

# Inference in Bayesian Networks

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10701-Recitation

# Convenient Probabilistic Model

- Compact Representation
  - Resistant to Over-fitting
  - Captures underlying structure
- Creates a “language” for describing probability relationships
- Permits the construction of generic inference techniques
- Variable Elimination, Variational Methods, Belief Propagation

# Joint Probability

- What can you compute with  $P(X_1, X_2, \dots, X_n)$ ?
  - Joint probability
  - Marginal Probability
  - Conditional distributions:  $P(X_1 | X_2, X_3)$  (Predictions)
  - Most likely explanations:  $\max P(X_1, X_4 | X_2, X_3)$
  - Samples from the distribution
  - Information gain ...

# What is a Bayesian Network?

$$P(A) = f_1(A)$$

$$P(B) = f_2(B)$$

$$P(C|A) = f_3(C,A)$$

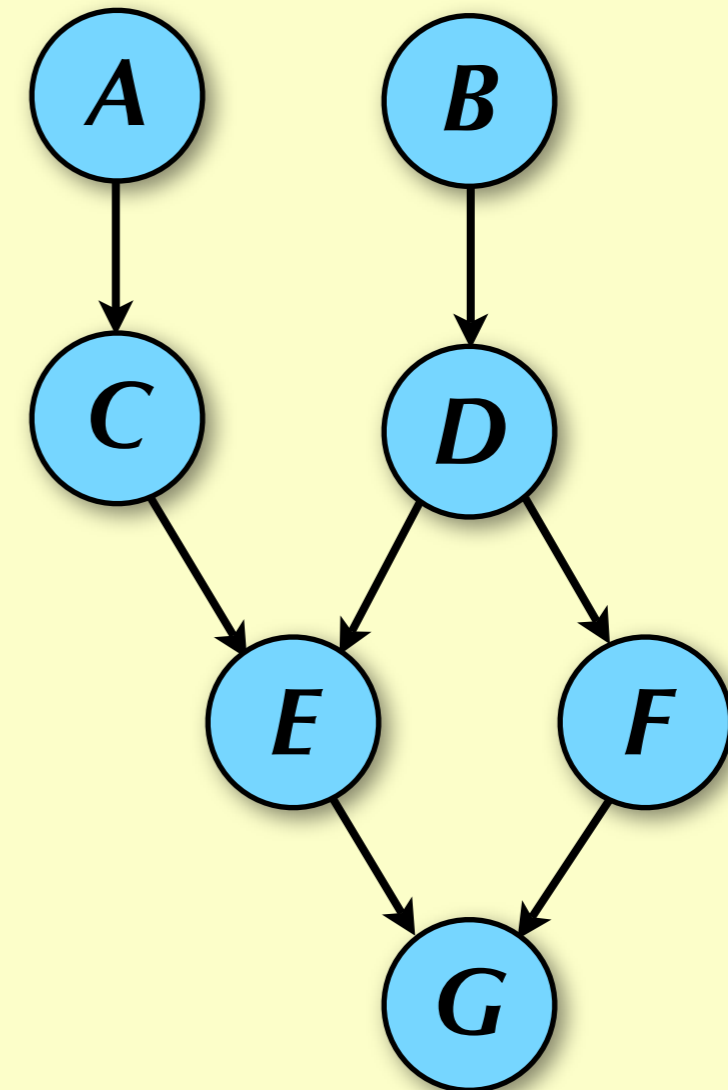
$$P(D|B) = f_4(D,B)$$

$$P(E|C,D) = f_5(E,C,D)$$

$$P(F|D) = f_6(F,D)$$

$$P(G|E,F) = f_7(G,E,F)$$

Conditional Probability  
Distributions / Tables /  
Functions



Directed Acyclic  
Graph

# Conditional Probability ... CPDs / CPTs

- Conditional Probability
  - CPD: Distributions
  - CPT: Tables
- Can be:
  - Discrete
  - Continuous

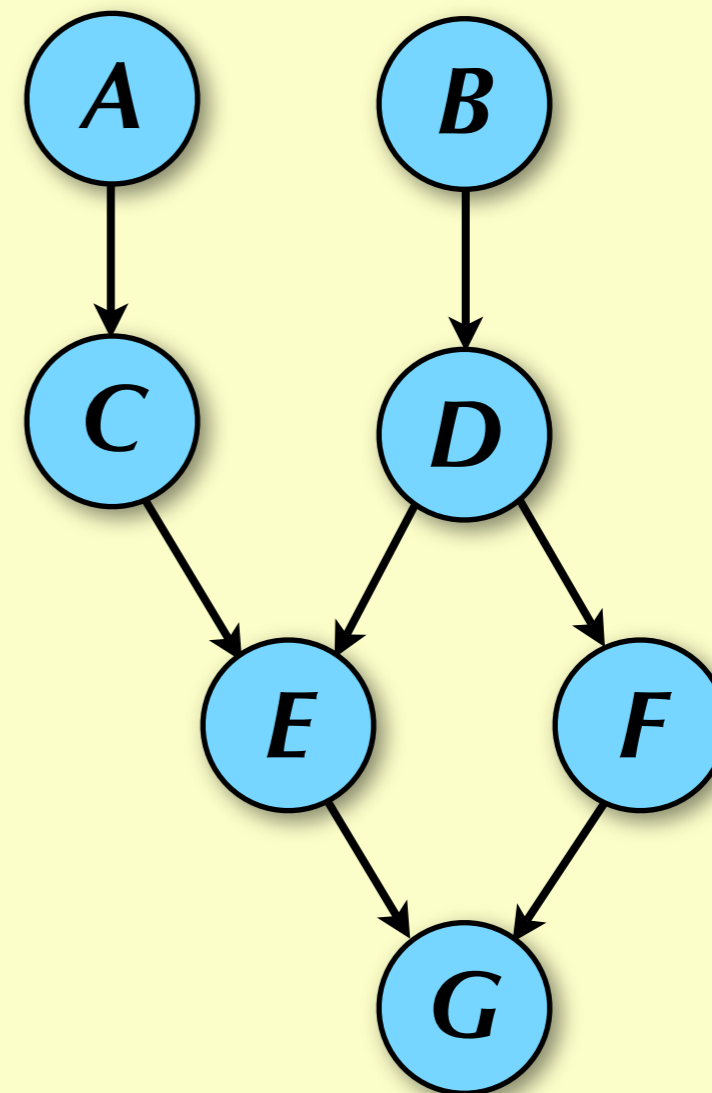
$P(C A)$	$A=T$	$A=F$
$C=T$	0.3	0.6
$C=F$	0.7	0.4

OR

$$P(C|A) = \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{(C-A)^2}{2\sigma^2}\right)}$$

# Graphs Describe Probability Distributions

$$P(A,B,C,D,E,F,G) = P(A) P(B) P(C|A) P(D|B) P(E|C,D) P(F|D) P(G|E,F)$$



# Writing the Joint Probability

## Bayesian Network

*For a directed acyclic graph  $G=(V,E)$*

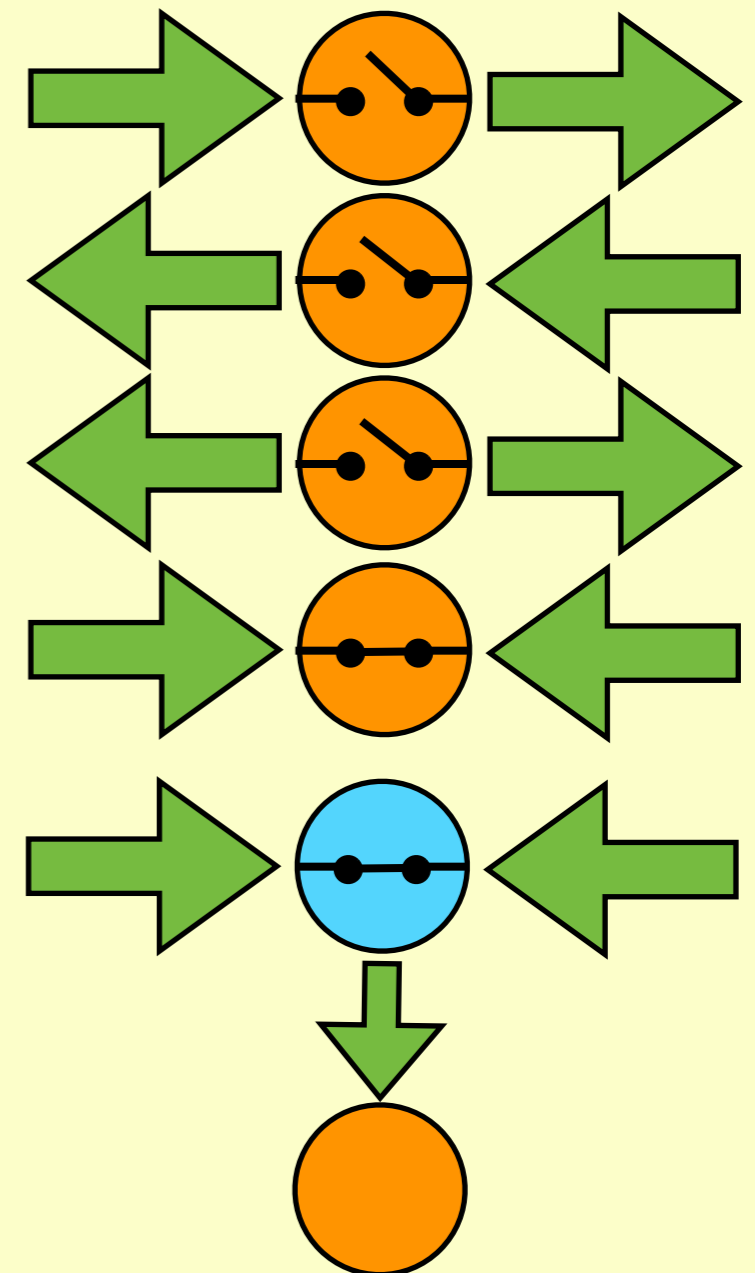
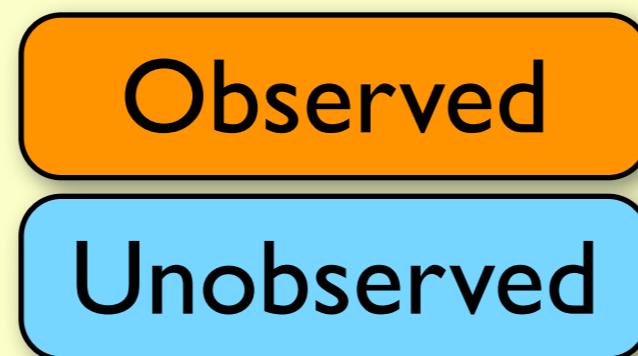
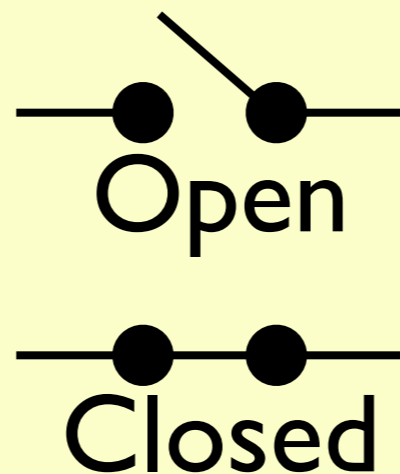
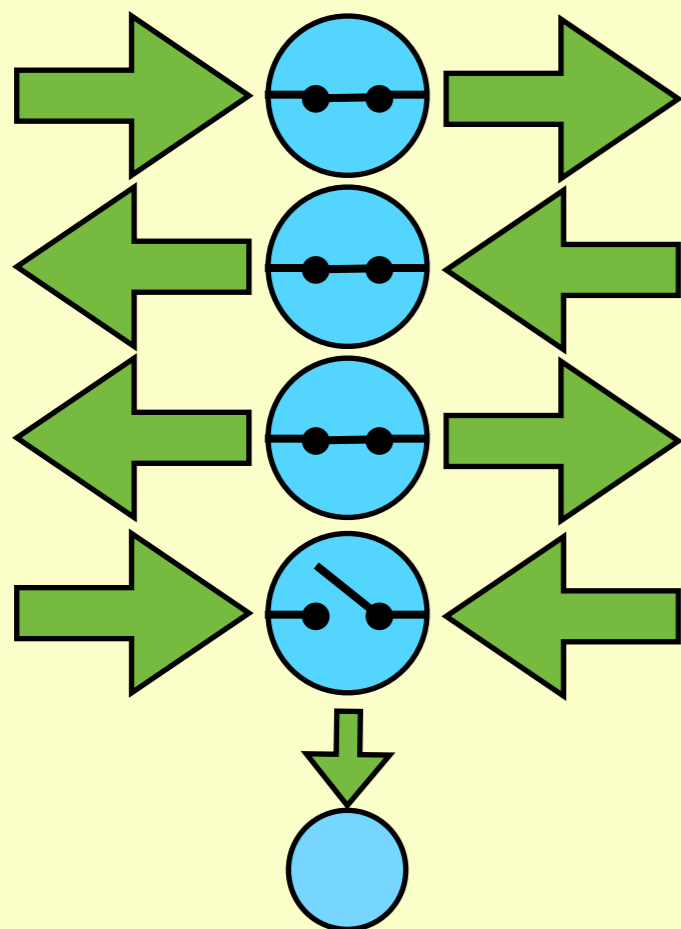
$$P(X_1, X_2, \dots, X_n) = \prod_{i=1}^n P(X_i | Pa_G[X_i])$$

