problem asks us to minimize the functional  $L$ . But  $y$  has an infinite number of degrees of freedom. After all, there is one value of  $y$  for each  $x$  in  $(x_0, x_1)$ . (Constraining  $y$  to be  $C^2$  say reduces the degrees of freedom a bit, but there are still infinitely many of them.)

For instance, in the case of periodic functions we can construct Fourier series. There the degrees of freedom correspond to choosing the coefficients  $\{c_j\}_{j=-\infty}^{\infty}$ . In short,  $L(y)$  depends on infinitely many parameters.

We "know" that the solution to the previous problem is given by a straight line. But how do we know this? And how can we generalize our solution to other similar problems?

That's what the calculus of variations is all about.

Some problems similar to the shortest curve problem are:

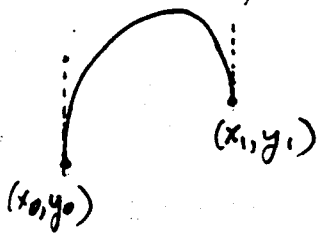
- Shortest connecting curve on a non-planar surface (for example on a sphere).
- Minimal surface of revolution generated by a connecting curve.
- Shortest curve with a given area below it.
- Closed curve of a given perimeter that encloses the greatest area.
- Shape of a string hanging from two points under the influence of gravity.
- Path of a ray of light travelling through an inhomogeneous medium.
- Shape of a wire that will cause a bead to move between two points in minimum time (gravity moves the bead).
- The Soap Bubble/Soap Film Problem (i.e., generalizations of the previous problems to higher dimensions).

## Caution

A functional may not have a minimizing solution. There are different reasons why this might happen. One analogy to keep in mind is that a function need not have a minimum either. For instance, the function may be unbounded towards  $-\infty$ , or it may asymptotically approach a minimum but never reach it.

Ex Consider the problem

"Find a curve of minimum length that has continuous curvature and that connects the points  $(x_0, y_0)$  &  $(x_1, y_1)$  in such a way that it is vertical at the end points."



Observe that there is a set of better and better solutions, but no best solution.

# Euler Equation

The principle "To minimize  $P$  is to solve  $P' = 0$ " suggest that if we try to minimize a functional  $L(y)$  given by an integral, we will do well to solve a differential equation. Indeed, this intuition is correct. The resulting differential equation is called the Euler Equation. We will derive it presently.

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First, some perspective.

Recall the following:

Suppose  $f: \mathbb{R}^n \rightarrow \mathbb{R}$ . What does it mean for  $x^*$  to be a local extremum of  $f$ ? (say a local minimum)

- 1) Definitionally we have that  $f(x) \geq f(x^*)$  for every  $x$  in some sufficiently small neighborhood of  $x^*$ .
- 2) A necessary condition is that  $\nabla f(x^*) = 0$ , i.e., that  $\frac{\partial f}{\partial x_i}(x^*) = 0$  for  $i=1, 2, \dots, n$ .

Now suppose instead that we are dealing with a functional  $P$ , say

$$P: C^2(\mathbb{R}^n) \rightarrow \mathbb{R}$$
$$f \mapsto P(f).$$

What does it mean for  $f^*$  to be an extremum of  $P$ ? (say a relative minimum)

Well, we can write down conditions similar to those we just wrote down for functions:

- 1) Definitionally we have that  $P(f) \geq P(f^*)$  for every function in some sufficiently small neighborhood of  $f^*$  (we are now talking about neighborhoods in function space).
- 2) We really want a kind of generalized gradient, i.e., we would like to be able to talk about  $\frac{\partial P}{\partial f}$  for every function  $f$ .

In other words, we want an expression that allows us to say:

"If  $f^*$  is an extremum of  $P$  then  $P(f^*)$  does not change much if we vary  $f^*$  in the direction of any function  $f$ ."

Informally we write

$$P(f^* + \delta f) \approx P(f^*)$$

for a small variation  $\delta f$ .

Note: We will never make the symbol  $\delta f$  as precise as one might like.

Think of  $f^* + \delta f$  as a point-wise differential displacement of the function  $f^*$  by the function  $f$ .

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Let's begin with the so-called "Simplest Problem":

$$\text{Minimize } J(y) = \int_{x_0}^{x_1} F(x, y, y') dx,$$

with  $x_0, x_1, y(x_0), y(x_1)$  specified.  
Both  $y$  &  $F$  are assumed to be  $C^2$ .

Suppose that  $y^*$  minimizes  $J$ .  
Our previous discussion says:

1) If  $y$  lies in a sufficiently small neighborhood of  $y^*$  then  $J(y) \geq J(y^*)$ .

2)  $\delta J = 0$  for any variation  $\delta y$ , where

$$\delta J(y^*) = J(y^* + \delta y) - J(y^*)$$

Observe that we should not consider all possible perturbations of  $y^*$  by some test function  $\delta y$ , merely those which are  $C^2$  and which vanish at the endpoints  $x_0, x_1$ .

We can use each of (1) and (2) to derive a necessary condition for  $y^*$  to be an extremizing function. This condition is called the Euler Equation (or Euler-Lagrange).



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Let's start with (1). This is the usual approach.  
Later we will then also consider a derivation based  
on (2), yielding the same condition, namely Euler's equation.

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Let  $\eta \in C^2[x_0, x_1]$  such that  $\eta(x_0) = \eta(x_1) = 0$ .

Let's fix this function for the moment.

Now let  $\varepsilon \in \mathbb{R}$  and consider the one-parameter family  
of functions.

$$y(x) = y^*(x) + \varepsilon \eta(x)$$

where  $y^*$  is the (unknown) optimal function.

Now define the function  $\Phi: \mathbb{R} \rightarrow \mathbb{R}$  by

$$\Phi(\varepsilon) = \int_{x_0}^{x_1} F(x, y, y') dx$$

$$\text{where } y = y^* + \varepsilon \eta$$

Since  $[x_0, x_1]$  is compact, if  $|\varepsilon|$  is small enough  
then all the functions  $y = y^* + \varepsilon \eta$  lie in an arbitrarily  
chosen small neighborhood of  $y^*$ . Therefore  $\Phi$   
attains a local minimum at  $\varepsilon = 0$ .

This means that we can apply standard old calculus  
to say that

$$\Phi'(0) = 0.$$

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So, what is  $\Phi'$ ?

well 
$$\Phi(\varepsilon) = \int_{x_0}^{x_1} F(x, y^* + \varepsilon \eta, y^{*'} + \varepsilon \eta') dx$$

So 
$$\Phi'(\varepsilon) = \frac{d}{d\varepsilon} \int_{x_0}^{x_1} F(x, y^* + \varepsilon \eta, y^{*'} + \varepsilon \eta') dx$$

since we are dealing with  $C^2$  functions on a compact interval we can interchange differentiation and integration.

So 
$$\begin{aligned} \Phi'(\varepsilon) &= \int_{x_0}^{x_1} \frac{d}{d\varepsilon} F(x, y^* + \varepsilon \eta, y^{*'} + \varepsilon \eta') dx \\ &= \int_{x_0}^{x_1} \left\{ \frac{\partial F}{\partial y}(x, y^* + \varepsilon \eta, y^{*'} + \varepsilon \eta') \eta \right. \\ &\quad \left. + \frac{\partial F}{\partial y'}(x, y^* + \varepsilon \eta, y^{*'} + \varepsilon \eta') \eta' \right\} dx \end{aligned}$$

So 
$$\Phi'(0) = \int_{x_0}^{x_1} \left\{ \frac{\partial F}{\partial y}(x, y^*, y^{*'}) \eta + \frac{\partial F}{\partial y'}(x, y^*, y^{*'}) \eta' \right\} dx$$

We write this compactly as

$$\Phi'(0) = \int_{x_0}^{x_1} \{ F_y \eta + F_{y'} \eta' \} dx$$

Integration by parts tells us that

$$\begin{aligned}\int_{x_0}^{x_1} F_{y'} y' dx &= F_{y'} y \Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} y \frac{d}{dx} F_{y'} dx \\ &= - \int_{x_0}^{x_1} y \frac{d}{dx} F_{y'} dx\end{aligned}$$

$$\text{So } \Phi'(0) = \int_{x_0}^{x_1} \left[ F_y - \frac{d}{dx} F_{y'} \right] y dx$$

But  $\Phi'(0) = 0$ . So, for every  $y$  as above we see that

$$\int_{x_0}^{x_1} \left[ F_y - \frac{d}{dx} F_{y'} \right] y dx = 0$$

Since  $y$  is arbitrary the only way this can happen is if the integrand is identically zero, that is, if

$$(*) \quad \boxed{F_y - \frac{d}{dx} F_{y'} = 0 \quad \text{on } [x_0, x_1]}$$

This is Euler's Equation. In short: A necessary condition for  $y^*$  to be an extremizing function of  $J(y)$  is that  $y^*$  satisfy the differential equation (\*) subject to the boundary conditions  $y(x_0) = y_0$  &  $y(x_1) = y_1$ .

For fun, let's also look at a derivation based on condition (2) on p. 8

We want to find  $y^*$  such that  $J(y^* + \delta y) \approx J(y^*)$ .

Again, let  $\eta \in C^2[x_0, x_1]$  with  $\eta(x_0) = \eta(x_1) = 0$ .

Let  $\varepsilon \in \mathbb{R}$ . We will write  $\delta y = \underset{\substack{\uparrow \\ \text{symbol}}}{\varepsilon} \underset{\substack{\uparrow \\ \text{number}}}{\eta}$ , thus finally giving some small meaning to the variational symbol  $\delta y$ . function.

In order to consider  $J(y^* + \varepsilon \eta)$ , let's look at the Taylor expansion of  $F$ .

$$F(x, y^* + \varepsilon \eta, y^{*'} + \varepsilon \eta') = F(x, y^*, y^{*'}) + F_y \varepsilon \eta + F_{y'} \varepsilon \eta' + \text{higher order terms.}$$

$$\text{So } J(y^* + \varepsilon \eta) = J(y^*) + \varepsilon \int_{x_0}^{x_1} (F_y \eta + F_{y'} \eta') dx + \dots$$

$$\text{So, for } J(y^* + \varepsilon \eta) \approx J(y^*)$$

$$\text{we must have that } \int_{x_0}^{x_1} (F_y \eta + F_{y'} \eta') dx \approx 0$$

This is exactly what we derived earlier, right before applying an integration by parts.

So, either way leads to the Euler Equation.

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A couple of side comments:

- The expression  $\int_{x_0}^{x_1} (F_y \eta + F_{y'} \eta') dx$  is essentially a "directional derivative" in the direction  $\eta$  (in function space).
- In saying that  $\delta J = 0$  we are really only finding functions  $y^*$  at which  $J$  is stationary. If we wish to optimize, we really would need to look at second order variational conditions. This is analogous to looking at second derivatives in the regular calculus. We will not consider second order conditions. Instead we will rely on physical intuition. Furthermore, for many mechanics problems it is in fact sufficient to find stationary points of  $J$ ; optimization isn't an issue.

Ex. Let's show that the shortest path in the plane between two points is a straight line.

Recall:  $L(y) = \int_{x_0}^{x_1} \sqrt{1+y'^2} dx$  with  $y_0 = y(x_0)$  &  $y_1 = y(x_1)$  given.

