# Approximate analysis of hybrid dynamics

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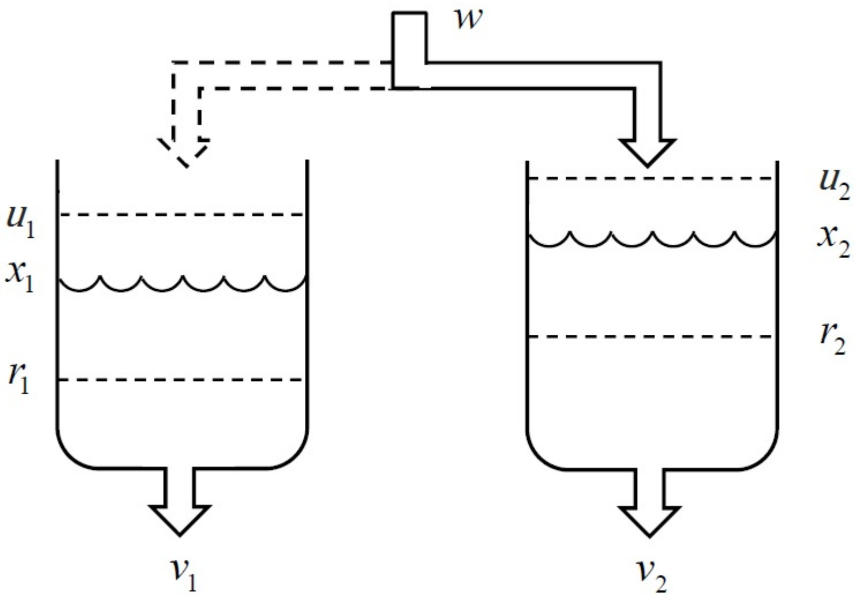
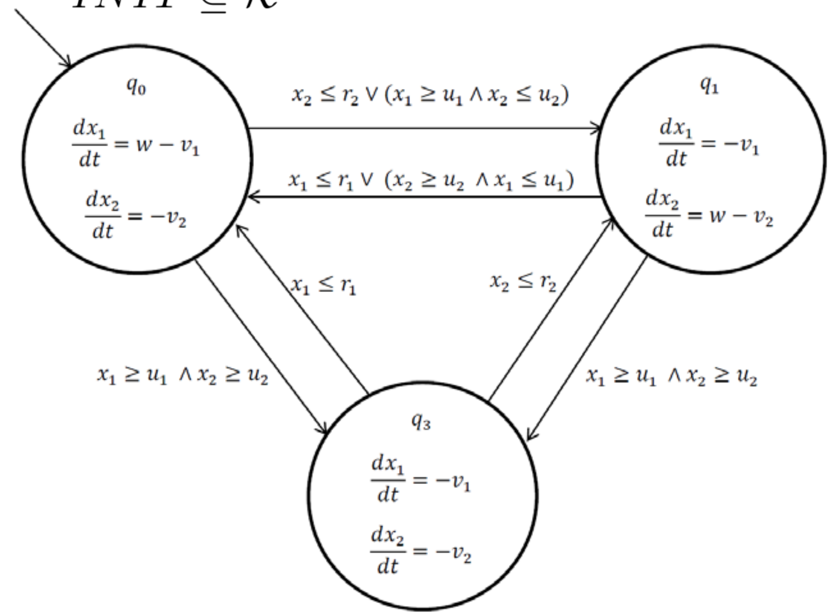
# Hybrid systems

$$X = \{x_1, x_2\}$$

$$INIT \subseteq \mathcal{R}^2$$




Hybrid system




Hybrid automaton

# Hybrid systems

- Hybrid systems are hard to analyze.
  - The solutions to the ODEs will not be available.
  - The way the continuous dynamics interacts with the guards is intractable.
- Restrict the mode dynamics (piecewise constant,..)?
- But in many situations, the mode dynamics are governed by ordinary differential equations.
  - Cyber-physical systems (automotive, avionics..), bio-pathways,..
- **A possible strategy:**
  - **Approximate analysis based on numerical simulations.**

# Main result

- For a ***rich*** class of hybrid systems:
  - we approximate a **hybrid system** H as a **Markov chain** M such that:
  - **Every trajectory of H satisfies a BLTL specification φ iff M meets the specification φ almost certainly. i.e with probability 1.**
  - **Kr: $Q \rightarrow 2^{AP}$**

# Main result

Instead of $H \models \varphi$ determine
whether $M \models \varphi$

We can not construct $M$ explicitly.