*Where  $Pa_G[X_i]$  are the parents of  $X_i$*

- We say that  $P(X_1, \dots, X_n)$  factors with respect to the graph  $G$

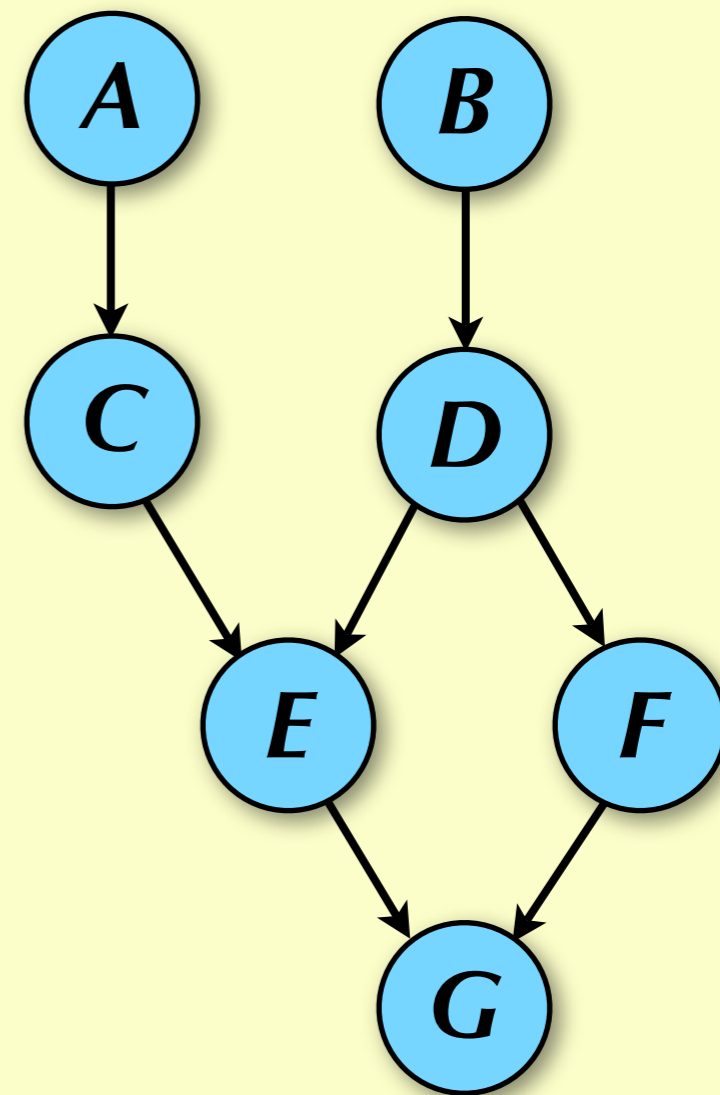
# Conditional Independence

If influence can flow from node A to node B (given observations) then nodes A and B are **not** (conditionally) independent.