So  $F(x, y, y') = \sqrt{1+y'^2}$

$$F_y = 0$$

$$F_{y'} = \frac{y'}{\sqrt{1+y'^2}}$$

So the Euler equation reduces to

$$\frac{d}{dx} \frac{y'}{\sqrt{1+y'^2}} = 0$$

So  $\frac{y'}{\sqrt{1+y'^2}} = c \leftarrow \text{constant}$

so  $y'^2 = c^2 (1+y'^2)$

$$y' = \pm \frac{c}{\sqrt{1-c^2}} \left. \vphantom{y'} \right\} \text{ call this } m, \text{ it is just another constant}$$

$$y' = m$$

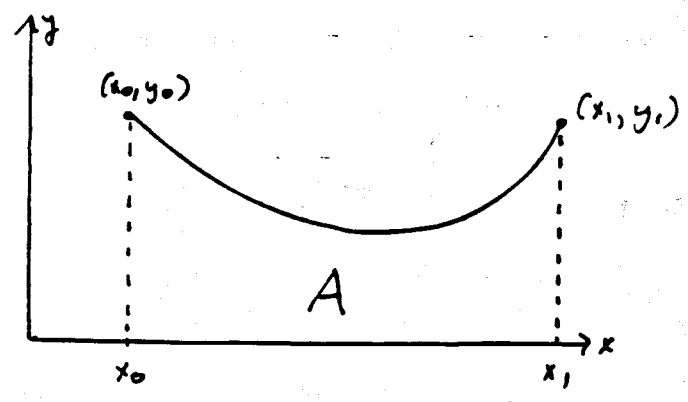
$$y = mx + b$$

So we see it is a straight line!

The boundary conditions specify  $m$  &  $b$ , in the usual way.

What about constrained optimization in this setting?

For instance, suppose we ask for the shortest curve with a given fixed area below it:



The area  $A$  is given.

The boundary conditions  $(x_0, y_0)$  &  $(x_1, y_1)$  are given.

So, we are trying to minimize  $L(y) = \int_{x_0}^{x_1} \sqrt{1+y'^2} dx$   
 subject to the constraint  $\int_{x_0}^{x_1} y dx = A$

This is an instance of a general class of problems of constraint called Isoperimetric Problems.

The simplest version is:

minimize  $J(y) = \int_{x_0}^{x_1} F(x, y, y') dx$   
 subject to  $K(y) = c$   
 where  $K(y) = \int_{x_0}^{x_1} G(x, y, y') dx$

We will see that a combination of variational techniques and Lagrange multipliers permits us to solve this type of problem.

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Following our earlier reasoning, let's define two functions:

$$\mathbb{I}(\varepsilon_1, \varepsilon_2) = \int_{x_0}^{x_1} F(x, y^* + \varepsilon_1 \eta + \varepsilon_2 \xi, y^{*'} + \varepsilon_1 \eta' + \varepsilon_2 \xi') dx$$

$$\mathbb{J}(\varepsilon_1, \varepsilon_2) = \int_{x_0}^{x_1} G(x, y^* + \varepsilon_1 \eta + \varepsilon_2 \xi, y^{*'} + \varepsilon_1 \eta' + \varepsilon_2 \xi') dx$$

Here  $y^*$  is our unknown minimizing function with  $y^*(x_0) = y_0$  &  $y^*(x_1) = y_1$  specified. And  $\eta$  &  $\xi$  are two  $C^2$  functions that vanish at the endpoints  $x_0$  &  $x_1$ .

We want to minimize  $\mathbb{I}$  subject to the constraint  $\mathbb{J} = c$ .  
We know that there is a local minimum at  $\varepsilon_1 = \varepsilon_2 = 0$ .

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Question:

Why do we introduce a two-parameter family of functions  $(y^* + \varepsilon_1 \eta + \varepsilon_2 \xi)$  instead of a one-parameter family as we did before?

Answer:

Because: With a one-parameter family, any



change in the parameter would cause us to violate the constraint  $\int_{x_0}^{x_1} G(x, y, y') dx = \text{constant}$ .

So, we need one parameter to allow us to optimize  $\int_{x_0}^{x_1} F(x, y, y') dx$  and another to satisfy the constraint.

Since we are minimizing  $\Phi$  subject to a constraint on  $\Psi$ , let's apply the technique of Lagrange multipliers

That approach tells us to form the function

$$E(\epsilon_1, \epsilon_2, \lambda) = \Phi(\epsilon_1, \epsilon_2) + \lambda(\Psi(\epsilon_1, \epsilon_2) - c)$$

At a local minimum,  $\nabla E = 0$ .

In other words, there exists  $\lambda_0$  such that

$$\frac{\partial}{\partial \epsilon_1} E(0, 0, \lambda_0) = 0 \quad \frac{\partial}{\partial \epsilon_2} E(0, 0, \lambda_0) = 0$$

$$\text{and } \frac{\partial}{\partial \lambda} E(0, 0, \lambda_0) = 0,$$

Let's compute.

[ Note that as usual with Lagrange multipliers, the last condition just repeats the constraint, i.e.,

$$\Psi(0, 0) = c. ]$$

Interchanging differentiation and integration as we did before, we see that

$$\frac{\partial}{\partial \varepsilon_1} E(0,0,\lambda_0) = \int_{x_0}^{x_1} (F_y \eta + F_{y'} \eta' + \lambda_0 G_y \eta + \lambda_0 G_{y'} \eta') dx$$

by the usual integration by parts, and since  $\eta$  vanishes at the endpoints, we see that

$$\frac{\partial}{\partial \varepsilon_1} E(0,0,\lambda_0) = \int_{x_0}^{x_1} \left\{ [F_y - \frac{d}{dx} F_{y'}] + \lambda_0 [G_y - \frac{d}{dx} G_{y'}] \right\} \eta dx$$

Similarly,

$$\frac{\partial}{\partial \varepsilon_2} E(0,0,\lambda_0) = \int_{x_0}^{x_1} \left\{ [F_y - \frac{d}{dx} F_{y'}] + \lambda_0 [G_y - \frac{d}{dx} G_{y'}] \right\} \xi dx$$

Comments:

The conditions  $\frac{\partial}{\partial \varepsilon_1} E(0,0,\lambda_0) = 0$  and  $\frac{\partial}{\partial \varepsilon_2} E(0,0,\lambda_0) = 0$

yield two corresponding integral constraints, one involving  $\eta$ , the other involving  $\xi$ . Since  $\eta$  &  $\xi$  are arbitrary these two constraints are essentially just one.

Why then did we need two constraints?

It has to do with the Lagrange multiplier

$\lambda_0$ . For all we know, its value could

depend on the perturbation functions  $\eta$  &  $\xi$ . But look at the two constraints:

$$(†) \int_{x_0}^{x_1} \left\{ [F_y - \frac{d}{dx} F_{y'}] + \lambda_0 [G_y - \frac{d}{dx} G_{y'}] \right\} \eta \, dx = 0$$

$$(‡) \int_{x_0}^{x_1} \left\{ [F_y - \frac{d}{dx} F_{y'}] + \lambda_0 [G_y - \frac{d}{dx} G_{y'}] \right\} \xi \, dx = 0$$

(†) tells us that  $\lambda_0$  does not depend on  $\xi$ .  
(And likewise, (‡) tells us that  $\lambda_0$  does not depend on  $\eta$ .)

So we conclude

$$(1) \text{ The value of } \lambda_0 \text{ is independent of the perturbing functions.}$$

Once we know (1), the arbitrariness of  $\eta$  &  $\xi$  allows us to conclude, as before, that

$$(2) [F_y - \frac{d}{dx} F_{y'}] + \lambda_0 [G_y - \frac{d}{dx} G_{y'}] = 0$$

So, a necessary condition for  $y^*$  to minimize  $J(y)$  subject to the constraint  $k(y) = c$  is that  $y^*$  satisfy the modified Euler Equation with  $\lambda_0$  determined by the constraint.

Back to the example:

$$\text{Minimize: } \int_{x_0}^{x_1} \sqrt{1+y'^2} dx$$

$$\text{Subject to: } y(x_0) = y_0$$

$$y(x_1) = y_1$$

$$\int_{x_0}^{x_1} y dx = A$$

$$\text{So } F(x, y, y') = \sqrt{1+y'^2}$$

$$G(x, y, y') = y$$

$$F_y = 0$$

$$F_{y'} = \frac{y'}{\sqrt{1+y'^2}}$$

$$G_y = 1$$

$$G_{y'} = 0$$

So we want to satisfy the differential equation

$$-\frac{d}{dx} \frac{y'}{\sqrt{1+y'^2}} + \lambda_0 = 0$$

In other words,  $\frac{y'}{\sqrt{1+y'^2}} = \lambda_0 x + c$ , for some constant  $c$ .

$$\text{So } \frac{y'^2}{1+y'^2} = (\lambda_0 x + c)^2$$

$$y'^2 = \frac{(\lambda_0 x + c)^2}{1 - (\lambda_0 x + c)^2}$$

$$\text{so } y' = \pm \frac{\lambda_0 x + c}{\sqrt{1 - (\lambda_0 x + c)^2}}$$

Integrating, we obtain

$$\begin{aligned}
 y(x) &= \pm \int \frac{\lambda_0 x + c}{\sqrt{1 - (\lambda_0 x + c)^2}} dx && u = \lambda_0 x + c \\
 &&& du = \lambda_0 dx \\
 &= \pm \frac{1}{\lambda_0} \int \frac{u}{\sqrt{1 - u^2}} du && = \pm \frac{1}{\lambda_0} \left[ -\sqrt{1 - u^2} + k \right] \\
 &= \pm \left[ -\frac{1}{\lambda_0} \sqrt{1 - (\lambda_0 x + c)^2} + \frac{k}{\lambda_0} \right], && \text{with } k \text{ an arbitrary constant.}
 \end{aligned}$$

For a given problem we would need to determine the three constants  $c, k, \lambda_0$  from the boundary conditions and the area constraint.

Notice however that  $y$  is the equation of a circle:

$$\left( y \mp \frac{k}{\lambda_0} \right)^2 + \left( x + \frac{c}{\lambda_0} \right)^2 = \frac{1}{\lambda_0^2}$$

Note: If the area constraint  $A$  is specified too large or too small relative to the boundary conditions  $(x_0, y_0), (x_1, y_1)$ , then it may be impossible to attain the constraint with a curve of the form  $y = y(x)$ , as  $x$  varies over  $[x_0, x_1]$ . A parameterized curve that strays outside the interval would instead do the trick. — This is similar to optimizing a function only to discover that the optimizing point lies outside the domain of interest.

Better alternative:  
Nonsmooth, e.g.,



Now let's look at some generalizations.

### Several Dependent Variables

This is very similar to the simplest problem.

Suppose we wish to determine a number of functions  $y(x), z(x), \dots$  that extremize the following integral (or at least are stationary points in function space)

$$J(y, z, \dots) = \int_{x_0}^{x_1} F(x, y, z, \dots, y', z', \dots) dx$$

The usual arguments imply that the functions  $y, z, \dots$  must satisfy the following differential equations:

$$F_y - \frac{d}{dx} F_{y'} = 0, \quad F_z - \frac{d}{dx} F_{z'} = 0, \dots$$

Only now, note that the derivatives  $\frac{d}{dx} [\cdot]$  are more complicated.

$$\text{E.g., } \frac{d}{dx} F_{y'} = F_{y'x} + F_{y'y'} y' + F_{y'z'} z' + \dots + F_{y'y''} y'' + F_{y'z''} z'' + \dots$$

Ex Find the shortest curve connecting two points in a multi-dimensional space.

Ex Suppose the propagation of light in a 3D medium is specified locally by the function  $\varphi(x, y, z, y', z')$ . [This is velocity as a function of location and angle of the light ray.]

Then the problem of finding the path of a light ray that minimizes the traversal time between two points is given by a variational problem with

$$F = \frac{\sqrt{1 + y'^2 + z'^2}}{\varphi(x, y, z, y', z')}$$

as integrand.