We can only check $M \models \varphi$ approximately by
sampling the paths of $M$.

We can not directly sample the paths of $M$.

Instead we sample the paths of $M$ indirectly.
By sampling the trajectories of $H$ through numerical simulations.

Set up a sequential hypothesis test and statistically model check
whether $M \models \varphi$.

# The SMC procedure

$H_0 : M \models_{\geq 1} \varphi$

$H_1 : M \models_{< 1-\delta} \varphi$

$\delta$ is the (one sided) indifference region.

Also fix $\alpha$, the false positives error.
i.e. when $H_0$ is accepted while $H_1$ happens to be the case.

We will never accept $H_1$ when $H_0$ happens to be the case.

$$N = \lceil \frac{log\, \alpha}{log\, 1-\delta} \rceil$$

$\delta = 0.01,\ \alpha = 0.01;\ N = 459$

A few case studies including the cardiac cell model
(non-linear ODEs, 4 modes, 4 dimensional) perform well.

# The class of systems

- The vector fields associated with each mode dynamics is a $C^1$ function. (polynomials, rational functions,)

- The state of the system is observed at discrete time-points.

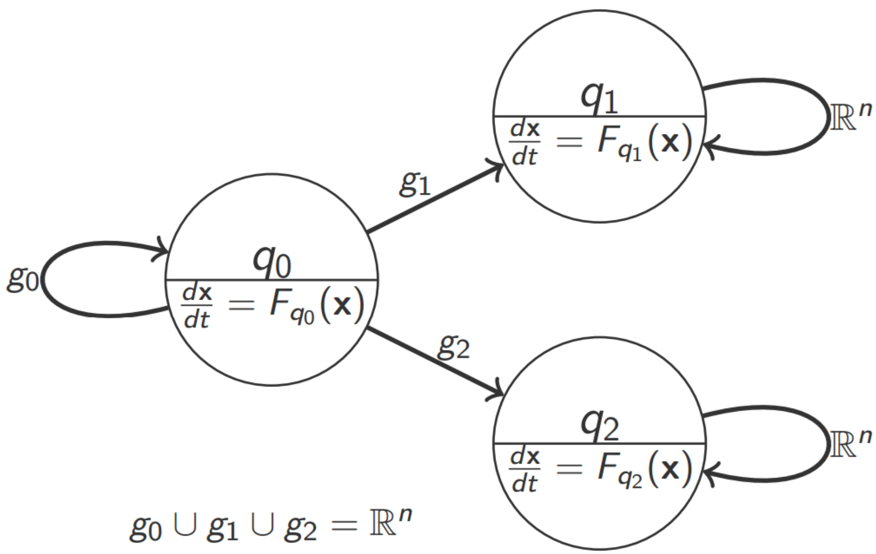
- The number of mode changes within a unit interval of time $(k, k+1)$ is bounded [non-zeno].
  - The interval is so chosen that at most one mode transition takes place per unit interval.

- The set of initial states is a bounded open set; all the guards are open sets.