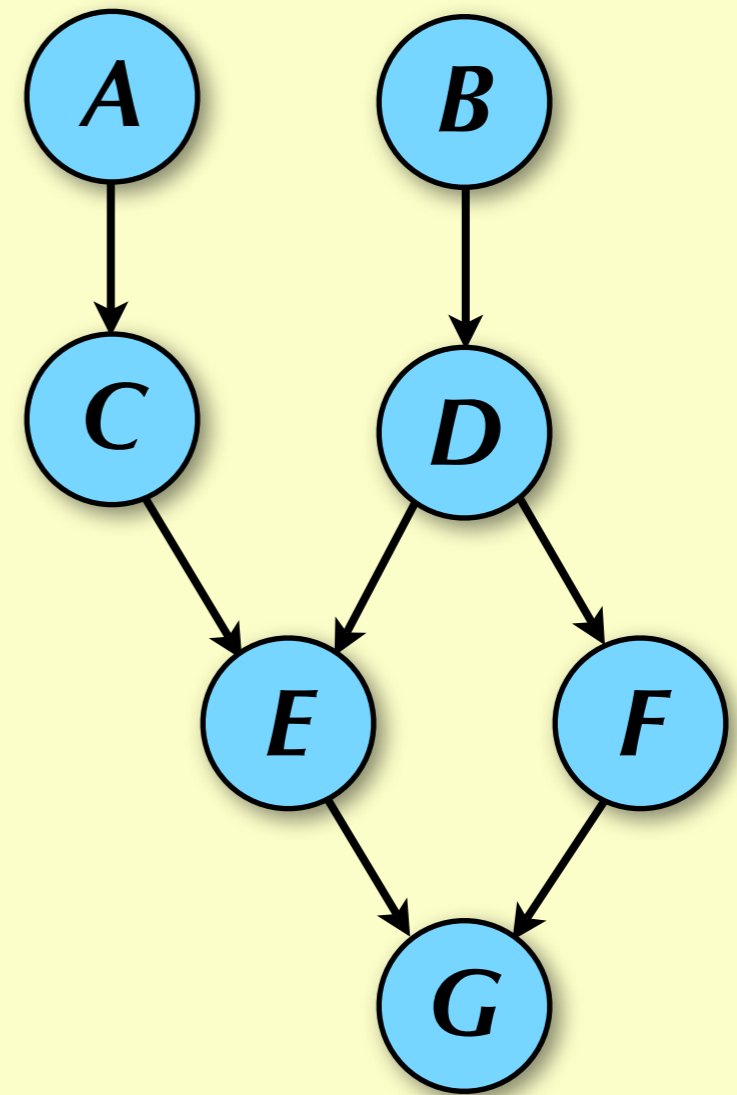


# Independence



# Independence

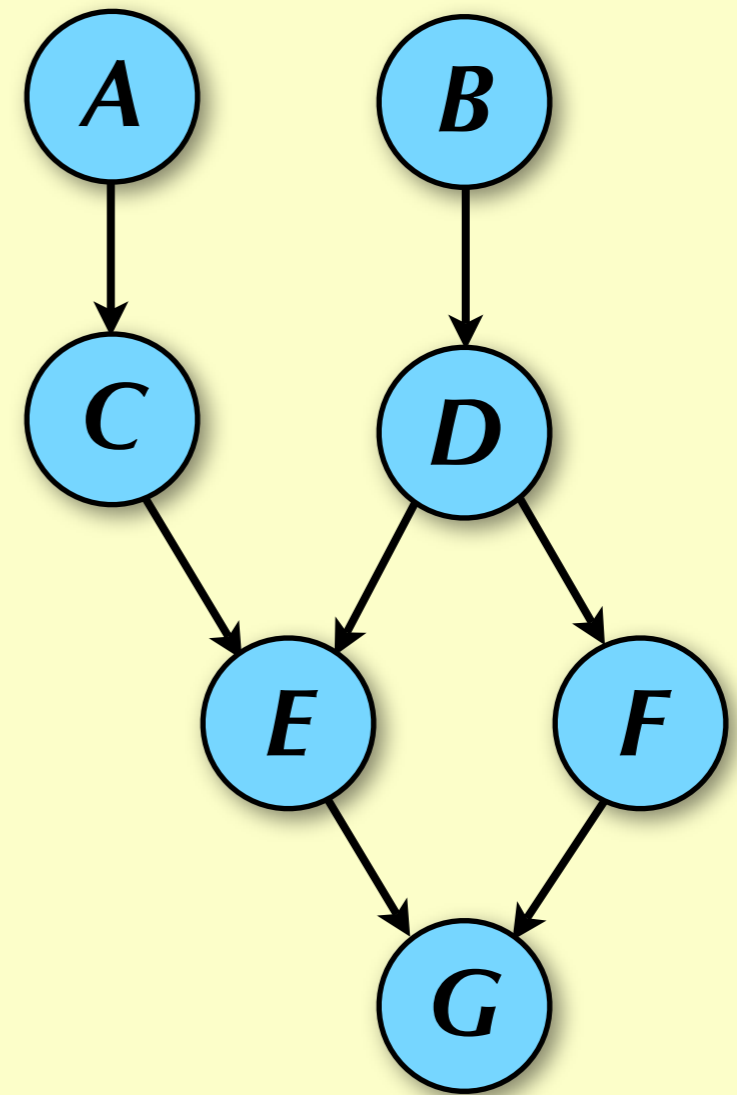
- A Indep B



# Independence

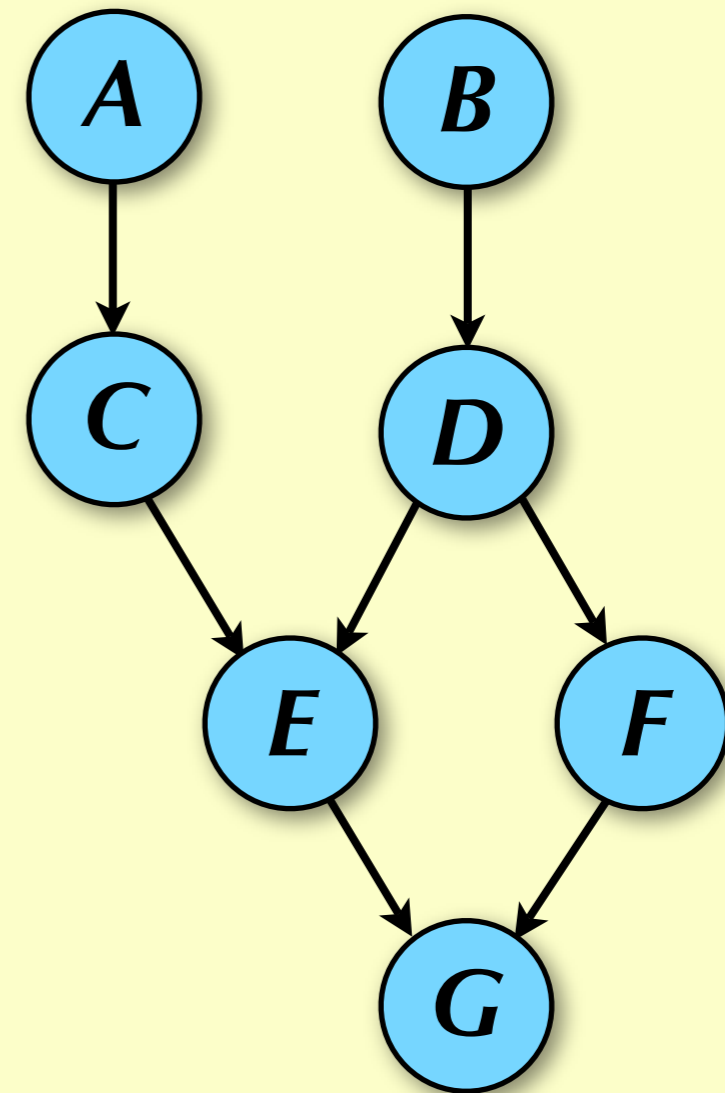
● A Indep B

● Yes



# Independence

- A Indep B
- Yes
- A Indep G



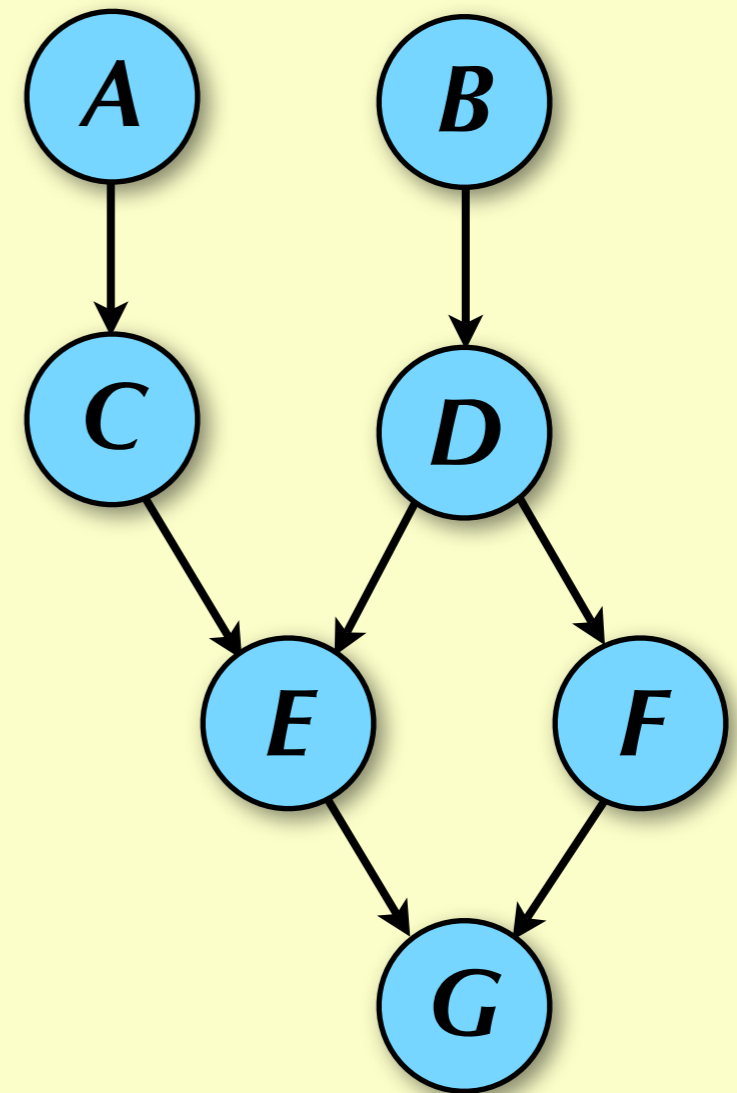
# Independence

● A Indep B

● Yes

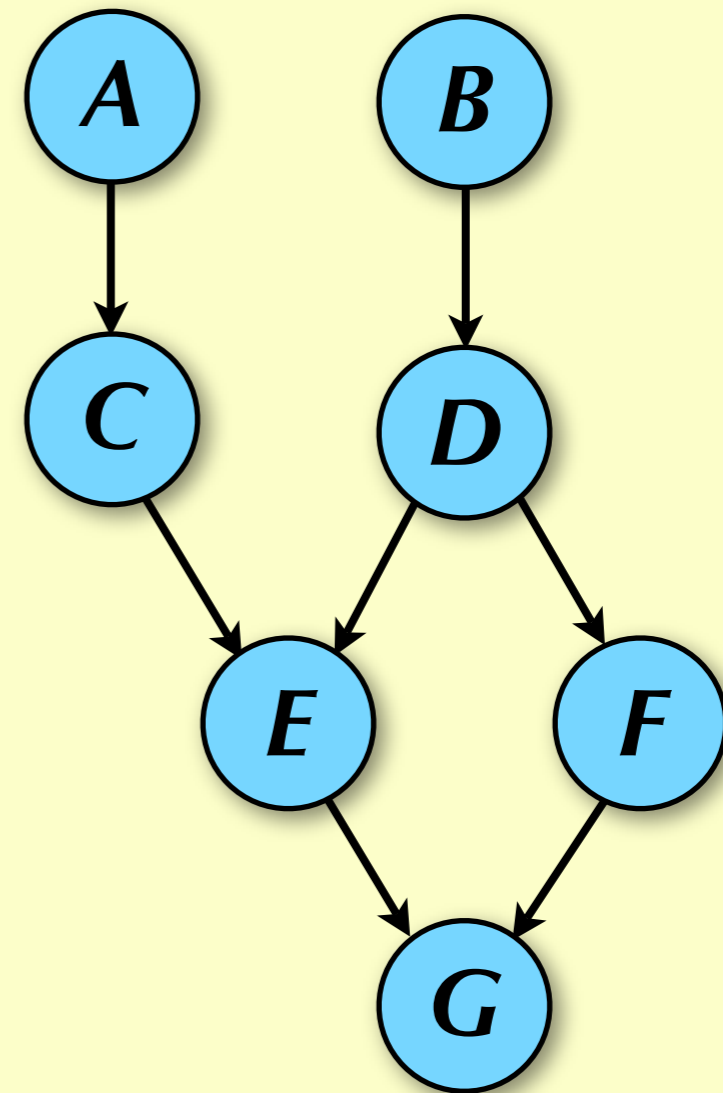
● A Indep G

● No



# Independence

- A Indep B
  - Yes
- A Indep G
  - No
- A Indep B given G



# Independence

● A Indep B

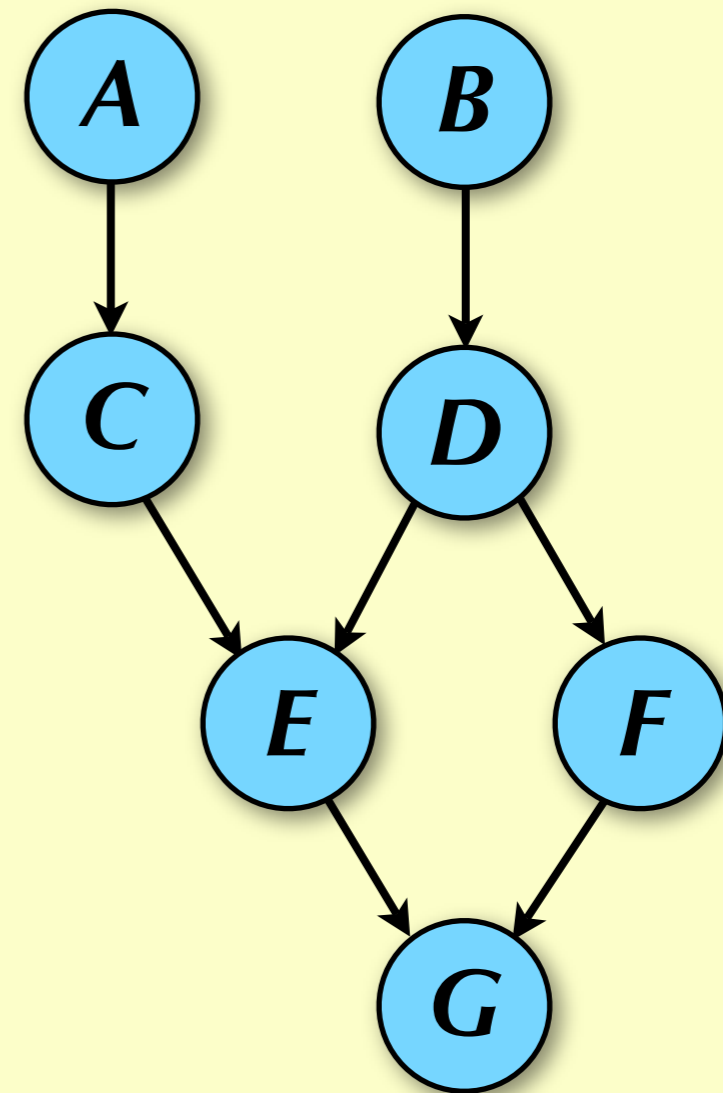
● Yes

● A Indep G

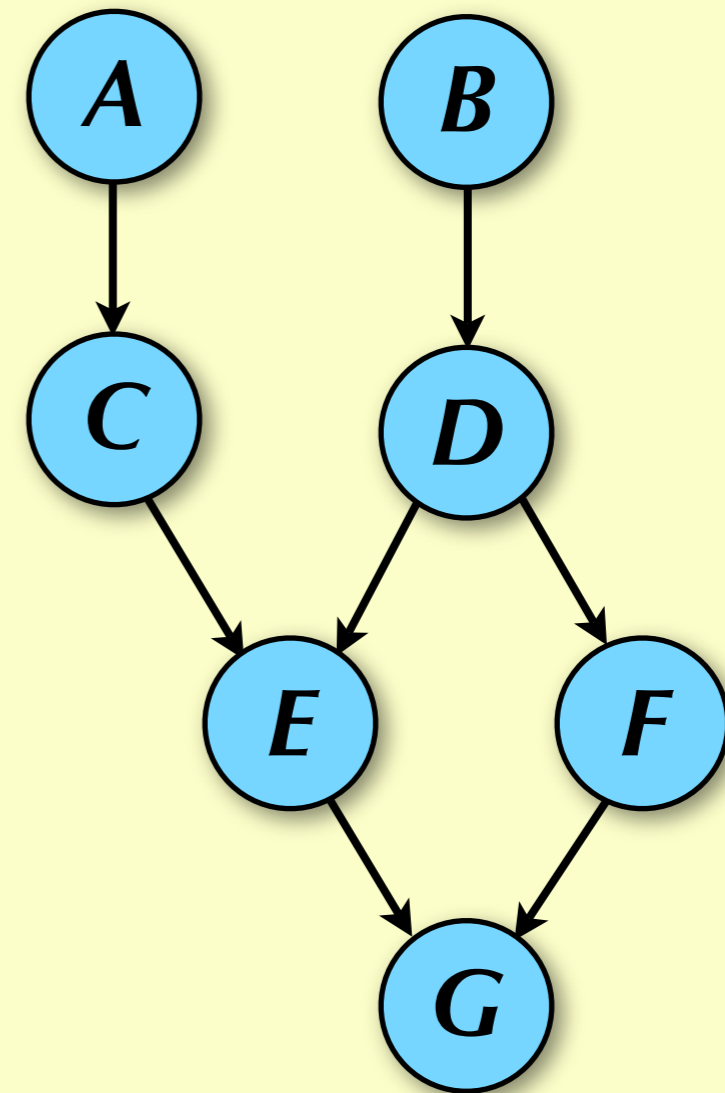
● No

● A Indep B given G

● No



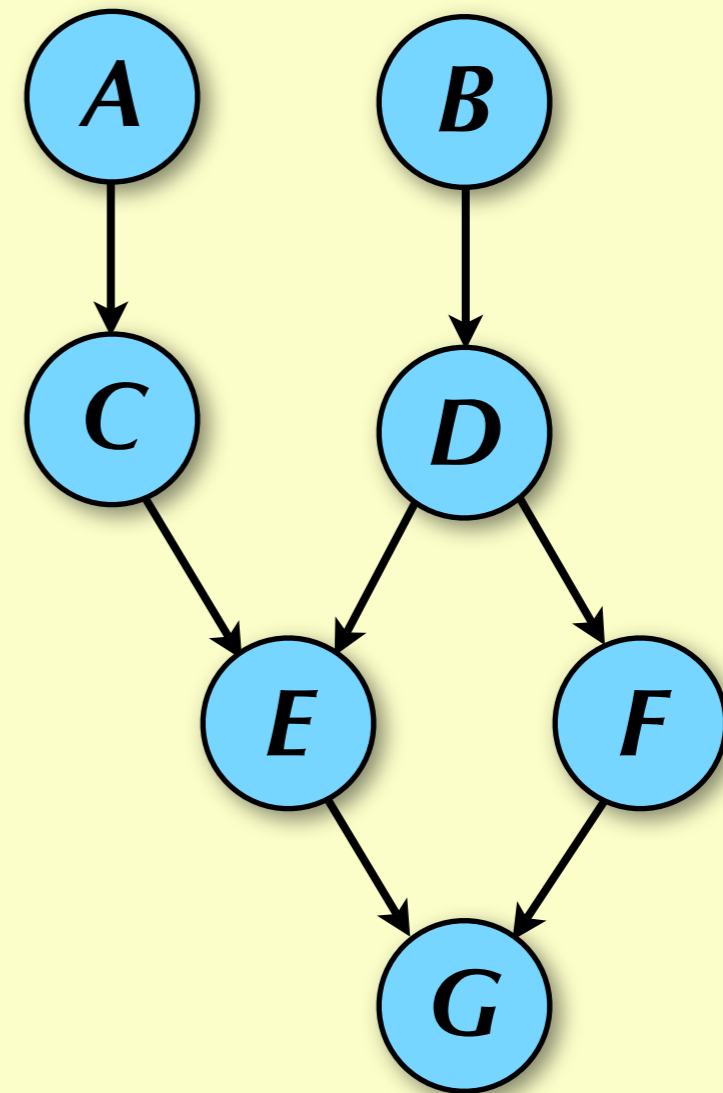
# Inference





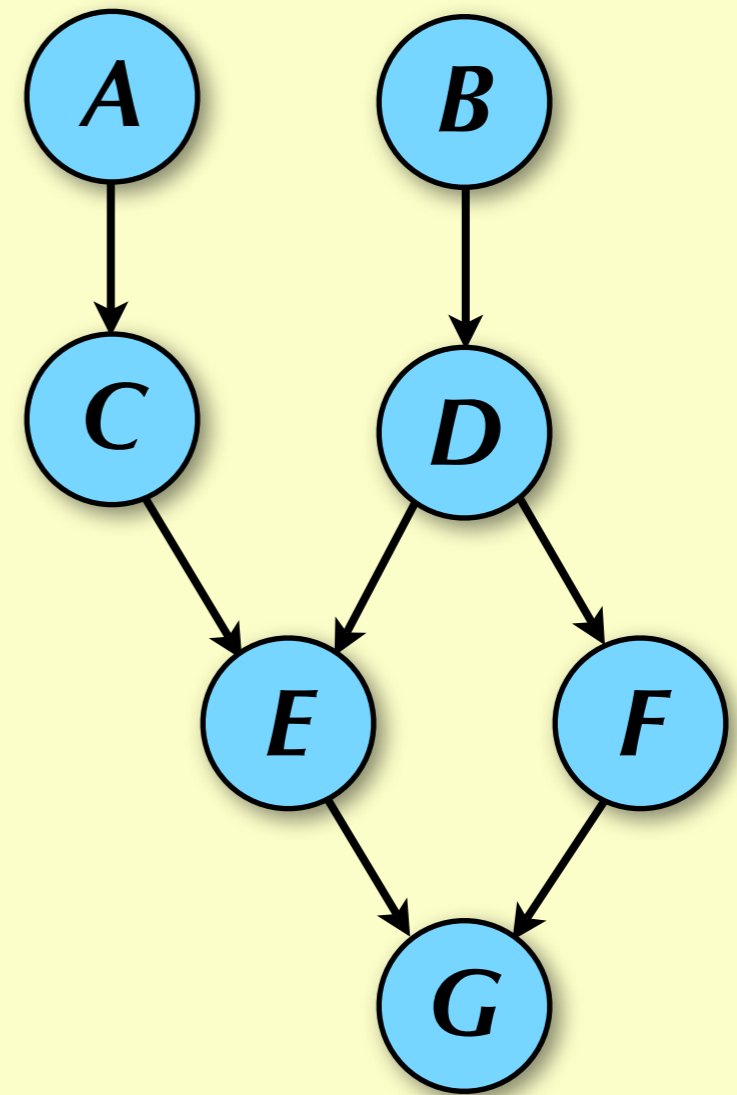
# Inference

- Computing Joint Assignment  
 $P(A=a, B=b, \dots, B=b)$



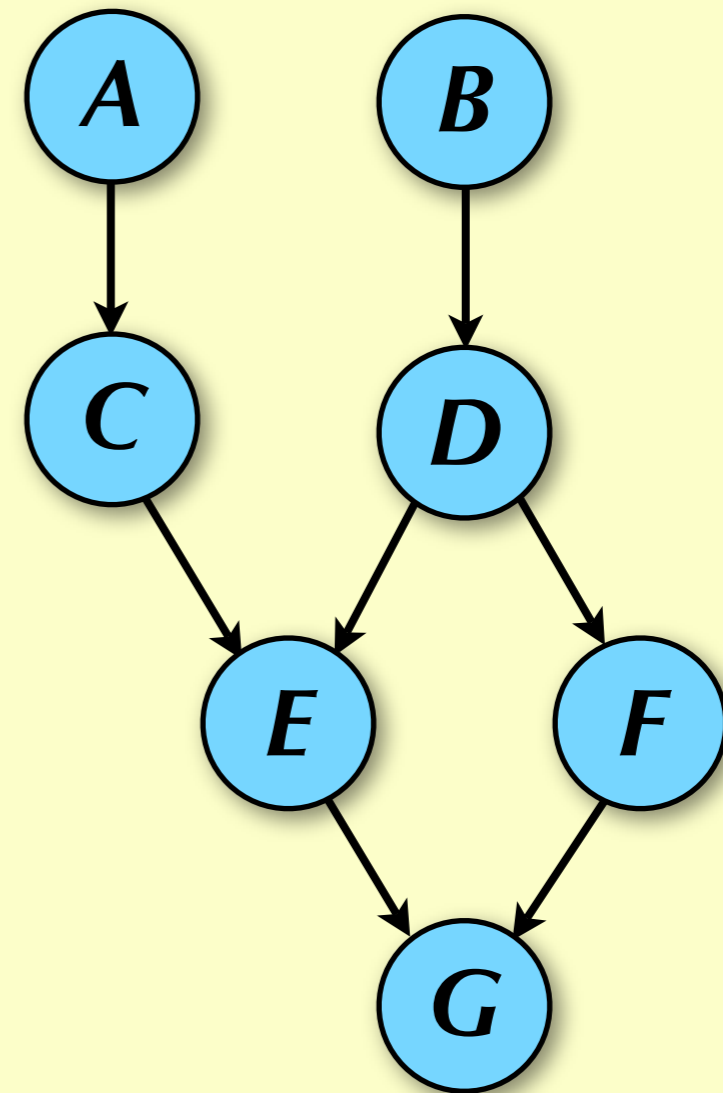
# Inference

- Computing Joint Assignment  
 $P(A=a, B=b, \dots, B=b)$
- Easy



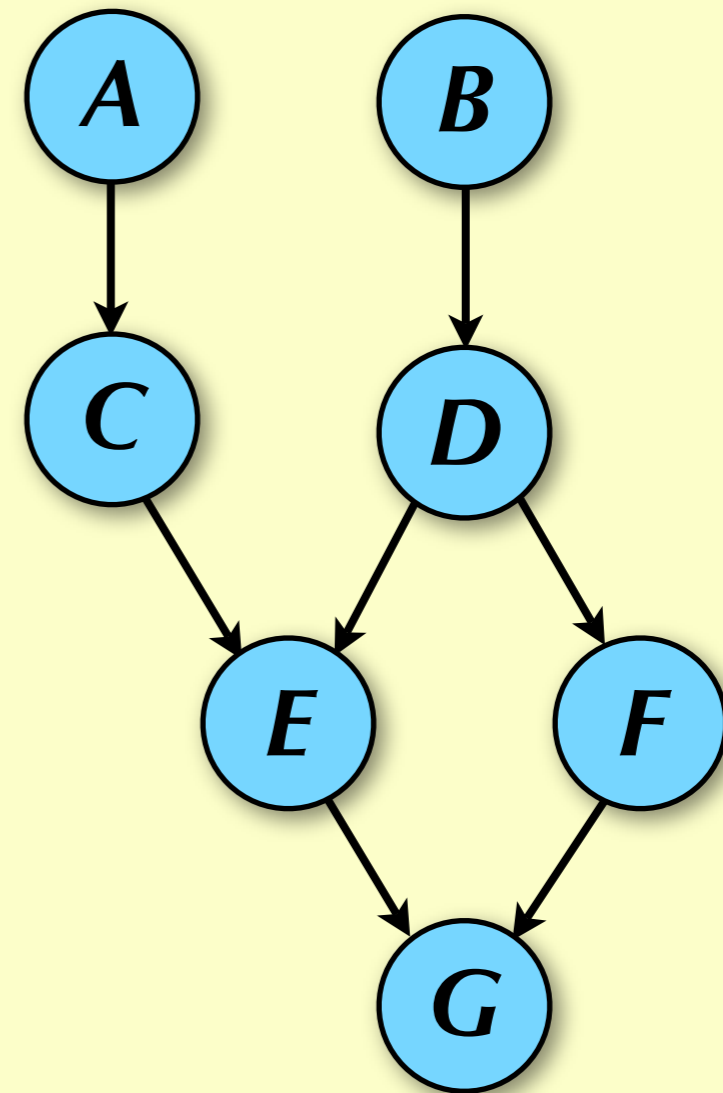
# Inference

- Computing Joint Assignment  
 $P(A=a, B=b, \dots, B=b)$
- Easy
- Computing Marginal



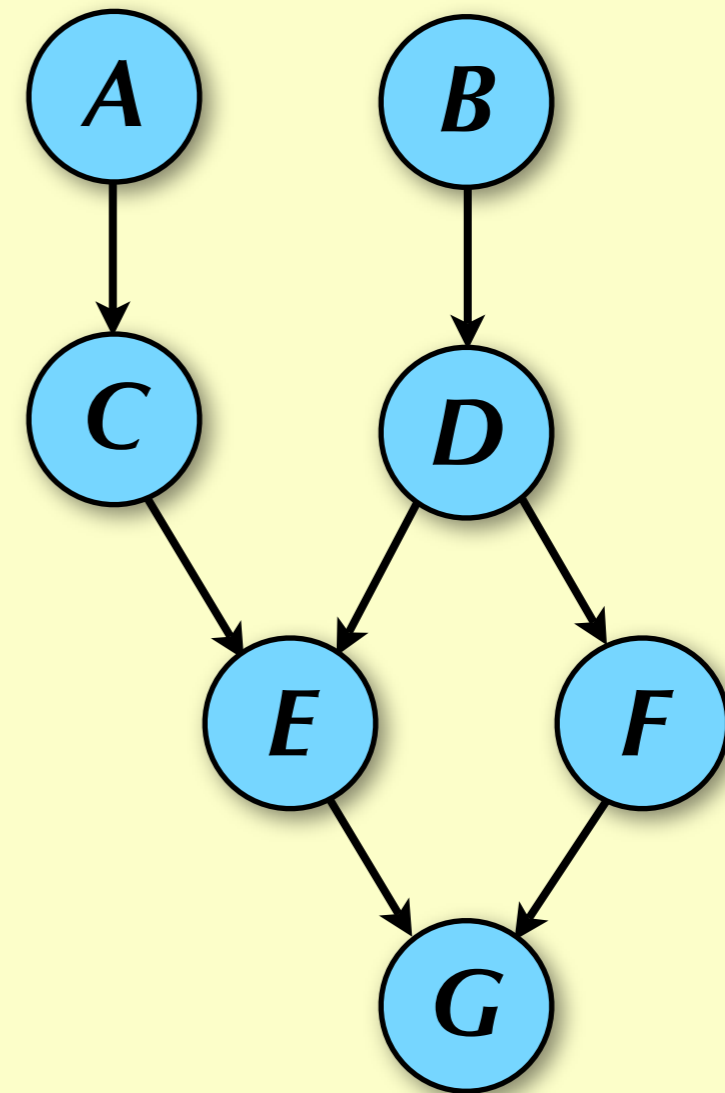
# Inference

- Computing Joint Assignment  
 $P(A=a, B=b, \dots, B=b)$
- Easy
- Computing Marginal
- Tricky to Hard



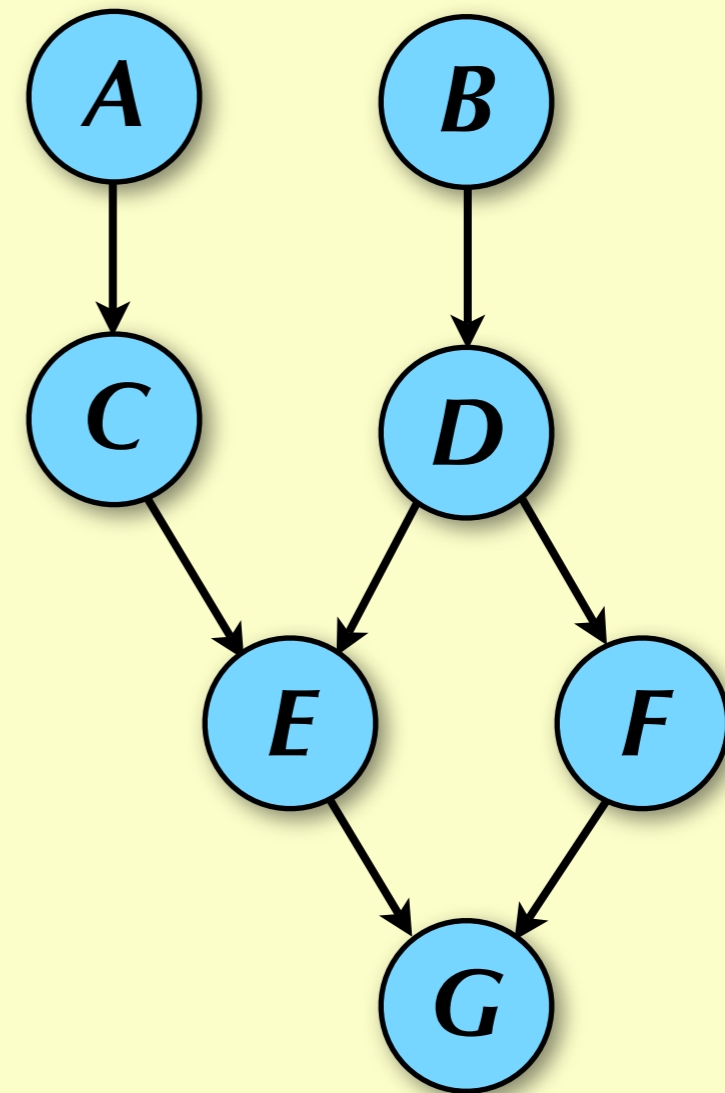
# Inference

- Computing Joint Assignment  