### Higher Derivatives

Suppose we seek  $y(x)$  that extremizes

$$J(y) = \int_{x_0}^{x_1} F(x, y, y', y'', \dots, y^{(n)}) dx,$$

Similar arguments as before show that  $y$  must satisfy:

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} - \dots + (-1)^n \frac{d^n}{dx^n} F_{y^{(n)}} = 0.$$

### Free Boundaries

Suppose we wish to minimize  $J(y) = \int_{x_0}^{x_1} F(x, y, y') dx$

but we no longer impose one or both of the endpoint conditions. Recall that in computing  $\delta J$  we found that

$$0 = \delta J = \int_{x_0}^{x_1} (F_y \eta + F_{y'} \eta') dx$$

where  $\eta$  is an arbitrary perturbation function. Before we insisted that  $\eta$  be  $C^2$  and that it vanish at the endpoints. We now still insist that  $\eta$  be  $C^2$  but remove one or both of the vanishing endpoint conditions. So

$$0 = \delta J = F_{y'} \eta \Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} [F_y - \frac{d}{dx} F_{y'}] \eta dx$$

Since  $y$  is arbitrary, we can of course also consider those  $y$  that vanish at  $x_0 \neq x_1$ . Thus we again obtain the necessary condition that

$$F_y - \frac{d}{dx} F_{y'} = 0.$$

In addition, since  $y$  is arbitrary, we obtain for each unspecified endpoint the condition that

$$F_{y'} \Big|_{x_i} = 0, \text{ if } y(x_i) \text{ is unspecified.}$$

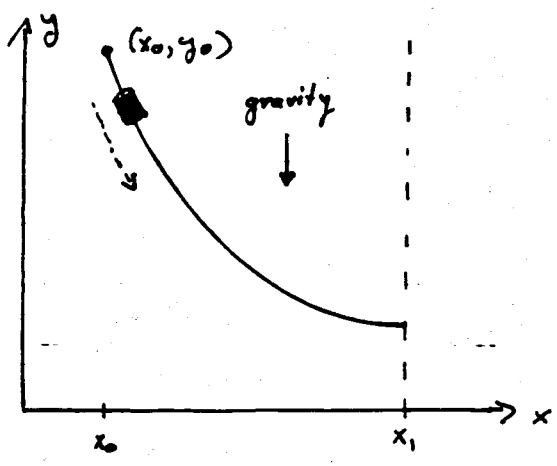
Higher dimensional generalizations exist, although they are a bit more complicated. See C&H, pp. 208 ff.

Also, it is possible to deal with semi-free boundaries

Ex This is a free boundary version of the brachistochrone problem:

	↙	↘
Greek:	brachistos	chronos
	(shortest)	(time)

We are given the problem of finding the curve of quickest descent from a given point to a given vertical line.



So, one endpoint  $(x_0, y_0)$  is specified and the other endpoint  $(x_1, y(x_1))$  is free.



Suppose  $y$  is a given curve. Let  $s$  denote arclength and let  $v = \frac{ds}{dt}$  be velocity along the curve. Then the time to reach  $x_1$  from  $(x_0, y_0)$  is given by:

$$J(y) = \int_{x=x_0}^{x=x_1} \frac{ds}{v}$$

$$= \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{v} dx$$

To compute  $v$ :

Conservation of energy tells us that

$$\Delta KE = -\Delta PE$$

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = mg(y - y_0)$$

$g$  is gravity,  
 $g = -9.8 \text{ m/s}^2$ .

Here  $v$  is the velocity along the curve after it has fallen from height  $y_0$  to height  $y$ , given that its initial velocity was  $v_0$ .

Suppose  $v_0 = 0$ . Then  $v = \sqrt{2g(y - y_0)}$ , so

$$J(y) = \frac{1}{\sqrt{-2g}} \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{y_0 - y}} dx$$

So the integrand is  $F(x, y, y') = \frac{1}{\sqrt{-2g}} \frac{\sqrt{1+y'^2}}{\sqrt{y_0 - y}}$

observe that the integrand does not depend on  $x$ . This simplifies the Euler Equation, as we shall see.

Observe that

$$\begin{aligned} \frac{d}{dx} (y' F_{y'} - F) &= y'' F_{y'} + y' \frac{d}{dx} F_{y'} - F_x - F_{y'} y' - F_{y'} y'' \\ &= y' \frac{d}{dx} F_{y'} - F_x - F_{y'} y' \\ &= -y' \left[ F_{y'} - \frac{d}{dx} F_{y'} \right] - F_x. \end{aligned}$$

Now the Euler Equation says  $F_{y'} - \frac{d}{dx} F_{y'} = 0$ , so

$$\frac{d}{dx} (y' F_{y'} - F) = -F_x$$

This is true always.

But notice what it implies: If  $F$  does not depend on  $x$  then we know that

$$\frac{d}{dx} (y' F_{y'} - F) = 0.$$

So we can replace the Euler Equation by the equation

$$y' F_{y'} - F = c \quad (\text{constant})$$

In other words:

We have changed the problem of determining  $y$  from a second-order differential equation to that of determining  $y$  from a first-order differential equation involving only  $y$  and  $y'$  (no  $x$ )!

Back to our example:

$$F(x, y, y') = \frac{1}{\sqrt{-2g}} \frac{\sqrt{1+y'^2}}{\sqrt{y_0-y}}$$

So:

$$F_y = \frac{1}{2} \frac{1}{\sqrt{-2g}} \frac{\sqrt{1+y'^2}}{(y_0-y)^{3/2}} \qquad F_{y'} = -\frac{1}{\sqrt{-2g}} \frac{y'}{\sqrt{(y_0-y)(1+y'^2)}}$$

We thus get the following three conditions:

(i) 
$$\frac{y'^2}{\sqrt{(y_0-y)(1+y'^2)}} - \frac{\sqrt{1+y'^2}}{\sqrt{y_0-y}} = C$$

(ii) 
$$\frac{y'}{\sqrt{(y_0-y)(1+y'^2)}} \Big|_{x=x_1} = 0$$

(iii)  $y(x_0) = y_0.$

---

(iii) is just the fixed endpoint condition at  $x_0$ .  
(ii) is the free endpoint condition at  $x_1$ .  
(i) is the differential equation we need to solve.  
Simplifying it, we see that it really says:

$$\frac{-1}{\sqrt{(y_0-y)(1+y'^2)}} = C$$

So  $(y_0 - y)(1 + y'^2) = \frac{1}{c^2}$ .

It will be convenient to write this as

$$(y_0 - y)(1 + y'^2) = 2a \quad \text{where } a \text{ is a non-negative constant.}$$

Then  $1 + y'^2 = \frac{2a}{y_0 - y}$

$$y' = \pm \frac{\sqrt{2a - (y_0 - y)}}{\sqrt{y_0 - y}}$$

We will choose the negative minus sign, since we want the curve to be pointing downward (this assumes  $x_1 > x_0$ ; otherwise choose "+").

Writing  $y' = \frac{dy}{dx}$ , let's invoke the Inverse Function Theorem, wave our hands a bit, and conclude that

$$\frac{dx}{dy} = \frac{1}{y'}$$

and therefore  $x = - \int \frac{\sqrt{y_0 - y}}{\sqrt{2a - (y_0 - y)}} dy$

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Let's make the substitution  $y_0 - y = 2a \sin^2 \frac{\theta}{2}$  ( $= a(1 - \cos \theta$

Then  $\sqrt{y_0 - y} = \sqrt{2a} \sin \frac{\theta}{2}$

$$\sqrt{2a - (y_0 - y)} = \sqrt{2a} \sqrt{1 - \sin^2 \frac{\theta}{2}} = \sqrt{2a} \cos \frac{\theta}{2}$$

$$a \quad -dy = 2a \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta$$

So

$$x = \int \frac{\sqrt{2a} \sin \frac{\theta}{2}}{\sqrt{2a} \cos \frac{\theta}{2}} 2a \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta$$

$$= 2a \int \sin^2 \frac{\theta}{2} d\theta$$

$$= a(\theta - \sin \theta) + K \quad (\text{some constant})$$

---

So, we have a parametrized representation of the curve as

$$C: \quad \theta \mapsto (a(\theta - \sin \theta) + K, -a(1 - \cos \theta) + y_0)$$

If we want the curve to start at  $(x_0, y_0)$  when  $\theta = 0$ , then the constant  $K$  is simply  $x_0$ .

How do we determine  $a$ ?

Well, if  $(x_1, y_1)$  were specified, then that condition would determine  $a$ . However,  $y$  is unspecified at  $x_1$ .

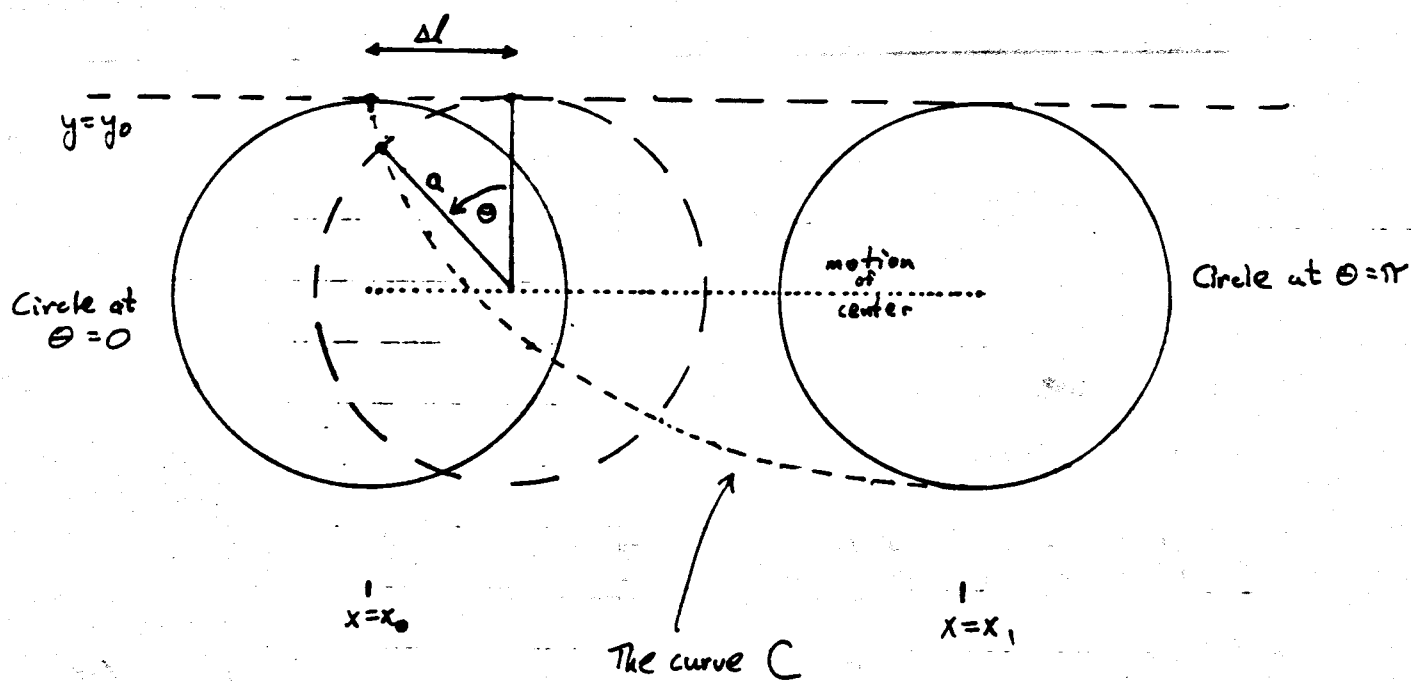
Instead, we have condition (ii) from page 27, namely

$$\frac{y'}{\sqrt{(y_0 - y)(1 + y'^2)}} = 0 \quad \text{at } x = x_1.$$

Assuming the denominator is non-zero (i.e.  $y < y_0$ ), this says that  $y'(x_1) = 0$ . In other words, the tangent to  $C$  at  $x = x_1$  is horizontal.

So,  $C$  is a cycloid that is horizontal at  $x = x_1$ .

Recall that a cycloid is generated by the motion of a fixed point on the circumference of a circle as the circle rolls along a line. In our example, the circle has radius  $a$  and rolls below the horizontal line  $y = y_0$ .



$\Delta l = a\theta$ . When the circle rotates by  $\theta$  along the line  $y = y_0$ ,  $y$  decreases by  $a(1 - \cos\theta)$  and  $x$  increases by  $\Delta l - a\sin\theta = a(\theta - \sin\theta)$ .