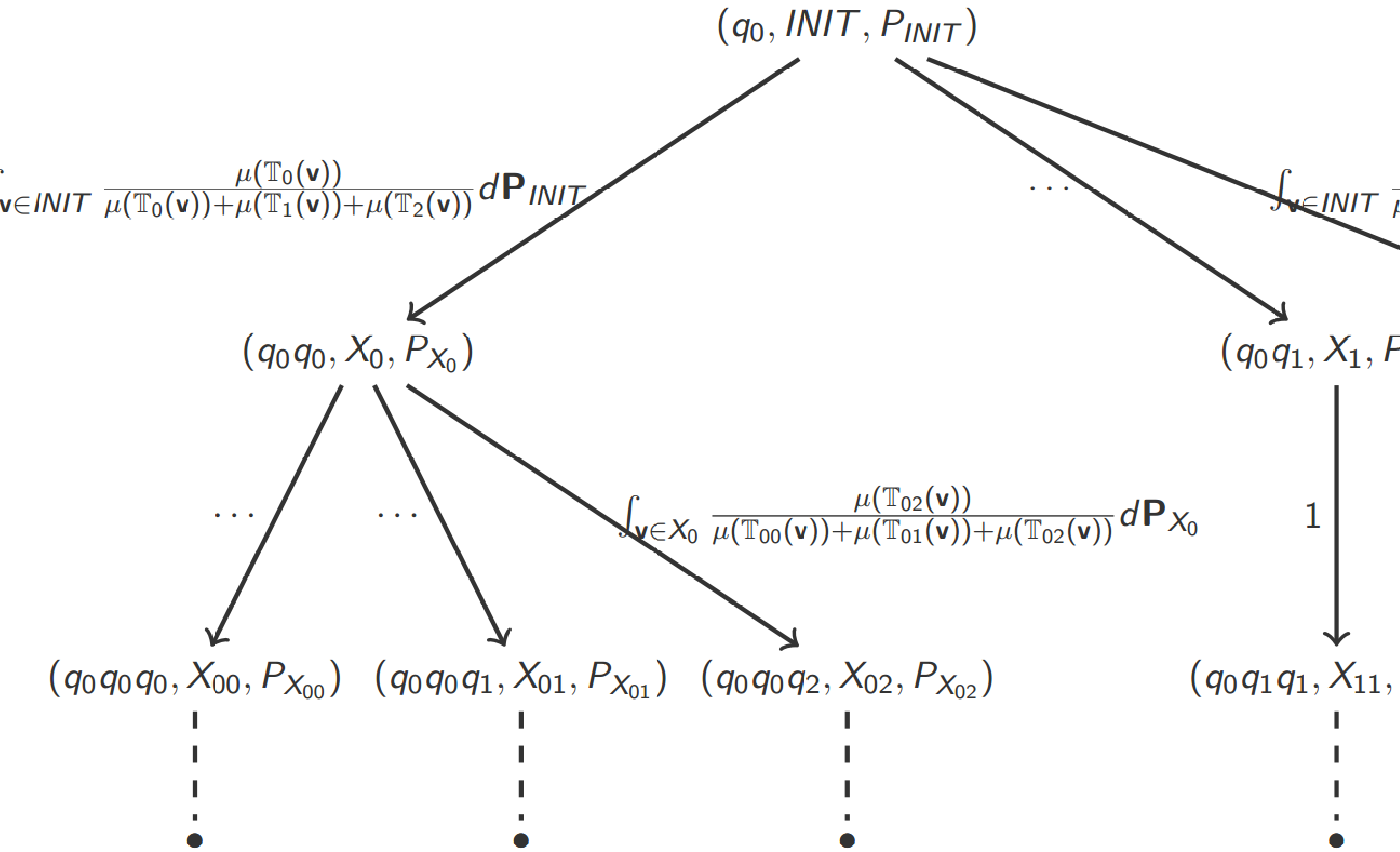
# The stochasticity hypothesis.

- Mode transitions are ***stochastic events***.
- The probability of a mode transition is proportional to how “often” its guard gets enabled.
  - Our assumptions ensure that this make measure theoretic sense.

# Approximating hybrid automata using Markov chain




A simple hybrid automaton



$(q_0, INIT, P_{INIT})$
$\int_{\mathbf{v}\in INIT} \frac{\mu(\mathbb{T}_0(\mathbf{v}))}{\mu(\mathbb{T}_0(\mathbf{v}))+\mu(\mathbb{T}_1(\mathbf{v}))+\mu(\mathbb{T}_2(\mathbf{v}))} d\mathbf{P}_{INIT}$
$\cdots$
$\int_{\mathbf{v}\in INIT}$
$(q_0 q_0, X_0, P_{X_0})$
$(q_0 q_1, X_1,$
$\cdots$
$\cdots$
$\int_{\mathbf{v}\in X_0} \frac{\mu(\mathbb{T}_{02}(\mathbf{v}))}{\mu(\mathbb{T}_{00}(\mathbf{v}))+\mu(\mathbb{T}_{01}(\mathbf{v}))+\mu(\mathbb{T}_{02}(\mathbf{v}))} d\mathbf{P}_{X_0}$
1
$(q_0 q_0 q_0, X_{00}, P_{X_{00}})$
$(q_0 q_0 q_1, X_{01}, P_{X_{01}})$
$(q_0 q_0 q_2, X_{02}, P_{X_{02}})$
$(q_0 q_1 q_1, X_{11},$

$k$
$\mathbb{T}_j(\mathbf{v})$
$k+1$

# An extension

- Currently the atomic propositions are qualitative; Kr: $Q \to 2^{AP}$
- Quantitative atomic propositions are harder to handle.
  - $(x < 0.5)$
- Must stick to ``smooth'' trajectories and rational mode switching time points.