 $P(A=a, B=b, \dots, B=b)$
- Easy
- Computing Marginal
  - Tricky to Hard
- Computing Conditional



# Inference

- Computing Joint Assignment  
 $P(A=a, B=b, \dots, B=b)$
- Easy
- Computing Marginal
  - Tricky to Hard
- Computing Conditional
  - Requires Marginal



# Why is it hard to Marginalize

## Marginalize Unobserved Variables

*You want  $P(X_1, X_2)$  but you have  $P(X_1, \dots, X_n)$*

$$P(X_1, X_2) = \sum_{x_3} \sum_{x_4} \dots \sum_{x_n} P(X_1, X_2, x_3, x_4, \dots, x_n)$$

- for  $X_3 = 1:b$ 
  - for  $X_4 = 1:b$ 
    - for  $X_5 = 1:b$ 
      - for  $X_6 = 1:b \dots$

# How can we save some work

## Simple Algebra Example

*Want to compute*

$$\sum_a \sum_b \sum_c \sum_d f_1(a) f_2(b) f_3(c) f_4(d)$$

## Rearrange the sums

*Much Easier*

$$\left( \sum_a f_1(a) \right) \left( \sum_b f_2(b) \right) \left( \sum_c f_3(c) \right) \left( \sum_d f_4(d) \right)$$



# Return functions

## Coupled Variables

*Suppose we want to compute*

$$\sum_a \sum_b \sum_c \sum_d f_1(a) f_2(b, a) f_3(c, b) f_4(d, c)$$

## Functions rather than constants

$$\sum_a \sum_b \sum_c f_1(a) f_2(b, a) f_3(c, b) \sum_d f_4(d, c)$$

$$g_1(c)$$

Cache Computation

# What is a Function

Assume variables are discrete

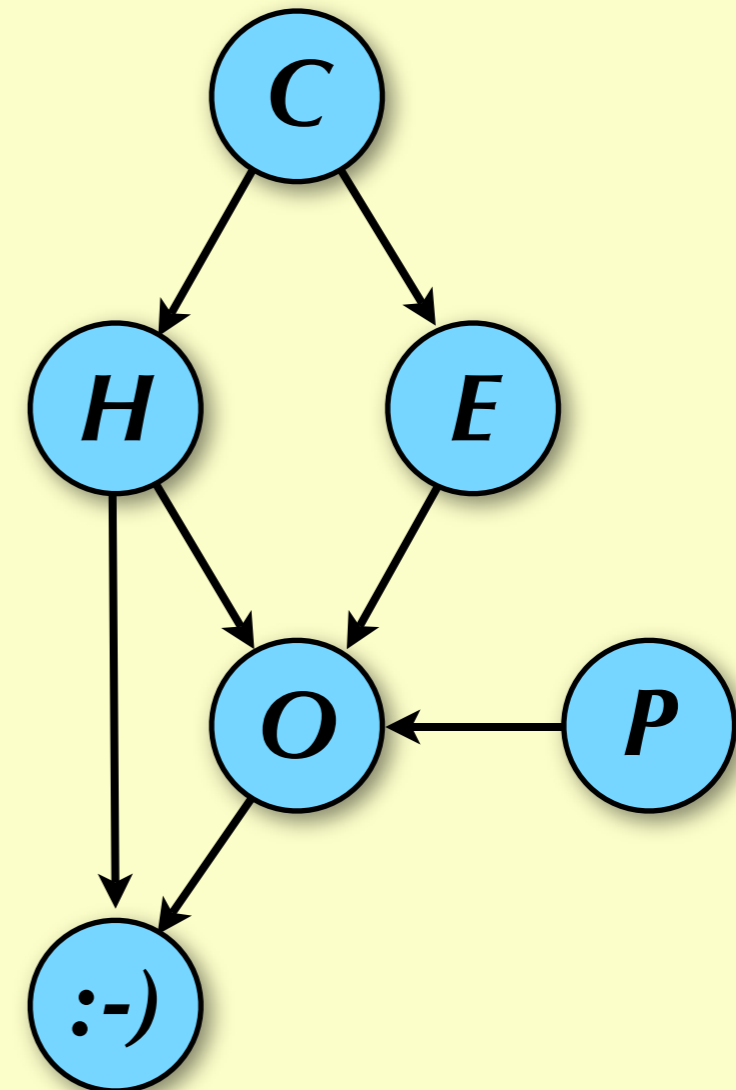
$$g(x_1, x_2, \dots, x_n) = \text{table}[x_1, x_2, \dots, x_n]$$