Note: As the figure suggests, the cycloid is horizontal when the circle has made half a rotation. Let's verify this:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a\sin\theta}{a(1-\cos\theta)} = \frac{-\sin\theta}{1-\cos\theta},$$

which is indeed zero at  $\theta = \pi$ .

So, we want  $x|_{\theta=\pi} = x_1$ ,

In other words  $a\pi + x_0 = x_1$ ,

$$a = \frac{x_1 - x_0}{\pi}$$

So, the final curve is given in parametric form as:

$$\left( \frac{x_1 - x_0}{\pi} (\theta - \sin\theta) + x_0, \frac{x_0 - x_1}{\pi} (1 - \cos\theta) + y_0 \right), \theta \in [0, \pi]$$

Note: If the endpoint  $(x_1, y_1)$  had been tied down, then we would have determined "a" by that boundary condition. This would have led to a possibly different interval for  $\theta$ . Indeed, the curve might achieve  $(x_1, y_1)$  before  $\theta$  is  $\pi$ , or even after. In the latter case, the curve  $C$  dips below  $y_1$ , then comes back up to hit the point  $(x_1, y_1)$ .

## Several Dimensions (several independent variables)

Let's focus on the 2D case. (Generalizations are clear.)

We are interested in finding a function  $u(x, y)$  for which the integral

$$J(u) = \iint_G F(x, y, u, u_x, u_y) dx dy$$

is an extremum (or at least stationary, i.e.,  $\delta J = 0$ ).

Here  $G$  is a region of integration.

We assume that it is well-behaved.

In particular,  $\partial G$  is a curve with a well-defined tangent that turns piecewise continuously.

$u(x, y)$  should be  $C^2$ , with prescribed values on the boundary  $\partial G$ . — It is possible to generalize these results somewhat to free boundary segments, but we will not do so.

In order to derive a differential equation for  $u$ , we can proceed much as we did in the one-dimensional case, and introduce a function  $\eta(x, y) \in C^2(G)$  that vanishes on  $\partial G$ . Next, we define  $\Phi(\epsilon)$  relative to the extremizing  $u^*$  as



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$$\Phi(\varepsilon) = \iint_G F(x, y, u^* + \varepsilon \eta, u_x^* + \varepsilon \eta_x, u_y^* + \varepsilon \eta_y) dx dy$$

and, as before, compute  $\delta J$  as  $\varepsilon \Phi'(0)$ . This yields

$$\delta J = \varepsilon \iint_G (F_u \eta + F_{u_x} \eta_x + F_{u_y} \eta_y) dx dy$$

Now what? — Well, we need a generalization of integration by parts. Stokes Theorem is the proper generalization. In 2D we know this as Green's Theorem:

Th<sup>m</sup> If  $P(x, y)$  and  $Q(x, y)$  are everywhere continuous in  $G$  and piecewise continuous on  $\partial G$ , and if  $G$  may be subdivided into a finite number of subdomains in each of which the first partials of  $P$  and  $Q$  are continuous, then

$$\iint_G \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx dy = \int_{\partial G} (P dy - Q dx)$$

Corollary: By writing  $P = \eta R$  and  $Q = \eta S$  (all are functions on  $G \in \mathbb{R}$ ) we obtain

$$\begin{aligned} & \iint_G (R \eta_x + S \eta_y) dx dy \\ &= - \iint_G [R_x + S_y] \eta dx dy + \int_{\partial G} [R dy - S dx] \eta \end{aligned}$$

Proof of Corollary

$$\frac{\partial P}{\partial x} = \frac{\partial \eta}{\partial x} R + \eta \frac{\partial R}{\partial x} = \eta_x R + \eta R_x$$

$$\frac{\partial Q}{\partial y} = \frac{\partial \eta}{\partial y} S + \eta \frac{\partial S}{\partial y} = \eta_y S + \eta S_y$$

$\therefore$  Green's theorem says that

$$\iint_G (\eta_x R + \eta R_x + \eta_y S + \eta S_y) dx dy = \int_{\partial G} (\eta R dy - \eta S dx)$$

which proves it. //

So, we see that

$$\delta J = 2 \iint_G \eta \left[ F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} \right] dx dy + \varepsilon \int_{\partial G} \eta [F_{u_x} dy - F_{u_y} dx]$$

Since  $\eta$  vanishes on  $\partial G$  and is otherwise arbitrary, we see that  $\delta J = 0$  implies that  $u$  satisfies the following PDE:

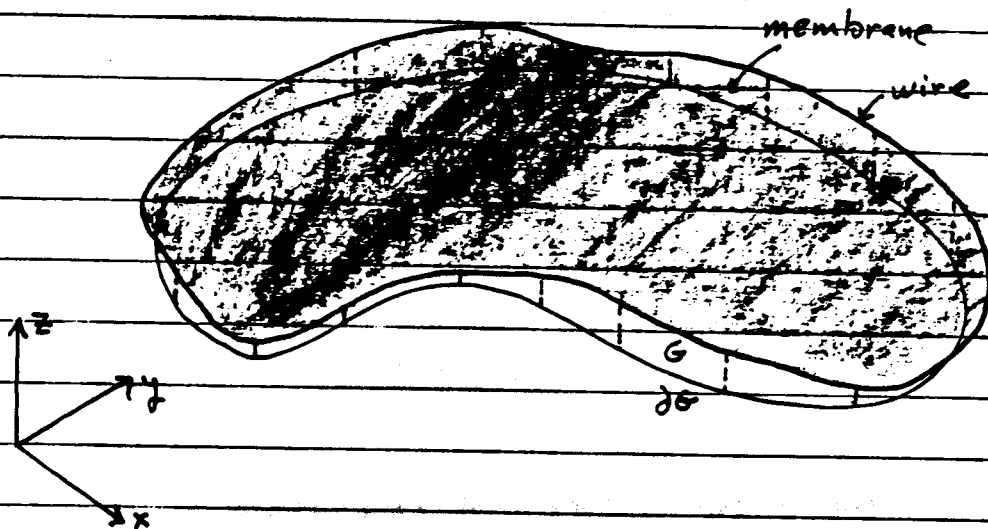
$$F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} = 0, \text{ subject to the prescribed boundary conditions for } u(x,y) \text{ on } \partial G.$$

So, this is a boundary value problem.

[We haven't talked about how to solve PDEs, but check out NRiC for some numerical techniques. And see, for instance, books on Fourier Series & series based on orthogonal functions for other techniques.]

### Ex The Soap Bubble Problem

Consider a membrane that covers a 2D region  $G$ .  
 A wire runs around the region  $G$ , along its boundary  $\partial G$ .  
 The membrane is attached to the wire.  
 Something like this:



Suppose the membrane consists of homogeneous material.  
 Then stretching the membrane requires energy  
 proportional to the change in area.

We let  $u(x,y)$  be the  $z$ -height of the membrane.  
 $u(x,y)$  is prescribed on  $\partial G$  by the height of the wire.  
 Let us assume that  $u$  is  $C^2$  and that the surface  
 area of the membrane is well-defined.

We may then think of the membrane as storing  
 potential energy proportional to its surface area.

So, with  $\mu$  as the proportionality constant, the potential energy of the membrane is

$$E(u) = \mu \int_G \int \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2} dx dy$$

If the height of the wire is 0 (i.e., if  $u|_{\partial G} = 0$ ), then  $E(u)$  is a minimum for  $u \equiv 0$ .

[Note: there is no force of gravity in the problem as we have formulated it so far.]

However, if the wire is bent, then  $u$  need not be identically zero. In other words, we can obtain a non-trivial soap bubble in that case.

The Euler-Lagrange equation for this problem is a non-linear PDE. (See Strang p. 249.)

That's a tad difficult to solve. Instead, one often assumes that the deformation of the membrane is small, in the sense that higher powers of  $u$ ,  $u_x$ , &  $u_y$  are small compared with lower ones. Then

$$(1 + u_x^2 + u_y^2)^{\frac{1}{2}} \quad \text{may be replaced with} \\ 1 + \frac{1}{2}(u_x^2 + u_y^2).$$

(3)

Then  $E(u)$  becomes  $\mu \iint_G \left[ 1 + \frac{1}{2}(u_x^2 + u_y^2) \right] dx dy$ .

Finally, since constant terms don't matter, it is enough for us to look at the change in potential energy. So we consider the following functional:

$$E(u) = \frac{1}{2} \mu \iint_G (u_x^2 + u_y^2) dx dy.$$

We're not done! Let us now add in one more feature. Suppose that the membrane has mass, with uniform density  $\sigma$  ( $\text{kg/m}^2$ ) and suppose the membrane lies in a vertical gravitational field. Then the total potential energy of the membrane is

$$E(u) = \iint_G \left[ \frac{\mu}{2} u_x^2 + \frac{\mu}{2} u_y^2 - g\sigma u \right] dx dy$$

where  $g = -9.8 \text{ m/s}^2$

whew! we've finally set up the problem:

We would like to find  $u(x,y)$  such that  $E(u)$  is stationary and  $u$  takes on prescribed values on  $\partial G$ .

Let's derive the Euler-Lagrange equation:

$$F(x,y,u,u_x,u_y) = \frac{\mu}{2} u_x^2 + \frac{\mu}{2} u_y^2 - g\sigma u$$

So  $F_u = -g\sigma$

$$F_{u_x} = \mu u_x$$

$$F_{u_y} = \mu u_y$$

$$\text{so } \frac{\partial}{\partial x} F_{u_x} = \mu u_{xx}$$

$$\frac{\partial}{\partial y} F_{u_y} = \mu u_{yy}$$

So  $u$  must satisfy the boundary value problem:

$$-g\sigma - \mu u_{xx} - \mu u_{yy} = 0$$

that is  $\nabla^2 u(x,y) = k$  where  $k = -\frac{g\sigma}{\mu} > 0$   
is a constant  
with  $u|_{\partial G}$  specified.

This is Poisson's equation.

Ex Suppose  $G$  is the disk of radius  $r$  about the origin and suppose  $u(x,y) = 0$  on the boundary.

Consider  $u(x,y) = A(x^2 + y^2 - r^2)$  with  $A = \frac{k}{4} = \frac{-g\sigma}{4\mu}$

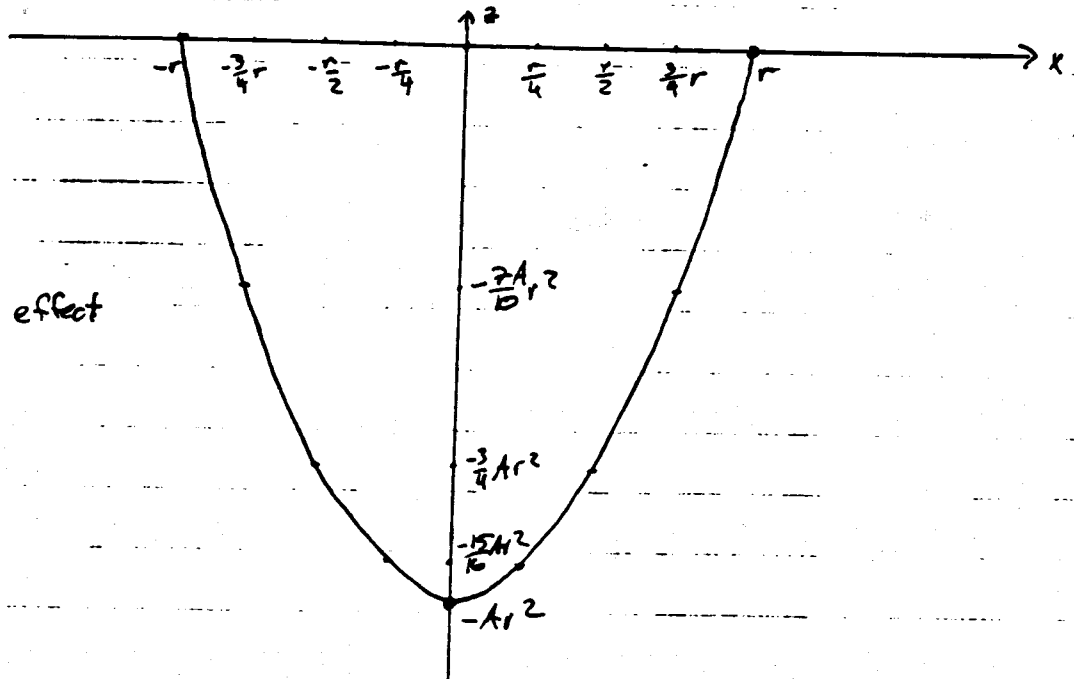
Then  $u$  satisfies the boundary condition, and

$$u_x = 2Ax \quad u_y = 2Ay$$

$$u_{xx} = 2A \quad u_{yy} = 2A$$

So  $\nabla^2 u = 4A = k$ , so  $u(x,y)$  is the solution we want.

Picture:



This is a  
 $y \geq 0$  cross  
section.

$A$  is large for effect

Aside: Do the dimensions make sense?

$g$  is  $m/s^2$

$\mu$  is  $N/m$  (i.e., membrane "tension")

$\sigma$  is  $kg/m^2$  (i.e., mass density)

$$\begin{aligned}
 \text{So } A &= \frac{-g}{4} \frac{\sigma}{\mu} \text{ is } \frac{m}{s^2} \cdot \frac{kg}{m^2} \cdot \frac{m}{N} \\
 &= \frac{kg}{Ns^2} \\
 &= \frac{kg}{\frac{kg \cdot m}{s^2} s^2} \\
 &= \frac{1}{m}
 \end{aligned}$$

So  $A(x^2+y^2-r^2)$  has units of

$$\frac{m^2}{m} = m$$

which is good.

# Mechanics

Much of mechanics is built upon variational principles. Perhaps the most important equations in physics, the Hamilton equations of motion follow from a very simple variational principle, called the Principle of Least Action.

We will discuss the foundational basis for this principle, derive from it Lagrange's equations of motion, and through them Hamilton's equations, then consider some simple applications.

## Lagrange's Equations

Just as a reminder (or a preview if you haven't already seen this), Lagrange's equations of motion describe the motions of a system of generalized coordinates. It is necessary to have such a description, since Newton's equations really are only formulated for point masses. In order to avoid writing down infinite numbers of point masses and all the constraint forces between them, it is convenient to deal with generalized coordinates.

Specifically, suppose that a system's configuration can



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be specified by the  $r$  coordinates  $q_1, \dots, q_r$ .  
Corresponding to these one can define generalized forces  
 $Q_1, \dots, Q_r$ .

Intuitively, the force  $Q_i$  measures the amount of work done in the  $q_i$  direction due to an infinitesimal displacement of the system. In other words,

$$\text{work done} = \sum_{i=1}^r Q_i \delta q_i.$$

For instance, suppose we have  $N$  particles (point masses) in 3 space. If these particles are free from constraint, then we may describe the state of the system with  $3N$  coordinates, say the positions  $\vec{r}_1, \dots, \vec{r}_N$  of the  $N$  particles. The forces in this case are simply the usual Newtonian forces acting on the particles. Call them  $\vec{F}_1, \dots, \vec{F}_N$ .

Now suppose that there are  $k$  (possibly time-varying) holonomic constraints that connect these  $N$  particles. [A holonomic constraint is of the form  $f(\vec{r}_1, \dots, \vec{r}_N, t) = 0$ . For instance, the rigid connection of two particles is expressed by the constraint  $|\vec{r}_i - \vec{r}_j|^2 - d_{ij}^2 = 0$ , where  $d_{ij}$  is the fixed distance between the particles.]

Then we can express the old coordinates of the particles in terms of  $3N - k$  generalized coordinates (and time), namely as:

$$\vec{r}_1 = \vec{r}_1(q_1, \dots, q_{3N-k}, t)$$

$$\vdots$$

$$\vec{r}_N = \vec{r}_N(q_1, \dots, q_{3N-k}, t)$$

Now suppose  $\vec{F}_i$  is the force acting on particle  $\#i$ .  
If we subject particle  $\#i$  to a virtual displacement  $\delta \vec{r}_i$ , then the total work done by the forces  $\{\vec{F}_i\}$  is

$$W = \sum_{i=1}^N \vec{F}_i \cdot \delta \vec{r}_i$$

(Note: of course constraints may exist between the  $\delta \vec{r}_i$  due to our  $k$  holonomic constraints.)

We can expand this as:

$$\begin{aligned} W &= \sum_{i=1}^N \vec{F}_i \cdot \sum_{j=1}^{3N-k} \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j \\ &= \sum_{j=1}^{3N-k} \left( \sum_{i=1}^N \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) \delta q_j \end{aligned}$$

By construction, the  $\delta q_j$  are constraint-free, i.e., independent of each other. So, if we let

$$Q_j = \sum_{i=1}^N \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

then  $W = \sum_j Q_j \delta q_j$ . So the  $\{Q_j\}$  are our desired generalized forces. They are obtained from the  $\{\vec{F}_i\}$  by what amounts to a Jacobian transformation.

With these definitions in hand, one could now derive Lagrange's equations from Newton's equations. The argument involves virtual work (more accurately: D'Alembert's principle). — We will not do so. Instead we will later derive Lagrange's equations in a more general setting from the principle of least action (which is yet to be defined).

(The philosophical difference between the two derivations is the difference between a local argument and a global variational argument.)

However, let us state Lagrange's equations:

If we write the kinetic energy  $T$  of a system in terms of the generalized coordinates  $q_j$  and their time derivatives  $\dot{q}_j$ , then the motions of the system must satisfy the following differential equations:

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j$$

$j=1, \dots, r$

where  $T = T(q_1, \dots, q_r, \dot{q}_1, \dots, \dot{q}_r)$

(you probably already see the strong resemblance to the Euler-Lagrange equation.)

Furthermore, if  $Q_j$  is derived from a potential  $-V(q_1, \dots, q_r)$ , that is,  $Q_j = -\frac{\partial V}{\partial q_j}$ ,

then the Lagrange equations take the compact form

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0 \quad j=1, \dots, r$$

where  $L = T - V$  is the Lagrangian of the system.

Ex For a 1D particle we have

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{x}^2, \quad x \text{ being our generalized coordinate}$$

so  $T$  depends on  $\dot{x}$ , but not on  $x$ .

$$\text{so } \frac{\partial T}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{x}} = m \dot{x}, \quad \text{while } \frac{\partial T}{\partial q_j} = 0$$

And  $Q_j$  is just some force  $F$ .

Therefore Lagrange says:  $\frac{d}{dt}(m \dot{x}) = F$

And that's exactly what Newton says.

Application

Computation of manipulator dynamics.

We now set about deriving Lagrange's Equations as a variational principle

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(see Y+M)

## The Principle of Least Action

The principle evolved over the years from a vague mystical statement to a precise mathematical axiom.

Simply stated, the principle says that a system will move in a manner that minimizes a certain quantity called "action." It is the definition of this quantity that has evolved over the years.

- Perhaps the first partial enunciation of this statement was by Fermat (1601-1665), who stated that light always travels from one point to another in such a way as to make the traversal time a minimum.
- In a 1740 paper, Maupertius<sup>us</sup> (1698-1759) contemplated minimal principles, albeit in a specialized case, namely that of gravitational attraction. However, in 1744 in an address to the Paris Academy he announced the principle of least action as stated above. Maupertius<sup>us</sup>' definition of action derived from his assertion that action must depend on mass, velocity, and the distance traversed by a body. — Unfortunately, Maupertius<sup>us</sup> was vague about the time interval for which the product was to be computed, and so his actual choice of action varied from problem to problem.

- Also in 1744, Euler (1707-1793<sup>(?)</sup>) published a paper that he had written in the second half of 1743. It said:

"When a particle travels between two fixed points, it takes a path for which

$$\int v ds$$

is a minimum,  $v$  being the velocity of the particle and  $ds$  the corresponding distance element of the curve."

Euler restricted himself to plane curves and single particles. He did make a metaphysical argument to suggest that the principle holds for multiple particles.

We now have a reasonably precise statement. But if we look carefully, we see that the methods employed by Euler merely show that the integral is stationary, not a minimum. One other observation: Euler implicitly assumes that only paths of equal energy are to be compared, that is, the energy of the system is fixed. But he never explicitly says this.

- Lagrange (1736-1813) seems to have been the first to formulate correctly a precise statement of the principle of least action. He considers a system of mutually interacting particles, with inter-particle forces based on a potential. He defines the action of each particle as  $m \int v ds$ .

He then asserts that "the system moves from one configuration to another in such a way as to make the total action (i.e., the sum of the actions of the individual particles) stationary as compared with adjacent virtual motions between the same two configurations and having the same energy as the actual motion."

In other words,

$$\delta_{E \text{ const}} \left( \sum_i \int_A^B m_i v_i ds_i \right) = 0$$

Here  $A$  is the initial configuration of the system, and  $B$  is the final configuration. Also,  $m_i$ ,  $v_i$ , and  $ds_i$  are the mass, velocity, and curve element of the  $i^{\text{th}}$  particle.

- Hamilton (1805-1865) generalized this idea further.

Consider again the Lagrangian  $L = T - V$  of a conservative system. Hamilton's principle asserts that a system moves so as to render the variation in the integral

$$\int_{t_i}^{t_f} L dt$$

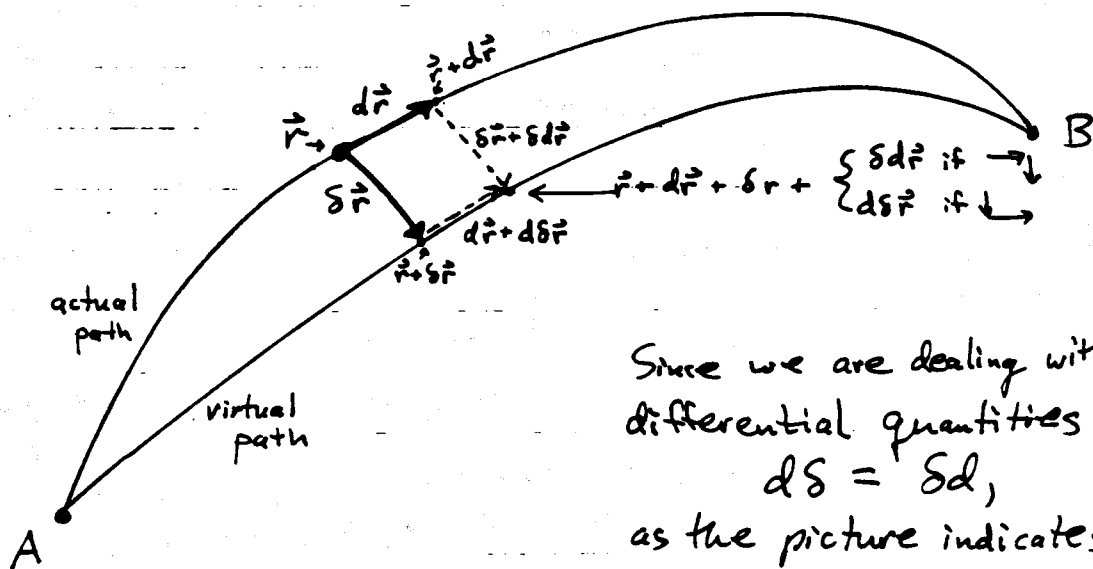
zero in comparison with neighboring virtual motions that are co-terminus in space and time.

Note that there is no explicit restriction that all paths have the same energy.

The importance of this principle is partly that it applies, with appropriate definitions of  $L$ , even in the relativistic and quantum cases.

Let us prove Lagrange's principle of least action from first principles (including Newton's laws). (The proof is taken from Y&M and is purportedly similar to Lagrange's own proof.)