- Assume variables are discrete
- Each entry in the table is a cached “marginalization”

# Example :: Graph

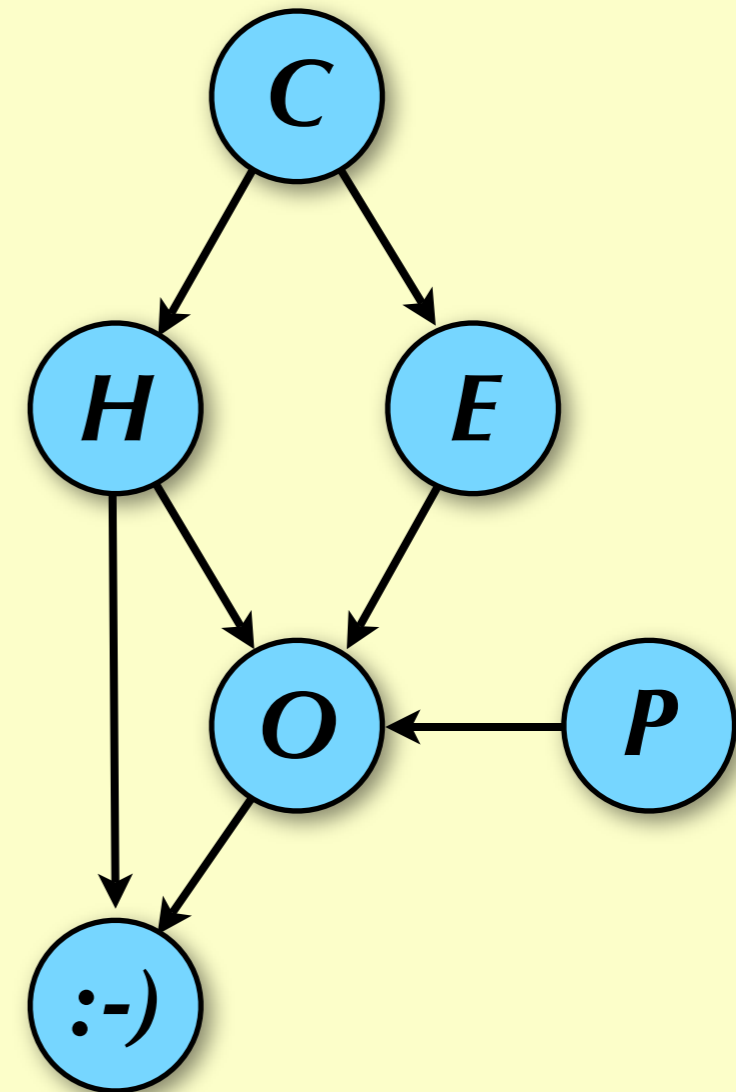
## Define Some Variables

- (H) HW Grade = {P, F}
- (E) Exam Grade = {P, F}
- (P) Project Grade = {P, F}
- (O) Overall Grade = {P, F}
- (C) Class Attend = {T, F}
- ( :- ) ) Happy = {T, F}



# Example :: Joint Probability

$$\begin{aligned} \mathbf{P}(C, H, E, O, P, :-)) = \\ \mathbf{P}(C)\mathbf{P}(P)\mathbf{P}(H | C)\mathbf{P}(E | C) \\ \mathbf{P}(O | H, E, P)\mathbf{P}(:-) | H, O) \end{aligned}$$



# Example :: Tables

P(O H,E,P)			O=p	O=f
H=p	E=p	P=p	0.95	0.05
H=p	E=p	P=f	0.6	0.4
H=p	E=f	P=p	0.6	0.4
H=p	E=f	P=f	0.3	0.7
H=f	E=p	P=p	0.3	0.7
H=f	E=p	P=f	0.2	0.8
H=f	E=f	P=p	0.1	0.9
H=f	E=f	P=f	0.01	0.99

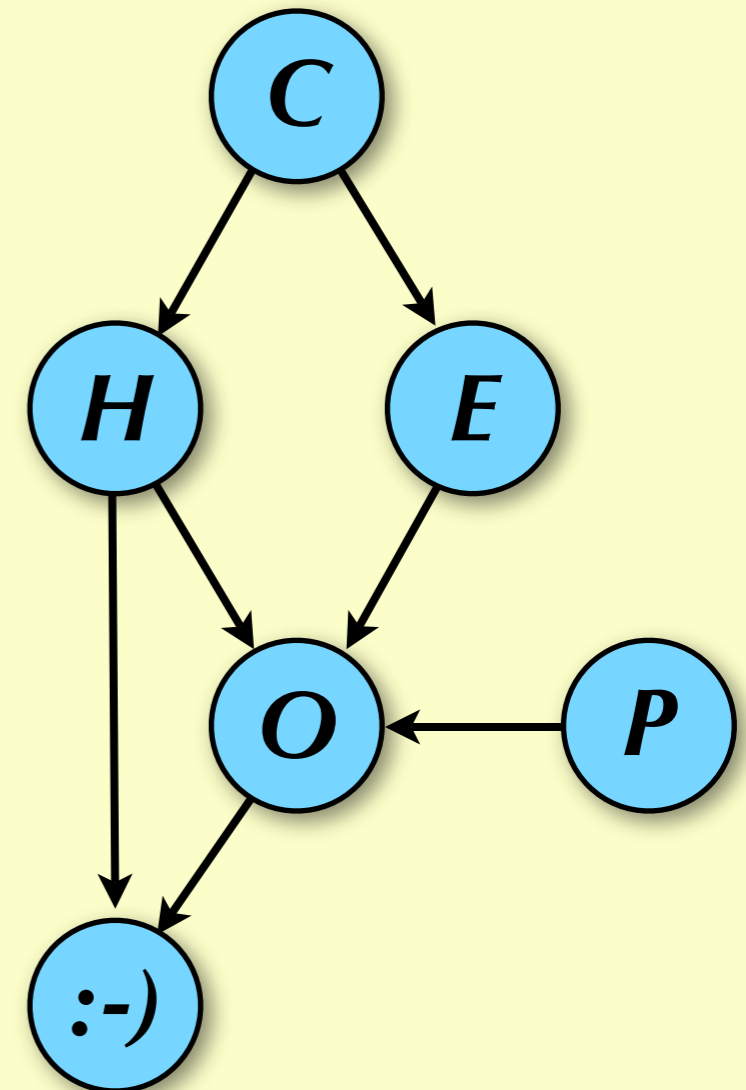
P(E C)	E=p	E=f
C=t	0.7	0.3
C=f	0.4	0.6

P(H C)	H=p	H=f
C=t	0.7	0.3
C=f	0.4	0.6

P(:-) O,H)		:-)=t	:-)=f
O=p	H=p	0.7	0.3
O=p	H=f	0.8	0.2
O=f	H=p	0.2	0.8
O=f	H=f	0.1	0.9

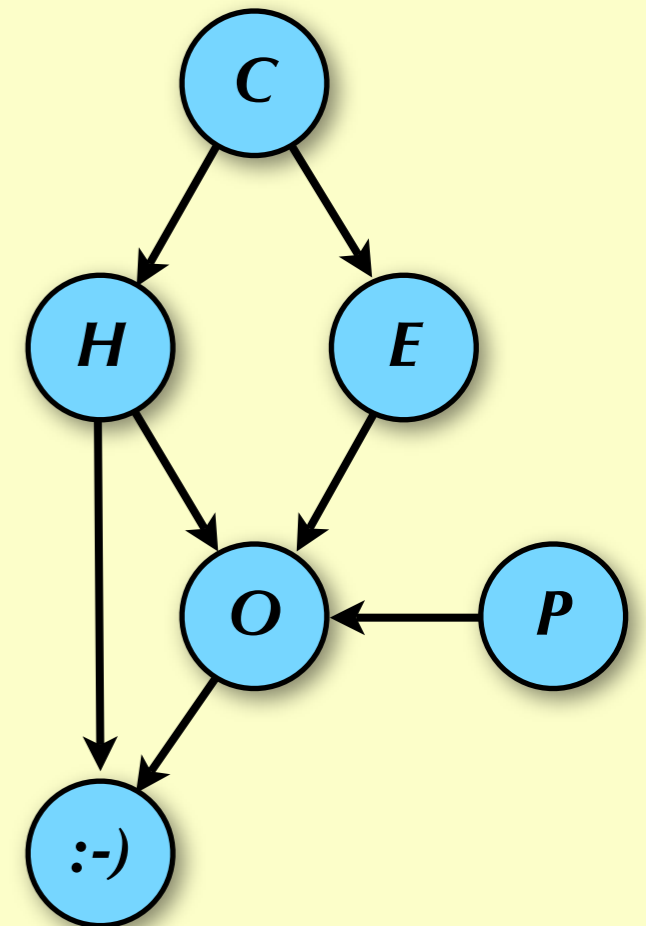
P(P)	P=p	P=f
	0.7	0.3

P(C)	C=t	C=f
	0.8	0.2



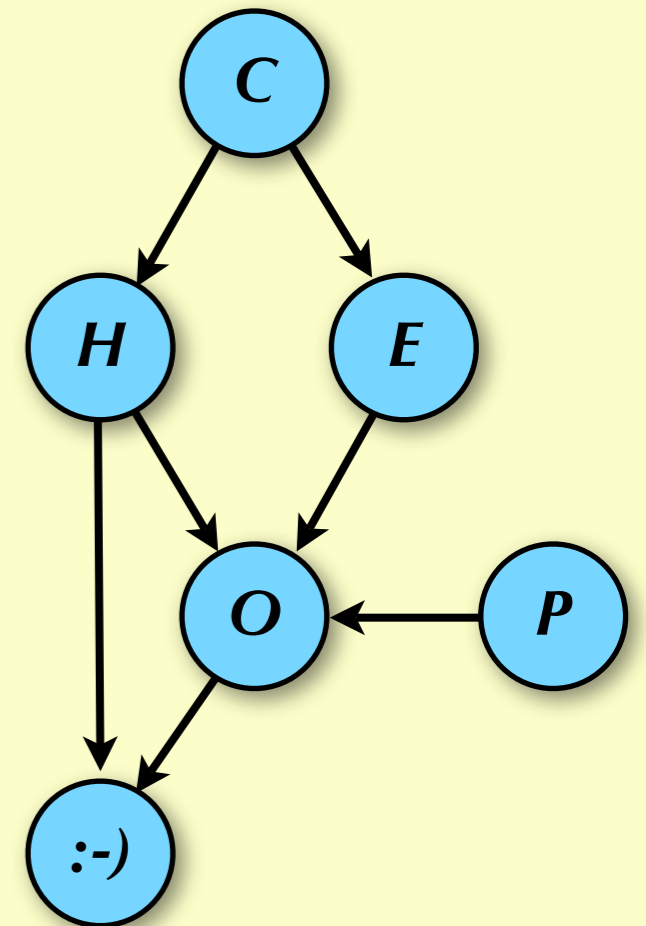
# Happiness and Class Attendance

$$\mathbf{P}(:-), C) = \sum_H \sum_O \sum_P \sum_E \mathbf{P}(C, H, E, O, P, :-)$$



# Happiness and Class Attendance

$$\begin{aligned} \mathbf{P}(:-), C) &= \sum_H \sum_O \sum_P \sum_E \mathbf{P}(C, H, E, O, P, :-)) \\ &= \sum_{H, O, P, E} Pr(C) \mathbf{P}(P) \mathbf{P}(H | C) \mathbf{P}(E | C) \\ &\quad \mathbf{P}(O | H, E, P) \mathbf{P}(:-) | H, O) \end{aligned}$$





# Happiness and Class Attendance

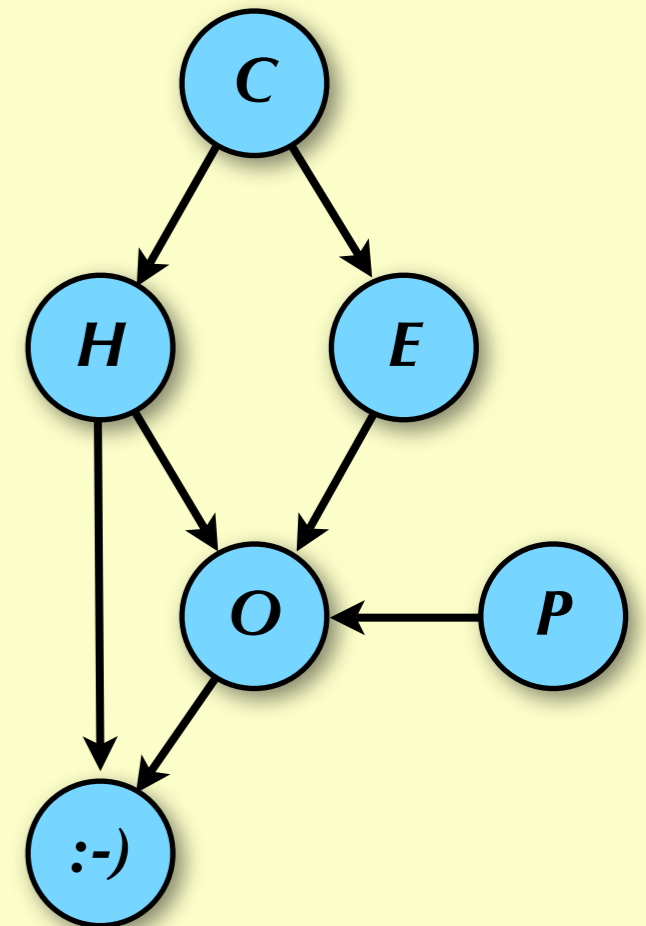
$$\mathbf{P}(:-), C) = \sum_H \sum_O \sum_P \sum_E \mathbf{P}(C, H, E, O, P, :-))$$