Notation: Consider the motion of a system of particles between two configurations  $A \neq B$ . Call this motion a path. Now construct any adjacent virtual path leading from  $A$  to  $B$ . Each point on the real path should correspond to a point on the neighboring path. The symbol " $d$ " represents the differential of some quantity along one path, while the symbol " $\delta$ " stands for the variation of a quantity between any point on one path and the corresponding point on the adjacent path. So:





So, consider

$$\delta_{E_{const}} \left( \sum_i \int_A^B m_i v_i ds_i \right) \quad \left( \text{Call this integral } J, \text{ for later reference} \right)$$

We wish to show this is zero.

$$= \delta_{E_{const}} \left( \sum_i m_i \int_A^B [x_i dx_i + y_i dy_i + z_i dz_i] \right)$$

↑ this is a line integral along the path.  
Note that the path varies with  $\delta$

By the way, there are several ways to see the last equality. One is to note that  $v ds$  is the dot product of the velocity vector with the curve element. Another is to write out the expressions for  $v$  &  $ds$  in terms of square roots.

$$= \sum_i m_i \sum_{u=x,y,z} \delta_{E_{const}} \left( \int_A^B u_i du_i \right)$$

This is true since the  $m_i$  are constant, [If they can vary, then they must remain inside the  $\delta$ .]

$$= \sum_i m_i \sum_{u=x,y,z} \int_A^B \delta_{E_{\text{cont}}}(\dot{u}_i du_i)$$

The interchange of  $\delta$  &  $\int$  in that last step is OK if we assume that we are dealing with one path and a neighboring virtual path between which we have established a correspondence, so that we can write integrals for the two paths in terms of an independent parameterization that is fixed. (So really  $du_i = \frac{du_i}{dw} dw$  where  $w$  is the parameterization.)

$$= \sum_i m_i \sum_{u=x,y,z} \int_A^B \left[ \delta(\dot{u}_i) du_i + \dot{u}_i \delta(du_i) \right] \quad (\text{chain rule})$$

$$= \sum_i m_i \sum_{u=x,y,z} \int_A^B \left[ \delta \dot{u}_i du_i + \dot{u}_i d(\delta u_i) \right] \quad (\text{commutativity of } \delta \text{ \& } d)$$

$$= \sum_i m_i \sum_{u=x,y,z} \int_A^B \left[ d(\dot{u}_i \delta u_i) - d\dot{u}_i \delta u_i + \delta \dot{u}_i du_i \right] \quad (\text{add \& subtract } m_i d\dot{u}_i \delta u_i)$$

We haven't yet used the assumption that both paths have the same energy.  
That assumption says

$$\delta_{E_{\text{const}}} \left( \sum_i \sum_{u=x,y,z} \frac{1}{2} m_i \dot{u}_i^2 + V \right) = 0,$$

where  $V$  is the the potential energy (as a function of the particles' coordinates).

In other words (symbols really),

$$\sum_i \sum_{u=x,y,z} \left( m_i \dot{u}_i \delta u_i + \frac{\partial V}{\partial u_i} \delta u_i \right) = 0$$


---

So, let us make two substitutions in our earlier expression:

- ① We substitute  $\dot{u}_i dt$  for  $du_i$ . In other words, we think of everything parametrized by time.
- ② Using the energy argument above, we substitute

$$\sum_i \sum_{u=x,y,z} \left( -\frac{\partial V}{\partial u_i} \delta u_i \right) \text{ for } \sum_i \sum_{u=x,y,z} m_i \dot{u}_i \delta u_i$$

So  $\delta E_{\text{const}} J$

$$= \sum_i \sum_{u=x,y,z} \int_A^B \left[ m_i d(\dot{u}_i \delta u_i) - m_i \dot{u}_i \delta u_i - \frac{\partial V}{\partial u_i} \delta u_i dt \right]$$

$$= \sum_i \sum_{u=x,y,z} m_i \dot{u}_i \delta u_i \Big|_{\text{initial}}^{\text{final}}$$

$$- \int_A^B \sum_i \sum_{u=x,y,z} \left( m_i \dot{u}_i + \frac{\partial V}{\partial u_i} dt \right) \delta u_i$$

The first term vanishes since we have assumed that the paths are co-terminus.

Furthermore, Newton's laws tell us that for each particle

$$m_i \ddot{u}_i = - \frac{\partial V}{\partial u_i}$$

So we see that the second term vanishes as well.

In short, we have proved that  $\delta E_{\text{const}} J = 0$ .

This establishes Lagrange's version of the principle of least action.

## Comments

- The proof of Lagrange's principle shows us how to generalize the principle to pairs of paths that do not necessarily have the same energy and/or that do not necessarily start and end at the same configuration.

We see that in general

$$\delta J = \sum_i \sum_{u=x,y,z} (m_i \dot{u}_{i,f} \delta u_{i,f} - m_i \dot{u}_{i,a} \delta u_{i,a}) + \int \delta E dt.$$

Here the subscript "f" denotes the final configurations of the paths and the subscript "a" denotes the initial configurations of the paths. Furthermore  $\delta E$  is the variation in energy between the paths, that is

$$\delta E = \sum_i \sum_{u=x,y,z} \left( m_i \dot{u}_i \delta u_i + \frac{\partial V}{\partial u_i} \delta u_i \right)$$

at each configuration.

---

- We can generalize Lagrange's principle to generalized coordinates,

$$\delta_{\text{const}} \left( \int_A^B \sum_{j=1}^r p_j dq_j \right) = 0$$

where  $q_1, \dots, q_r$  are the generalized coordinates, and  $p_1, \dots, p_r$  are the generalized momenta.

(We won't prove this, but it follows from a change-of-coordinates argument, as usual.)

Note: Generalized momentum is defined as

$$p_j = \frac{\partial L}{\partial \dot{q}_j}, \text{ where } L \text{ is the system Lagrangian.}$$

For a point mass we get

$$\begin{aligned}
p_j &= \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j} \left( \frac{1}{2} m v^2 \right) \\
&= \frac{\partial}{\partial \dot{q}_j} \left( \frac{1}{2} m \dot{q}_1^2 + \frac{1}{2} m \dot{q}_2^2 + \frac{1}{2} m \dot{q}_3^2 \right) \\
&= m \dot{q}_j,
\end{aligned}$$

$\uparrow$   
 $q_1 = x$   
 $q_2 = y$   
 $q_3 = z$

which agrees with the usual definition of momentum.

### Hamiltonians

It is convenient to define a function  $H$ , called the Hamiltonian, as the energy of a system, expressed as a function of generalized coordinates and generalized momenta.

So, for classical mechanics we have that

$$L = T - V \quad \text{expressed in terms of } \{q_j\} \text{ \& } \{\dot{q}_j\}$$

$$H = T + V \quad \text{expressed in terms of } \{q_j\} \text{ \& } \{p_j\}$$

where  $T$  is the kinetic energy of the system, and  $V$  is the potential energy of the system.

We observe the following:

$$2T = \sum_{j=1}^r \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j$$

(think, for example, in terms of the rectangular coordinate case)

$$L + H = 2T$$

$$L - H = -2V$$

and

$$H = \sum_{j=1}^r p_j \dot{q}_j - L$$

(This formula also holds in the relativistic sense, but there  $L = F - V$  and  $2T$  gets replaced by  $T + F$ , where now

$$F = m_0 c^2 \left( 1 - \sqrt{1 - \frac{v^2}{c^2}} \right)$$

$$T = m_0 c^2 \left( \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right)$$

We would now like to derive Hamilton's principle.

Recall that this principle is very similar to Lagrange's, the difference being that it places no restriction on the path energies. — This is desirable since it frees us from the prior assumption of energy conservation.

Without energy restrictions on the virtual paths, we see from Lagrange that

$$\delta \int \sum_{j=1}^r p_j dq_j = \int \delta H dt + \text{other terms if the paths are not co-terminus in space}$$

(The paths need not start at the same time or end at the same time.)

Now let us suppose that we are comparing paths that are co-terminus in space. Consider the variation in  $\int H dt$ :

$$\delta \int H dt = \int \delta H dt + \int H \delta t dt$$

$\swarrow$  variation of energy between the real & virtual paths       $\searrow$  variation originating from the time variation between the two paths

If the paths are co-terminus in both space and time, then

$$\int H \delta t dt = 0.$$

[Why? Well, the argument is a little handwavy, but the idea is that time runs the same everywhere, so there is no variation across paths, except possibly at the endpoints. So,

$$\int H \delta t dt = \int H d(\delta t) = H_f \delta t_f - H_a \delta t_a = 0$$

$\uparrow$  since  $\delta t = 0$  except possibly at the endpoints. ]

In other words, for paths co-terminus in space and time,

$$\int \delta H dt = \delta \int H dt.$$



So, Lagrange tells us that the motions of a system evolve in such a way that

$$\delta \int_A^B \left( \sum_{j=1}^r p_j dq_j - H dt \right) = 0,$$

where the variation is <sup>now</sup> computed over all virtual paths that are co-terminus in space and time with the real path, including those that might have different energies.

Now observe that 
$$\sum_{j=1}^r p_j dq_j = \sum_{j=1}^r \frac{\partial L}{\partial \dot{q}_j} dq_j = \sum_{j=1}^r \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j dt = 2T dt$$

So we get 
$$\delta \int_{t_a}^{t_f} (2T - H) dt = 0,$$

that is, 
$$\delta \int_{t_a}^{t_f} L dt = 0$$

This is Hamilton's Principle.

We have derived it from Newton's Equations. However, the principle can be taken as the starting point of mechanics.

We can now apply our usual calculus of variations technique to determine a set of differential equations that  $L$  must satisfy. We shall do so in two different ways:

1. Suppose we write  $L$  as  $L(q_1, \dots, q_r, \dot{q}_1, \dots, \dot{q}_r)$ .

Then we get:

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} = 0 \quad j=1, \dots, r$$

These are precisely Lagrange's equations of motion.

2. Suppose we write  $L$  as  $\sum p_j \dot{q}_j - H$  with  $H$  a function of the form  $H(q_1, \dots, q_r, p_1, \dots, p_r)$ .

In this case the state variables are  $q_1, \dots, q_r, p_1, \dots, p_r$ .

Their time derivatives are  $\dot{q}_1, \dots, \dot{q}_r, \underbrace{\dot{p}_1, \dots, \dot{p}_r}$

These don't appear in the integrand.

Hence the Euler Equations become:

$$\frac{\partial (\sum p_j \dot{q}_j - H)}{\partial u} - \frac{d}{dt} \frac{\partial (\sum p_j \dot{q}_j - H)}{\partial \dot{u}} = 0 \quad \text{where } u \text{ is a state variable.}$$

In other words:

with  $u \in \{q_1, \dots, q_r\}$ :

$$\frac{\partial H}{\partial q_j} = -\dot{p}_j$$

$j=1, \dots, r$

with  $u \in \{p_1, \dots, p_r\}$ :

$$\frac{\partial H}{\partial p_j} = \dot{q}_j$$

These are Hamilton's equations of motion.

We can now deduce conservation of energy:

Recall that  $H$  is the total energy, i.e.,  $H = T + V$ .

$$\frac{dH}{dt} = \sum_{j=1}^r \frac{\partial H}{\partial q_j} \dot{q}_j + \sum_{j=1}^r \frac{\partial H}{\partial p_j} \dot{p}_j$$

By Hamilton's equations,

$$\begin{aligned} \frac{dH}{dt} &= - \sum_{j=1}^r \dot{p}_j \dot{q}_j + \sum_{j=1}^r \dot{q}_j \dot{p}_j \\ &= 0. \end{aligned}$$

(In our formulation in these notes,  $H$  has no explicit dependence on  $t$ . If it did, then we would see an additional term:

$$\begin{aligned} \frac{dH}{dt} &= \frac{\partial H}{\partial t} + \dots \text{(as before)} \\ &= \frac{\partial H}{\partial t} + 0. \end{aligned}$$

In such a setting, energy could be entering or exiting the system, explicitly so.)