$$= \sum_{H, O, P, E} Pr(C) \mathbf{P}(P) \mathbf{P}(H | C) \mathbf{P}(E | C)$$

$$\mathbf{P}(O | H, E, P) \mathbf{P}(:-) | H, O)$$

$$= \sum_{H, O, E} Pr(C) \mathbf{P}(H | C) \mathbf{P}(E | C) \mathbf{P}(:-) | H, O)$$

$$\left( \sum_P \mathbf{P}(P) \mathbf{P}(O | H, E, P) \right)$$





# Happiness and Class Attendance

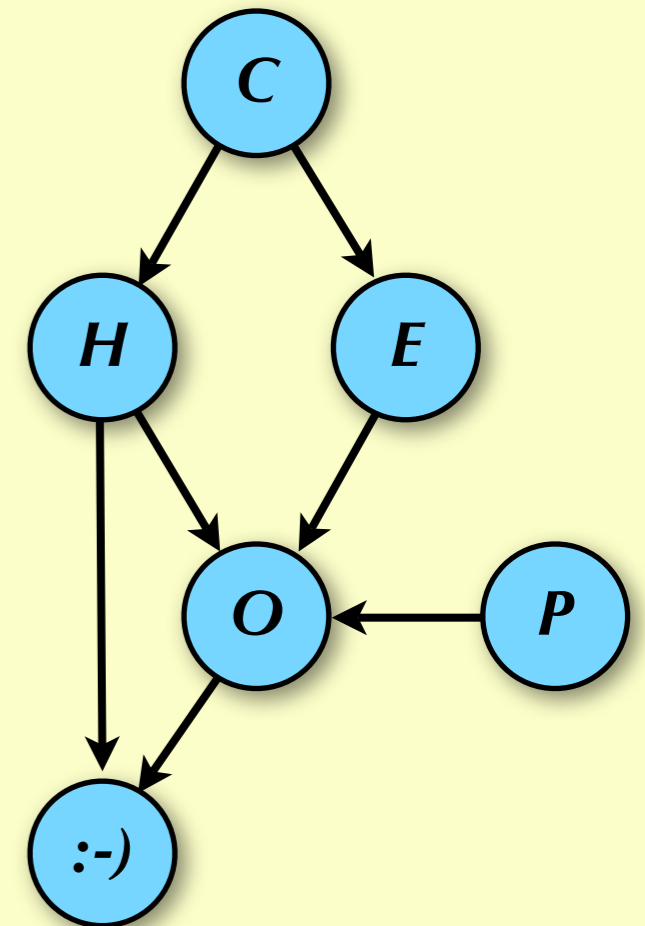
$$\mathbf{P}(:-), C) = \sum_H \sum_O \sum_P \sum_E \mathbf{P}(C, H, E, O, P, :-))$$

$$= \sum_{H, O, P, E} Pr(C) \mathbf{P}(P) \mathbf{P}(H | C) \mathbf{P}(E | C)$$

$$\mathbf{P}(O | H, E, P) \mathbf{P}(:- | H, O)$$

$$= \sum_{H, O, E} Pr(C) \mathbf{P}(H | C) \mathbf{P}(E | C) \mathbf{P}(:- | H, O)$$

$$\left( \sum_P \mathbf{P}(P) \mathbf{P}(O | H, E, P) \right)$$



This becomes a “function” table  $g_1(O, H, E)$ .

# The Function $g_1$

$$g_1(O, H, E) = \sum_P \mathbf{P}(P) \mathbf{P}(O | H, E, P)$$

$\mathbf{P}(O H,E,P)$			O=p	O=f
H=p	E=p	P=p	0.95	0.05
H=p	E=p	P=f	0.6	0.4
H=p	E=f	P=p	0.6	0.4
H=p	E=f	P=f	0.3	0.7
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H=f	E=f	P=p	0.1	0.9
H=f	E=f	P=f	0.01	0.99

$\mathbf{P}(P)$	P=p	P=f
	0.7	0.3

# The Function $g_1$

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H=f	E=f	P=f	0.01	0.99

P(P)	P=p	P=f
	0.7	0.3

$g_1(O, H, E)$

# The Function $g_1$

$$g_1(O, H, E) = \sum_P \mathbf{P}(P) \mathbf{P}(O | H, E, P)$$

$\mathbf{P}(O H,E,P)$			O=p	O=f
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$\mathbf{P}(P)$	P=p	P=f
	0.7	0.3

$g_1(O,H,E)$		
O=p	H=p	E=p

# The Function $g_1$

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P(P)	P=p	P=f
	0.7	0.3

$g_1(O,H,E)$			
O=p	H=p	E=p	$0.7(0.95)+0.3(0.6)=0.845$

# The Function $g_1$

$$g_1(O, H, E) = \sum_P \mathbf{P}(P) \mathbf{P}(O | H, E, P)$$

P(O H,E,P)			O=p	O=f
H=p	E=p	P=p	0.95	0.05
H=p	E=p	P=f	0.6	0.4
H=p	E=f	P=p	0.6	0.4
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H=f	E=p	P=p	0.3	0.7
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H=f	E=f	P=p	0.1	0.9
H=f	E=f	P=f	0.01	0.99

P(P)	P=p	P=f
	0.7	0.3

$g_1(O,H,E)$			
O=p	H=p	E=p	0.7(0.95)+0.3(0.6)=0.845
O=p	H=p	E=f	

# The Function $g_1$

$$g_1(O, H, E) = \sum_P \mathbf{P}(P) \mathbf{P}(O | H, E, P)$$

P(O H,E,P)			O=p	O=f
H=p	E=p	P=p	0.95	0.05
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H=p	E=f	P=f	0.3	0.7
H=f	E=p	P=p	0.3	0.7
H=f	E=p	P=f	0.2	0.8
H=f	E=f	P=p	0.1	0.9
H=f	E=f	P=f	0.01	0.99

P(P)	P=p	P=f
	0.7	0.3

$g_1(O,H,E)$			
O=p	H=p	E=p	$0.7(0.95)+0.3(0.6)=0.845$
O=p	H=p	E=f	$0.7(0.6)+0.3(0.3)=0.510$

# The Function $g_1$

$$g_1(O, H, E) = \sum_P \mathbf{P}(P) \mathbf{P}(O | H, E, P)$$

P(O H,E,P)			O=p	O=f
H=p	E=p	P=p	0.95	0.05
H=p	E=p	P=f	0.6	0.4
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H=f	E=f	P=p	0.1	0.9
H=f	E=f	P=f	0.01	0.99

P(P)	P=p	P=f
	0.7	0.3

$g_1(O,H,E)$			
O=p	H=p	E=p	$0.7(0.95)+0.3(0.6)=0.845$
O=p	H=p	E=f	$0.7(0.6)+0.3(0.3)=0.510$
O=p	H=f	E=p	



# The Function $g_1$

$$g_1(O, H, E) = \sum_P \mathbf{P}(P) \mathbf{P}(O \mid H, E, P)$$

P(O H,E,P)			O=p	O=f
H=p	E=p	P=p	0.95	0.05
H=p	E=p	P=f	0.6	0.4
H=p	E=f	P=p	0.6	0.4
H=p	E=f	P=f	0.3	0.7
H=f	E=p	P=p	0.3	0.7
H=f	E=p	P=f	0.2	0.8
H=f	E=f	P=p	0.1	0.9
H=f	E=f	P=f	0.01	0.99

P(P)	P=p	P=f
	0.7	0.3

$g_1(O,H,E)$			
O=p	H=p	E=p	$0.7(0.95)+0.3(0.6)=0.845$
O=p	H=p	E=f	$0.7(0.6)+0.3(0.3)=0.510$
O=p	H=f	E=p	$0.7(0.3)+0.3(0.2)=0.270$

# The Function $g_1$

$$g_1(O, H, E) = \sum_P \mathbf{P}(P) \mathbf{P}(O | H, E, P)$$

P(O H,E,P)			O=p	O=f
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H=p	E=f	P=p	0.6	0.4
H=p	E=f	P=f	0.3	0.7
H=f	E=p	P=p	0.3	0.7
H=f	E=p	P=f	0.2	0.8
H=f	E=f	P=p	0.1	0.9
H=f	E=f	P=f	0.01	0.99

P(P)	P=p	P=f
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$g_1(O,H,E)$			
O=p	H=p	E=p	$0.7(0.95)+0.3(0.6)=0.845$
O=p	H=p	E=f	$0.7(0.6)+0.3(0.3)=0.510$
O=p	H=f	E=p	$0.7(0.3)+0.3(0.2)=0.270$
O=p	H=f	E=f	

# The Function $g_1$

$$g_1(O, H, E) = \sum_P \mathbf{P}(P) \mathbf{P}(O | H, E, P)$$

P(O H,E,P)			O=p	O=f
H=p	E=p	P=p	0.95	0.05
H=p	E=p	P=f	0.6	0.4
H=p	E=f	P=p	0.6	0.4
H=p	E=f	P=f	0.3	0.7
H=f	E=p	P=p	0.3	0.7
H=f	E=p	P=f	0.2	0.8
H=f	E=f	P=p	0.1	0.9
H=f	E=f	P=f	0.01	0.99

P(P)	P=p	P=f
	0.7	0.3

$g_1(O,H,E)$			
O=p	H=p	E=p	$0.7(0.95)+0.3(0.6)=0.845$
O=p	H=p	E=f	$0.7(0.6)+0.3(0.3)=0.510$
O=p	H=f	E=p	$0.7(0.3)+0.3(0.2)=0.270$
O=p	H=f	E=f	$0.7(0.1)+0.3(0.01)=0.073$

# The Function $g_1$

$$g_1(O, H, E) = \sum_P \mathbf{P}(P) \mathbf{P}(O | H, E, P)$$

P(O H,E,P)			O=p	O=f
H=p	E=p	P=p	0.95	0.05
H=p	E=p	P=f	0.6	0.4
H=p	E=f	P=p	0.6	0.4
H=p	E=f	P=f	0.3	0.7
H=f	E=p	P=p	0.3	0.7
H=f	E=p	P=f	0.2	0.8
H=f	E=f	P=p	0.1	0.9
H=f	E=f	P=f	0.01	0.99

P(P)	P=p	P=f
	0.7	0.3

$g_1(O,H,E)$			
O=p	H=p	E=p	$0.7(0.95)+0.3(0.6)=0.845$
O=p	H=p	E=f	$0.7(0.6)+0.3(0.3)=0.510$
O=p	H=f	E=p	$0.7(0.3)+0.3(0.2)=0.270$
O=p	H=f	E=f	$0.7(0.1)+0.3(0.01)=0.073$
O=f	H=p	E=p	

# The Function $g_1$

$$g_1(O, H, E) = \sum_P \mathbf{P}(P) \mathbf{P}(O | H, E, P)$$

P(O H,E,P)			O=p	O=f
H=p	E=p	P=p	0.95	0.05
H=p	E=p	P=f	0.6	0.4
H=p	E=f	P=p	0.6	0.4
H=p	E=f	P=f	0.3	0.7
H=f	E=p	P=p	0.3	0.7
H=f	E=p	P=f	0.2	0.8
H=f	E=f	P=p	0.1	0.9
H=f	E=f	P=f	0.01	0.99

P(P)	P=p	P=f
	0.7	0.3

$g_1(O,H,E)$			
O=p	H=p	E=p	$0.7(0.95)+0.3(0.6)=0.845$
O=p	H=p	E=f	$0.7(0.6)+0.3(0.3)=0.510$
O=p	H=f	E=p	$0.7(0.3)+0.3(0.2)=0.270$
O=p	H=f	E=f	$0.7(0.1)+0.3(0.01)=0.073$
O=f	H=p	E=p	$0.7(0.05)+0.3(0.4)=0.155$

# The Function $g_1$

$$g_1(O, H, E) = \sum_P \mathbf{P}(P) \mathbf{P}(O | H, E, P)$$

P(O H,E,P)			O=p	O=f
H=p	E=p	P=p	0.95	0.05
H=p	E=p	P=f	0.6	0.4
H=p	E=f	P=p	0.6	0.4
H=p	E=f	P=f	0.3	0.7
H=f	E=p	P=p	0.3	0.7
H=f	E=p	P=f	0.2	0.8
H=f	E=f	P=p	0.1	0.9
H=f	E=f	P=f	0.01	0.99

P(P)	P=p	P=f
	0.7	0.3

$g_1(O,H,E)$			
O=p	H=p	E=p	$0.7(0.95)+0.3(0.6)=0.845$
O=p	H=p	E=f	$0.7(0.6)+0.3(0.3)=0.510$
O=p	H=f	E=p	$0.7(0.3)+0.3(0.2)=0.270$
O=p	H=f	E=f	$0.7(0.1)+0.3(0.01)=0.073$
O=f	H=p	E=p	$0.7(0.05)+0.3(0.4)=0.155$
O=f	H=p	E=f	

# The Function $g_1$

$$g_1(O, H, E) = \sum_P \mathbf{P}(P) \mathbf{P}(O | H, E, P)$$

P(O H,E,P)			O=p	O=f
H=p	E=p	P=p	0.95	0.05
H=p	E=p	P=f	0.6	0.4
H=p	E=f	P=p	0.6	0.4
H=p	E=f	P=f	0.3	0.7
H=f	E=p	P=p	0.3	0.7
H=f	E=p	P=f	0.2	0.8
H=f	E=f	P=p	0.1	0.9
H=f	E=f	P=f	0.01	0.99

P(P)	P=p	P=f
	0.7	0.3

$g_1(O,H,E)$			
O=p	H=p	E=p	$0.7(0.95)+0.3(0.6)=0.845$
O=p	H=p	E=f	$0.7(0.6)+0.3(0.3)=0.510$
O=p	H=f	E=p	$0.7(0.3)+0.3(0.2)=0.270$
O=p	H=f	E=f	$0.7(0.1)+0.3(0.01)=0.073$
O=f	H=p	E=p	$0.7(0.05)+0.3(0.4)=0.155$
O=f	H=p	E=f	$0.7(0.4)+0.3(0.7)=0.490$



# The Function $g_1$

$$g_1(O, H, E) = \sum_P \mathbf{P}(P) \mathbf{P}(O | H, E, P)$$

P(O H,E,P)			O=p	O=f
H=p	E=p	P=p	0.95	0.05
H=p	E=p	P=f	0.6	0.4
H=p	E=f	P=p	0.6	0.4
H=p	E=f	P=f	0.3	0.7
H=f	E=p	P=p	0.3	0.7
H=f	E=p	P=f	0.2	0.8
H=f	E=f	P=p	0.1	0.9
H=f	E=f	P=f	0.01	0.99

P(P)	P=p	P=f
	0.7	0.3

$g_1(O,H,E)$			
O=p	H=p	E=p	$0.7(0.95)+0.3(0.6)=0.845$
O=p	H=p	E=f	$0.7(0.6)+0.3(0.3)=0.510$
O=p	H=f	E=p	$0.7(0.3)+0.3(0.2)=0.270$
O=p	H=f	E=f	$0.7(0.1)+0.3(0.01)=0.073$
O=f	H=p	E=p	$0.7(0.05)+0.3(0.4)=0.155$
O=f	H=p	E=f	$0.7(0.4)+0.3(0.7)=0.490$
O=f	H=f	E=p	



# The Function $g_1$

$$g_1(O, H, E) = \sum_P \mathbf{P}(P) \mathbf{P}(O | H, E, P)$$

P(O H,E,P)			O=p	O=f
H=p	E=p	P=p	0.95	0.05
H=p	E=p	P=f	0.6	0.4
H=p	E=f	P=p	0.6	0.4
H=p	E=f	P=f	0.3	0.7
H=f	E=p	P=p	0.3	0.7
H=f	E=p	P=f	0.2	0.8
H=f	E=f	P=p	0.1	0.9
H=f	E=f	P=f	0.01	0.99

P(P)	P=p	P=f
	0.7	0.3

$g_1(O,H,E)$			
O=p	H=p	E=p	$0.7(0.95)+0.3(0.6)=0.845$
O=p	H=p	E=f	$0.7(0.6)+0.3(0.3)=0.510$
O=p	H=f	E=p	$0.7(0.3)+0.3(0.2)=0.270$
O=p	H=f	E=f	$0.7(0.1)+0.3(0.01)=0.073$
O=f	H=p	E=p	$0.7(0.05)+0.3(0.4)=0.155$
O=f	H=p	E=f	$0.7(0.4)+0.3(0.7)=0.490$
O=f	H=f	E=p	$0.7(0.7)+0.3(0.8)=0.730$

# The Function $g_1$

$$g_1(O, H, E) = \sum_P \mathbf{P}(P) \mathbf{P}(O | H, E, P)$$

P(O H,E,P)			O=p	O=f
H=p	E=p	P=p	0.95	0.05
H=p	E=p	P=f	0.6	0.4
H=p	E=f	P=p	0.6	0.4
H=p	E=f	P=f	0.3	0.7
H=f	E=p	P=p	0.3	0.7
H=f	E=p	P=f	0.2	0.8
H=f	E=f	P=p	0.1	0.9
H=f	E=f	P=f	0.01	0.99

P(P)	P=p	P=f
	0.7	0.3

$g_1(O,H,E)$			
O=p	H=p	E=p	$0.7(0.95)+0.3(0.6)=0.845$
O=p	H=p	E=f	$0.7(0.6)+0.3(0.3)=0.510$
O=p	H=f	E=p	$0.7(0.3)+0.3(0.2)=0.270$
O=p	H=f	E=f	$0.7(0.1)+0.3(0.01)=0.073$
O=f	H=p	E=p	$0.7(0.05)+0.3(0.4)=0.155$
O=f	H=p	E=f	$0.7(0.4)+0.3(0.7)=0.490$
O=f	H=f	E=p	$0.7(0.7)+0.3(0.8)=0.730$
O=f	H=f	E=f	

# The Function $g_1$

$$g_1(O, H, E) = \sum_P \mathbf{P}(P) \mathbf{P}(O | H, E, P)$$

P(O H,E,P)			O=p	O=f
H=p	E=p	P=p	0.95	0.05
H=p	E=p	P=f	0.6	0.4
H=p	E=f	P=p	0.6	0.4
H=p	E=f	P=f	0.3	0.7
H=f	E=p	P=p	0.3	0.7
H=f	E=p	P=f	0.2	0.8
H=f	E=f	P=p	0.1	0.9
H=f	E=f	P=f	0.01	0.99

$g_1(O,H,E)$			
O=p	H=p	E=p	$0.7(0.95)+0.3(0.6)=0.845$
O=p	H=p	E=f	$0.7(0.6)+0.3(0.3)=0.510$
O=p	H=f	E=p	$0.7(0.3)+0.3(0.2)=0.270$
O=p	H=f	E=f	$0.7(0.1)+0.3(0.01)=0.073$
O=f	H=p	E=p	$0.7(0.05)+0.3(0.4)=0.155$
O=f	H=p	E=f	$0.7(0.4)+0.3(0.7)=0.490$
O=f	H=f	E=p	$0.7(0.7)+0.3(0.8)=0.730$
O=f	H=f	E=f	$0.7(0.9)+0.3(0.99)=0.927$

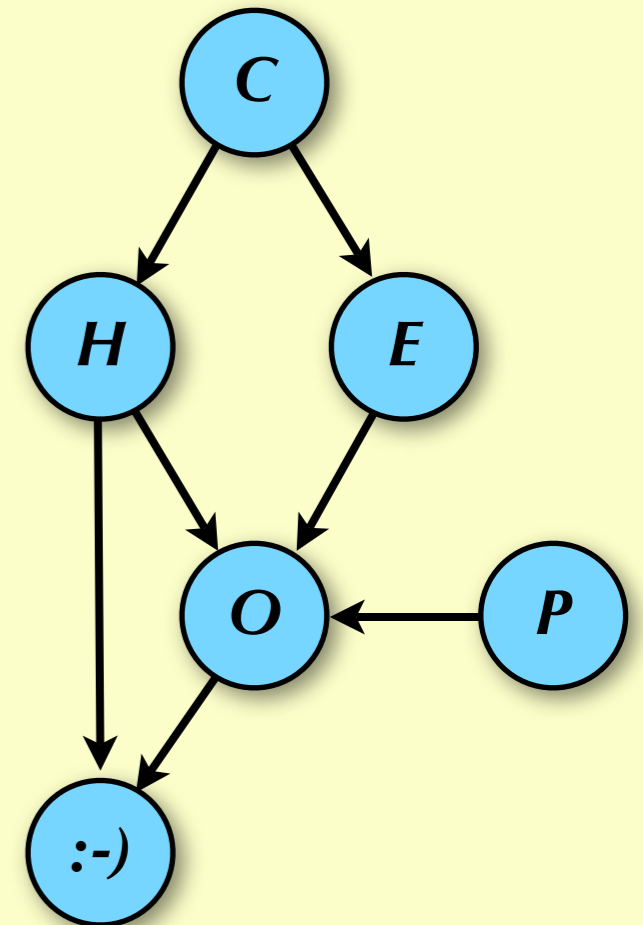
P(P)	P=p	P=f
	0.7	0.3

# Happiness and Class Attendance

$$\begin{aligned}\mathbf{P}(:-), C) &= \sum_{H,O,E} Pr(C)\mathbf{P}(H | C)\mathbf{P}(E | C)\mathbf{P}(:-) | H, O)g_1(O, H, E) \\ &= \sum_{H,O} Pr(C)\mathbf{P}(H | C)\mathbf{P}(:-) | H, O) \left( \sum_E \mathbf{P}(E | C)g_1(O, H, E) \right)\end{aligned}$$

$$g_2(O, H, C) = \sum_E \mathbf{P}(E | C)g_1(O, H, E)$$

Another table to compute



# The Function $g_2$

$$g_2(O, H, C) = \sum_E P(E | C) g_1(O, H, E)$$

$g_1(O, H, E)$			
O=p	H=p	E=p	0.845
O=p	H=p	E=f	0.510
O=p	H=f	E=p	0.270
O=p	H=f	E=f	0.073
O=f	H=p	E=p	0.155
O=f	H=p	E=f	0.490
O=f	H=f	E=p	0.730
O=f	H=f	E=f	0.927

$g_2(O, H, C)$			
O=p	H=p	C=t	$0.7(0.845)+0.3(0.510)=0.7445$
O=p	H=p	C=f	$0.4(0.845)+0.6(0.510)=0.644$
O=p	H=f	C=t	$0.7(0.270)+0.3(0.073)=0.2109$
O=p	H=f	C=f	$0.4(0.270)+0.6(0.073)=0.1518$
O=f	H=p	C=t	$0.7(0.155)+0.3(0.490)=0.2555$
O=f	H=p	C=f	$0.4(0.155)+0.6(0.490)=0.356$
O=f	H=f	C=t	$0.7(0.730)+0.3(0.927)=0.7891$
O=f	H=f	C=f	$0.4(0.730)+0.6(0.927)=0.8482$

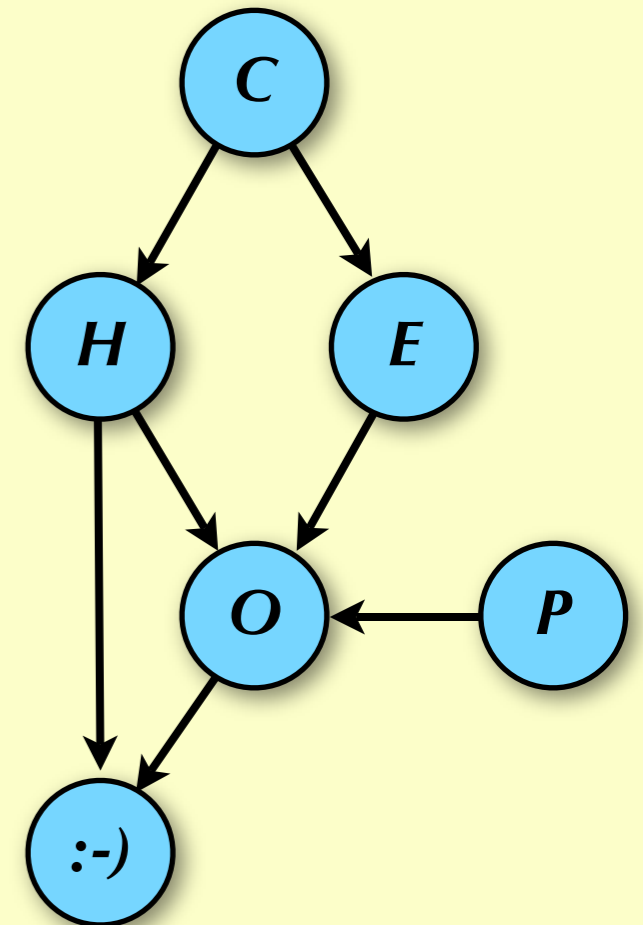
$P(E C)$	E=p	E=f
C=t	0.7	0.3
C=f	0.4	0.6

# Happiness and Class Attendance

$$\begin{aligned} \mathbf{P}(:-), C) &= \sum_{H,O} Pr(C) \mathbf{P}(H | C) \mathbf{P}(:-) | H, O) g_2(O, H, C) \\ &= \sum_H Pr(C) \mathbf{P}(H | C) \left( \sum_O \mathbf{P}(:-) | H, O) g_2(O, H, C) \right) \end{aligned}$$

$$g_3(:-), H, C) = \sum_O \mathbf{P}(:-) | H, O) g_2(O, H, C)$$

Another table to compute





# The Function $g_3$

P(:-) O,H)		:-)=t	:-)=f
O=p	H=p	0.7	0.3
O=p	H=f	0.8	0.2
O=f	H=p	0.2	0.8
O=f	H=f	0.1	0.9

$$g_3(:-), H, C) = \sum_O \mathbf{P}(:- | H, O) g_2(O, H, C)$$

$g_2(O, H, C)$			
O=p	H=p	C=t	0.7445
O=p	H=p	C=f	0.644
O=p	H=f	C=t	0.2109
O=p	H=f	C=f	0.1518
O=f	H=p	C=t	0.2555
O=f	H=p	C=f	0.356
O=f	H=f	C=t	0.7891
O=f	H=f	C=f	0.8482