Examples (mainly taken from Strang)

1. A ball of mass m attracted by gravity.

Let  $h$  be the height of the ball, let  $g = -9.8 \text{ m/s}^2$ .  
Then  $T = \frac{1}{2} m \dot{h}^2$  and  $V = -mgh$

So  $L = T - V = \frac{1}{2} m \dot{h}^2 + mgh$

$H = T + V = \frac{1}{2m} p^2 - mgh$

( $p = \text{momentum} = m \dot{h}$ )

Lagrange's Equations of Motion

$\frac{\partial L}{\partial h} - \frac{d}{dt} \frac{\partial L}{\partial \dot{h}} = 0 \implies mg - \frac{d}{dt} (m \dot{h}) = 0$

i.e.,  $m \ddot{h} = mg$ .

This is of course just Newton.

Hamilton's Equations of Motion

$\left. \begin{aligned} \frac{\partial H}{\partial h} &= -\dot{p} \\ \frac{\partial H}{\partial p} &= \dot{h} \end{aligned} \right\} \implies \begin{aligned} -mg &= -\dot{p} \\ \frac{p}{m} &= \dot{h} \end{aligned}$

The first equation is just Newton.  
The second equation is just the definition of momentum.  
Combined, the equations give  $m \ddot{h} = mg$ , as before.

## Some Comments:

- Lagrange's method requires integration of  $n$  second-order PDEs in the  $n$  unknowns  $q_1, \dots, q_r$ .

Hamilton's method requires integration of  $2n$  first-order PDEs in the  $2n$  unknowns  $q_1, \dots, q_r, p_1, \dots, p_r$ .

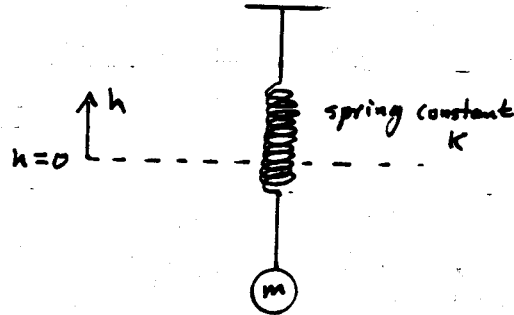
Often in solving Hamilton's equations one ends up producing Lagrange's equations. For this reason, many people regard Lagrange's equations as more convenient for calculations. However, because of their simplicity, Hamilton's equations are often more useful in theoretical inquiries.

- Technically, in deriving Hamilton's equations we cannot regard the variables  $p_j$  &  $\dot{q}_j$  as independent. Thus we really have a constrained variational problem. However, these constraints simply duplicate the second set of Hamilton's equations. So, fortuitously, we can ignore the dependency between  $p_j$  &  $\dot{q}_j$ , that is, we can regard the unconstrained variational problem.

Indeed, this suggests that general Hamiltonians in which the  $p_j$  and  $\dot{q}_j$  really are independent are equally covered by Hamilton's principle (e.g. relativity, quantum).

Examples continued.

2. Ball hanging from a spring in a uniform gravitational field



$h$  is zero when the spring is in its relaxed state.

$$g = -9.8 \text{ m/s}^2$$

Lagrange

$$T = \frac{1}{2} m \dot{h}^2 \quad V = -mgh + \frac{1}{2} k h^2$$

$$L = T - V = \frac{1}{2} m \dot{h}^2 + mgh - \frac{1}{2} k h^2$$

$$\frac{\partial L}{\partial h} - \frac{d}{dt} \frac{\partial L}{\partial \dot{h}} = 0 \Rightarrow mg - kh - \frac{d}{dt} m \dot{h} = 0$$

$$\text{i.e., } m \ddot{h} = -kh + mg$$

This is the usual Newton equation  $ma = \sum \text{Forces}$

Hamilton

$$H = T + V = \frac{1}{2m} p^2 - mgh + \frac{1}{2} k h^2$$

$$\left. \begin{aligned} \frac{\partial H}{\partial h} &= -\dot{p} \\ \frac{\partial H}{\partial p} &= \dot{h} \end{aligned} \right\} \Rightarrow \begin{aligned} -mg + kh &= -\dot{p} \\ \frac{p}{m} &= \dot{h} \end{aligned}$$

$$\text{so } m \ddot{h} = -kh + mg \text{ as before}$$

So, in both cases we get the usual result, namely

$$\underbrace{m \ddot{h} + kh}_{\text{fundamental equation of oscillation}} = mg \leftarrow \text{forcing function}$$

This is an easy equation to solve.

The homogeneous solution is

$$h_h(t) = C_1 \cos(\sqrt{\frac{k}{m}} t) + C_2 \sin(\sqrt{\frac{k}{m}} t),$$

and the particular solution is  $h_p(t) = \frac{mg}{k}$ , so the general solution is

$$h(t) = C_1 \cos(\sqrt{\frac{k}{m}} t) + C_2 \sin(\sqrt{\frac{k}{m}} t) + \frac{mg}{k}.$$

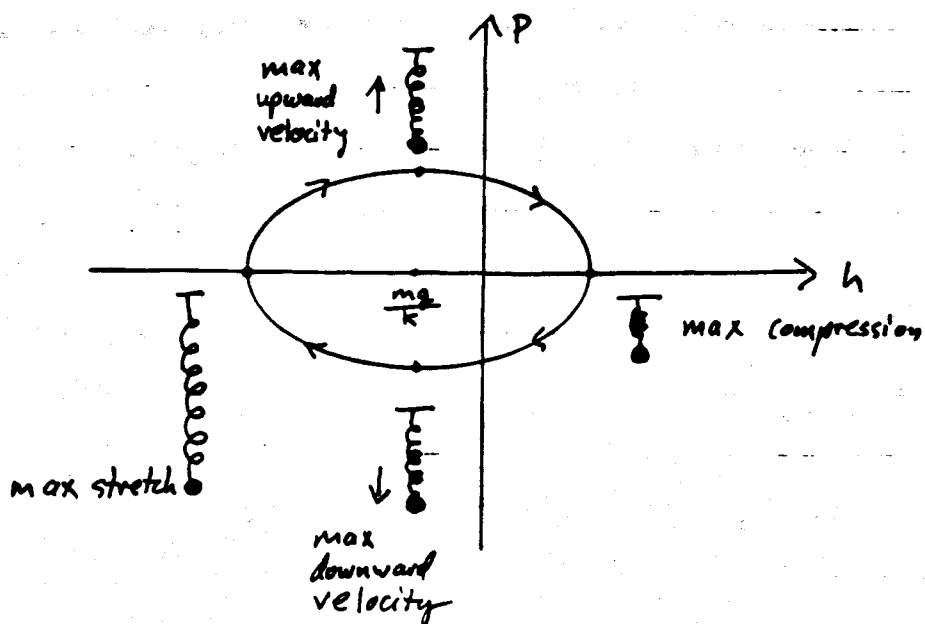
The constants  $C_1$  &  $C_2$  are determined by initial conditions.

The initial conditions also determine the total energy of the system. In the  $(h, p)$  phase space,

the motion lies on an ellipse, specified by the condition  $H(h, p) = \text{constant}$  (= total energy).

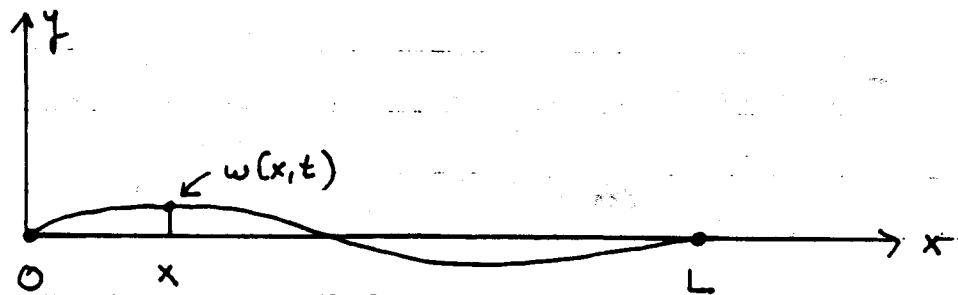
To see this observe that  $H = \frac{1}{2} m p^2 - mgh + \frac{1}{2} k h^2$ ,  
so

$$\frac{(h - \frac{mg}{k})^2}{\frac{2}{k}} + \frac{p^2}{2m} = \text{constant} \quad (= \text{initial energy} + \frac{m^2 g^2}{2k})$$



### 3. The Vibrating String

So far we have only considered finite particle problems. Let us now look at a continuous distribution of mass. The vibrating string is a classic such problem. Consider a string tied down at two ends:



The string is vibrating. Its height above a given location  $x$  at time  $t$  is given by  $w(x,t)$ . For specified initial conditions, we would like to determine the resulting motion of the string.

We assume that the string has uniform mass density  $\rho$  (mass per unit length). We assume further that the string has a constant tension  $\mu$ . As the string vibrates it stretches, thereby storing potential energy. For low amplitude vibrations this potential energy is proportional to the increase in the string's length, with proportionality constant  $\mu$ .



We therefore have:

$$T = \frac{1}{2} \int_0^L \rho w_t^2 dx \quad (\text{i.e., } \frac{1}{2}mv^2 \text{ for each differential element of mass})$$

$$V = \mu \left[ \int_0^L \sqrt{1+w_x^2} dx - L \right] \quad (\text{i.e., the change in length})$$

(This is without gravity.)

For small vibrations we can write  $V = \frac{1}{2} \int_0^L \mu w_x^2 dx$

Hamilton's principle says that we should find a function  $w(x,t)$  for which the integral

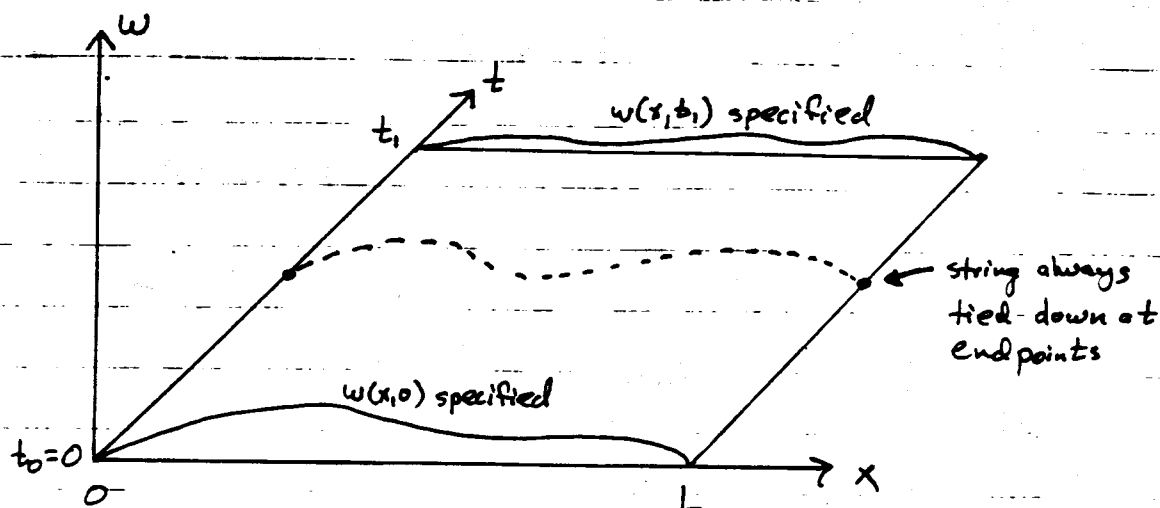
$$\int_{t_0}^{t_1} (T - V) dt \quad \text{is stationary, for any } t_0, t_1 \text{ of interest.}$$

In other words, we want  $w(x,t)$  for which the double integral

$$\int_{t_0}^{t_1} \int_0^L (\rho w_t^2 - \mu w_x^2) dx dt \quad \text{is stationary.}$$

We consider those  $w(x,t)$  which describe the state of the system at  $t=t_0$  and  $t=t_1$ , and which vanish for all  $t$  at  $x=0$  and at  $x=L$ .

A picture:



Note:

More generally, rather than specifying  $w(x, t_1)$ , one might specify  $w_t(x, 0)$  instead.

This makes more sense, since one usually doesn't know  $w(x, t_1)$ , but does know the initial configuration and velocity. — We thus have a mixed boundary value and initial conditions problem.

Observe that this is as well an example of a multi-dimensional variation. Recall, for a function  $F(x, y, u, u_x, u_y)$  in the integrand, the differential equation

$$F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} = 0$$

must hold.

In our case we have  $F(x, t, w, w_x, w_t) = \rho w_t^2 - \mu w_x^2$ .

So,  $F_w = 0$ ,  $F_{w_x} = -2\mu w_x$ , and  $F_{w_t} = 2\rho w_t$ .

So, we get the classic wave equation:

$$\boxed{w_{xx} = c^2 w_{tt}}$$

with  $c = \sqrt{\frac{\rho}{\mu}} > 0$ .

Since it is a famous equation, let us quickly review the relevant PDE techniques for solving this equation.

First, we suppose that  $w$  is separable, that is,

$$w(x,t) = \phi(x)q(t)$$

The PDE then becomes

$$\phi''(x)q(t) = c^2 \phi(x)\ddot{q}(t)$$

In other words

$$\frac{\phi''(x)}{\phi(x)} = c^2 \frac{\ddot{q}(t)}{q(t)}$$

↑
↑  
 depends only on  $x$ 
depends only on  $t$

So, the LHS and the RHS must be equal to some constant, call it  $-\lambda$  ( $-\lambda$  is as yet undetermined).

We thus have two ODEs:

$$\phi''(x) + \lambda \phi(x) = 0 \quad \text{with boundary conditions } \phi(0) = \phi(L) = 0$$

and

$$c^2 \ddot{q}(t) + \lambda q(t) = 0 \quad \text{with initial condition dependent on } x.$$

The general solution to the first ODE is given by

$$\phi(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$$

The boundary condition  $\phi(0)=0$  says that  $c_1 + c_2 = 0$ , so

$$\phi(x) = c_1 (e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x})$$

The boundary condition  $\phi(L)=0$  implies either that  $c_1=0$  or that  $e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L} = 0$ . We will ignore the case  $c_1=0$  since that implies that  $\phi(x)$  and hence  $w(x,t)$  is identically zero.

Let's examine  $\lambda$  more closely.

Suppose  $\lambda \leq 0$

Then  $e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L} = 0$  is of the form  $e^a - e^{-a} = 0$  for a real and  $a \geq 0$ . Observe that  $e^a - e^{-a} > 0$  if  $a > 0$ , so this expression is zero iff  $\lambda$  is zero. But then  $\phi \equiv 0$  again. We conclude that  $\lambda$  cannot be zero or negative.

Suppose  $\lambda > 0$

Then  $\sqrt{-\lambda}$  is imaginary, so our boundary condition becomes

$$\{ \cos(\sqrt{\lambda}L) + i \sin(\sqrt{\lambda}L) \} - \{ \cos(-\sqrt{\lambda}L) + i \sin(-\sqrt{\lambda}L) \} = 0$$

In other words,  $2i \sin(\sqrt{\lambda}L) = 0$

This is only possible if  $\lambda$  is of the form  $\frac{k^2 \pi^2}{L^2}$ ,  $k=1, 2, \dots$

We conclude that the boundary value problem

$$\phi''(x) + \lambda \phi(x) = 0, \quad \phi(0) = 0 = \phi(L)$$

has a solution iff  $\lambda$  is of the form  $\frac{k^2 \pi^2}{L^2}$ ,  $k=1, 2, \dots$

(Aside: If you think about this for a second, this should seem very puzzling. And yet this abstract mathematical constraint says something about the world — the  $\lambda$  can only appear in quantized form

For  $\lambda = \lambda_k = \frac{k^2 \pi^2}{L^2}$  the solution is given by

$$\phi_k(x) = \sin\left(\frac{k\pi}{L}x\right) \text{ times any scalar.}$$

We thus have an eigenvalue problem.

The numbers  $\lambda_1, \lambda_2, \dots$  are called the eigenvalues of this problem, and the functions  $\phi_k$  are called the eigenfunctions. (The connection to linear algebra is in the form of linear operator theory.)

Looking ahead

The lowest eigenvalue provides the fundamental frequency  $\frac{1}{2L} \frac{\sqrt{\lambda}}{\pi}$  ( $= \frac{1}{2L}$ ) of the string, while the other eigenvalues provide the harmonics ( $\frac{1}{2L} \frac{k\pi}{L}$   $k \geq 2$ ).

Turning to the time domain, the general solution of the ODE

$$c^2 \ddot{q}(t) + \lambda_k q(t) = 0 \quad \left( \omega / \lambda_k = \frac{k^2 \pi^2}{L^2} \right)$$

is  $q_k(t) = a_k \cos\left(\frac{k\pi}{cL} t\right) + b_k \sin\left(\frac{k\pi}{cL} t\right)$

with  $a_k$  &  $b_k$  being arbitrary constants.

As a result we can write  $w(x, t)$  in the form

$$w(x, t) = \sum_{k=1}^{\infty} \phi_k(x) q_k(t)$$

[Why? Because each  $\phi_k(x) q_k(t)$  is a solution to the original linear homogenous PDE.]

So  $w(x, t) = \sum_{k=1}^{\infty} \phi_k(x) \left[ a_k \cos\left(\frac{k\pi}{cL} t\right) + b_k \sin\left(\frac{k\pi}{cL} t\right) \right]$

with the constants  $\{a_k\}$  &  $\{b_k\}$  yet to be determined.

Note the Fourier Series set up.

Indeed, solving boundary value problems in several dimensions is precisely one of the motivations/applications that spurred the development of Fourier techniques.

(For other types of differential equations one obtains other basis functions.)

For instance,

suppose  $w(x, 0)$  is specified by the function  $w_0(x)$ , and  
suppose  $w_x(x, 0)$  is specified by the function  $w_1(x)$ ,

$$\text{Then } w_0(x) = w(x, 0) = \sum_{k=1}^{\infty} a_k \phi_k(x)$$

$$\text{and } w_1(x) = w_x(x, 0) = \sum_{k=1}^{\infty} \frac{k\pi}{cL} b_k \phi_k(x)$$

So, the trick is to compute  $\{a_k\}$  &  $\{b_k\}$  by computing the Fourier coefficients of  $w_0(x)$  and  $w_1(x)$  relative to the basis functions  $\phi_k(x)$ .

In particular,

$$a_k = \frac{2}{L} \int_0^L w_0(x) \sin\left(\frac{k\pi}{L}x\right) dx$$

$$\text{and } \frac{k\pi}{cL} b_k = \frac{2}{L} \int_0^L w_1(x) \sin\left(\frac{k\pi}{L}x\right) dx$$

Note: It is easy (though algebraically tedious) to see that

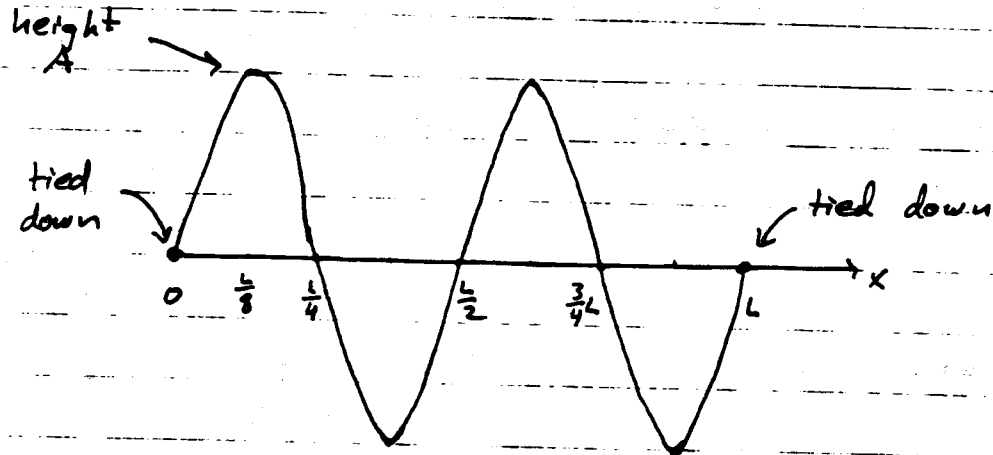
$$(i) \int_0^L \sin^2\left(\frac{k\pi}{L}x\right) dx = \frac{L}{2}$$

$$(ii) \int_0^L \left(\sin\frac{k\pi}{L}x\right) \left(\sin\frac{j\pi}{L}x\right) dx = 0 \quad \text{if } k \neq j$$

Hence the  $\{\phi_k\}$  are indeed an orthogonal set.

## Ex of specific initial conditions

Suppose we hold the string at time  $t=0$  so that it is perfectly shaped in a sine curve with spatial frequency some multiple of the fundamental frequency determined by  $L$ . For example,



(that's supposed to be a sine curve)

In this picture  $w(x,0) = w_0(x) = A \sin\left(\frac{4\pi}{L}x\right)$

And since we are holding it, then simply let go, we have

$$w_t(x,0) = w_1(x) \equiv 0$$

This says there is no initial velocity in the string. The string moves as a result of internal forces, that is, potential energy stored in the deformation. Of course, the height  $A$  is enormously exaggerated in the figure.



(7)

So we see that all  $b_k = 0$ .

$$\text{And } a_k = \frac{2A}{L} \int_0^L \sin\left(\frac{4\pi}{L}x\right) \sin\left(\frac{k\pi}{L}x\right) dx$$

$$= \begin{cases} 0 & \text{if } k \neq 4 \\ A & \text{if } k = 4 \end{cases}$$

Thus  $w(x,t) = \phi_4(x) A \cos\left(\frac{4\pi}{cL}t\right)$

i.e.,  $w(x,t) = A \sin\left(\frac{4\pi}{L}x\right) \cos\left(\frac{4\pi}{cL}t\right)$

So, in the time domain the wave has period  $\frac{cL}{2}$ .

Every odd  $\frac{cL}{4}$  time units the string is in a mirror image from its initial configuration.

At times  $\frac{cL}{8}, \frac{3cL}{8}, \frac{5cL}{8}, \frac{7cL}{8}, \dots$

the string is completely vertical, horizontal.

$w(x,t)$  is a standing wave with node points at

$$x = 0, \frac{L}{4}, \frac{L}{2}, \frac{3L}{4}, L.$$

Let's check that energy is conserved

$$w_t(x, t) = -A \frac{4\pi}{cL} \sin\left(\frac{4\pi}{L}x\right) \sin\left(\frac{4\pi}{cL}t\right)$$

$$w_x(x, t) = A \frac{4\pi}{cL} \cos\left(\frac{4\pi}{L}x\right) \cos\left(\frac{4\pi}{cL}t\right)$$

Assuming  $A$  is small:

$$V(t) = \frac{1}{2} \int_0^L \mu w_x^2 dx = \mu A^2 \frac{4\pi^2}{L} \cos^2\left(\frac{4\pi}{cL}t\right)$$

$$T(t) = \frac{1}{2} \int_0^L \rho w_t^2 dx = \rho A^2 \frac{4\pi^2}{c^2 L} \sin^2\left(\frac{4\pi}{cL}t\right)$$

so  $E(t) = T(t) + V(t)$

$$= A^2 \frac{4\pi^2}{L} \left[ \mu \cos^2\left(\frac{4\pi}{cL}t\right) + \frac{\rho}{c^2} \sin^2\left(\frac{4\pi}{cL}t\right) \right]$$

note that  $c^2 = \frac{\rho}{\mu}$

$$= A^2 \frac{4\pi^2}{L} \mu$$

which is indeed constant.

(Hmmm. Why does only  $\mu$  appear?)

Some omitted topics

- Second-Order Sufficiency Conditions

- Hamilton-Jacobi Equation

Schrödinger Wave Equation