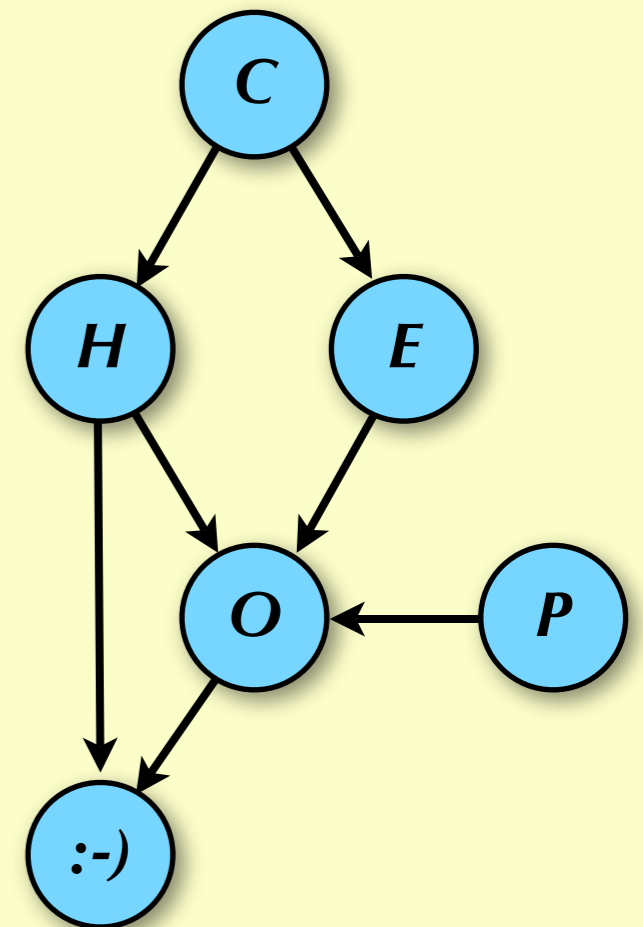
$g_3(:-), H, C)$			
:-)=t	H=p	C=t	$0.7(0.7445)+0.2(0.2555)=0.5723$
:-)=t	H=p	C=f	$0.7(0.644)+0.2(0.356)=0.522$
:-)=t	H=f	C=t	$0.8(0.2109)+0.1(0.7891)=0.2468$
:-)=t	H=f	C=f	$0.8(0.1518)+0.1(0.8482)=0.2063$
:-)=f	H=p	C=t	$0.3(0.7445)+0.8(0.2555)=0.4278$
:-)=f	H=p	C=f	$0.3(0.644)+0.8(0.356)=0.478$
:-)=f	H=f	C=t	$0.2(0.2109)+0.9(0.7891)=0.752$
:-)=f	H=f	C=f	$0.2(0.1518)+0.9(0.8482)=0.793$

# Happiness and Class Attendance

$$\begin{aligned}\mathbf{P}(:-), C) &= \sum_H Pr(C) \mathbf{P}(H | C) g_3(:-), H, C) \\ &= Pr(C) \sum_H \mathbf{P}(H | C) g_3(:-), H, C)\end{aligned}$$

$$g_4(:-), C) = \sum_H \mathbf{P}(H | C) g_3(:-), H, C)$$

Last table to compute





# The Function $g_4$

$$g_4(\text{:}-), C) = \sum_H \mathbf{P}(H | C) g_3(\text{:}-), H, C)$$

$g_3(\text{:}-), H, C)$			
$\text{:})=t$	$H=p$	$C=t$	0.5723
$\text{:})=t$	$H=p$	$C=f$	0.522
$\text{:})=t$	$H=f$	$C=t$	0.2468
$\text{:})=t$	$H=f$	$C=f$	0.2063
$\text{:})=f$	$H=p$	$C=t$	0.4278
$\text{:})=f$	$H=p$	$C=f$	0.478
$\text{:})=f$	$H=f$	$C=t$	0.752
$\text{:})=f$	$H=f$	$C=f$	0.793

$g_4(\text{:}-), C)$		
$\text{:})=t$	$C=t$	$0.7(0.5723)+0.3(0.2468)=0.4747$
$\text{:})=t$	$C=f$	$0.4(0.522)+0.6(0.2063)=0.3326$
$\text{:})=f$	$C=t$	$0.7(0.4278)+0.3(0.752)=0.5251$
$\text{:})=f$	$C=f$	$0.4(0.478)+0.6(0.793)=0.667$

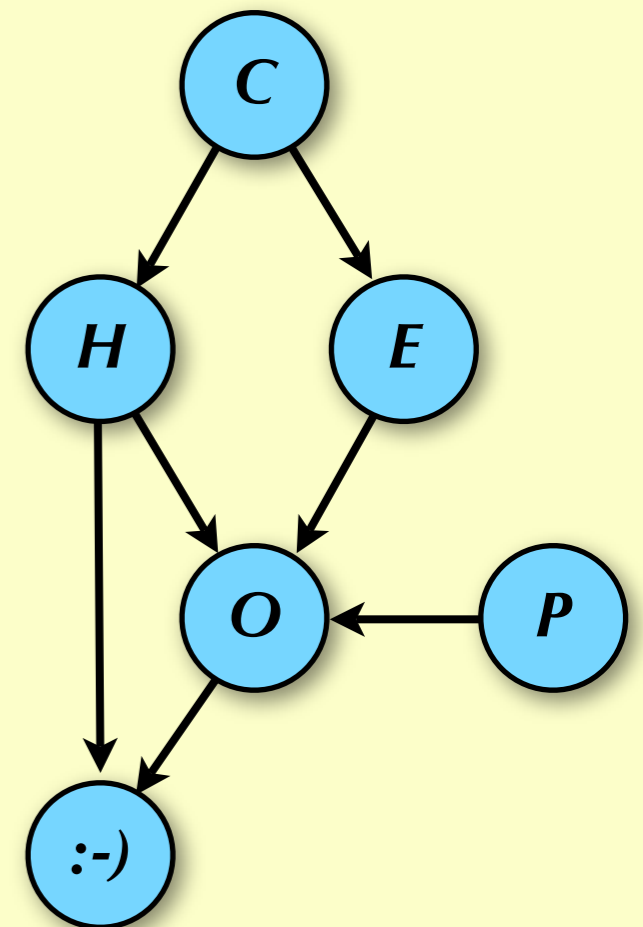
$\mathbf{P}(H C)$	$H=p$	$H=f$
$C=t$	0.7	0.3
$C=f$	0.4	0.6

# Happiness and Class Attendance

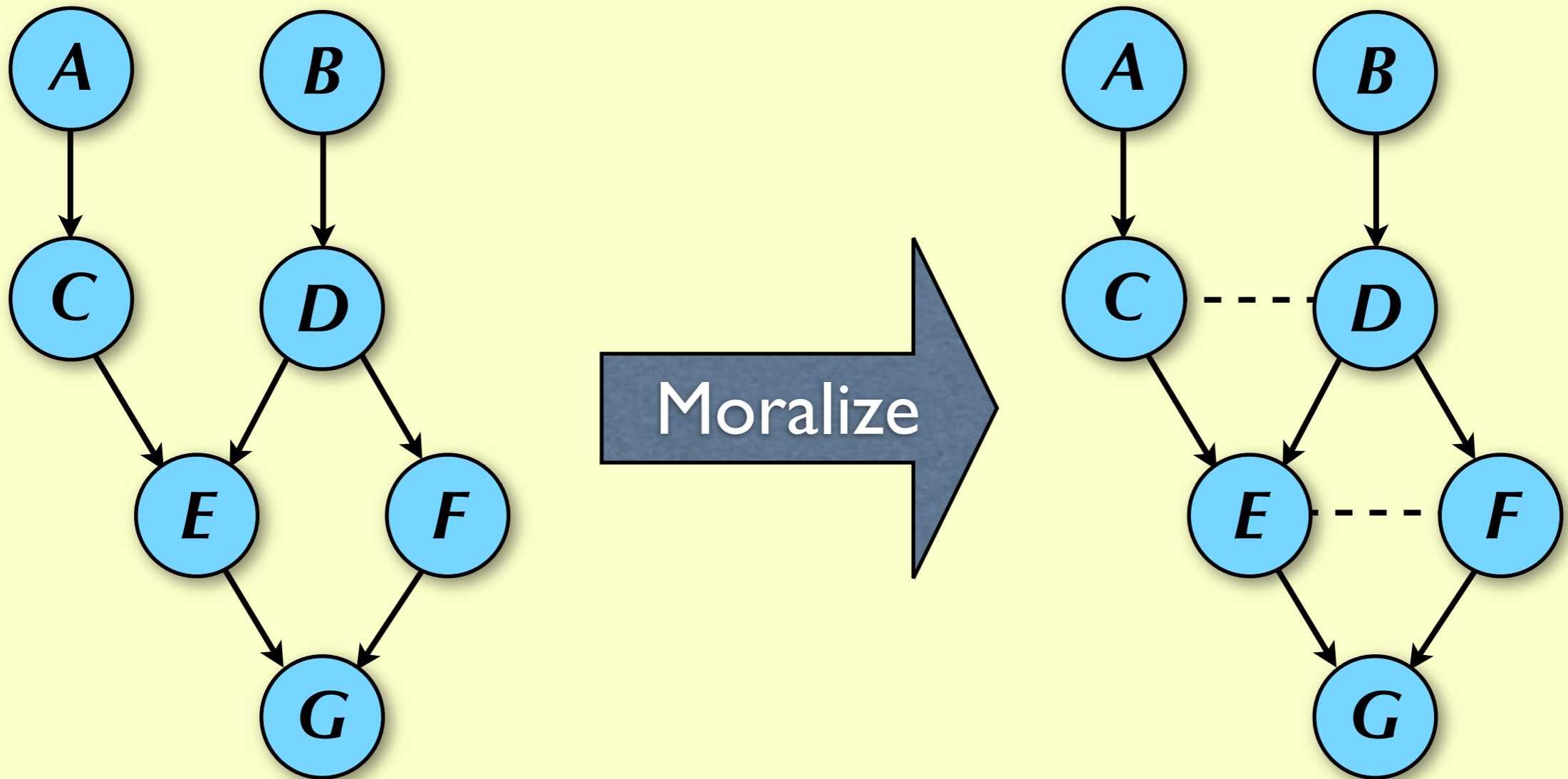
$$\mathbf{P}(\text{:}), C) = Pr(C)g_4(\text{:}), C)$$

$$\begin{aligned} \mathbf{P}(\text{:}) | C) &= \frac{Pr(C)g_4(\text{:}), C)}{\sum_{\text{:})} Pr(C)g_4(\text{:}), C)} \\ &= \frac{Pr(C)g_4(\text{:}), C)}{Pr(C) \sum_{\text{:})} g_4(\text{:}), C)} \\ &= \frac{Pr(C)g_4(\text{:}), C)}{Pr(C)} \\ &= g_4(\text{:}), C) \end{aligned}$$

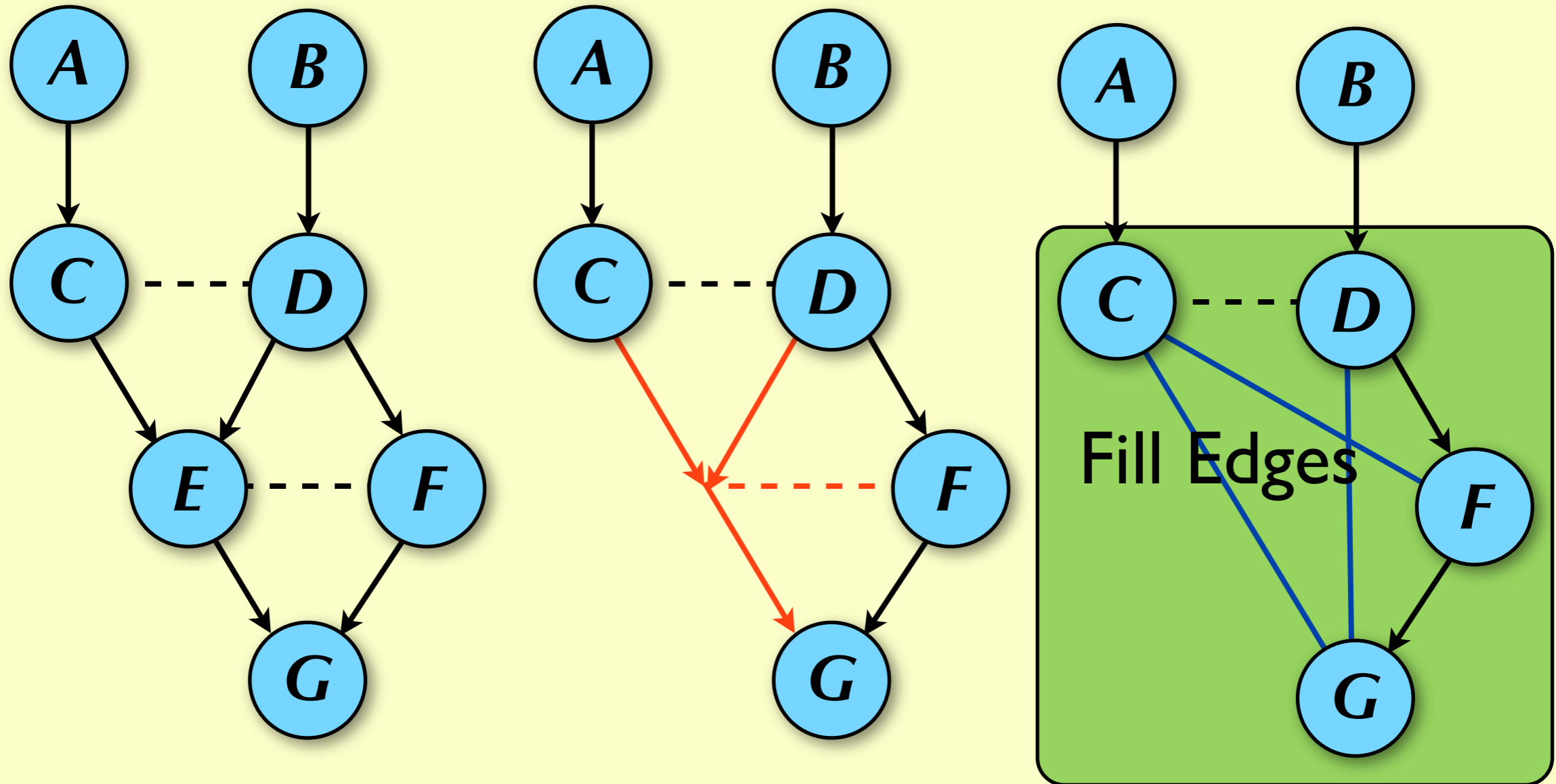
$g_4(\text{:}), C)$		
$\text{:})=t$	$C=t$	0.4747
$\text{:})=t$	$C=f$	0.3326
$\text{:})=f$	$C=t$	0.5251
$\text{:})=f$	$C=f$	0.6667



# Variable Elimination as Pictures

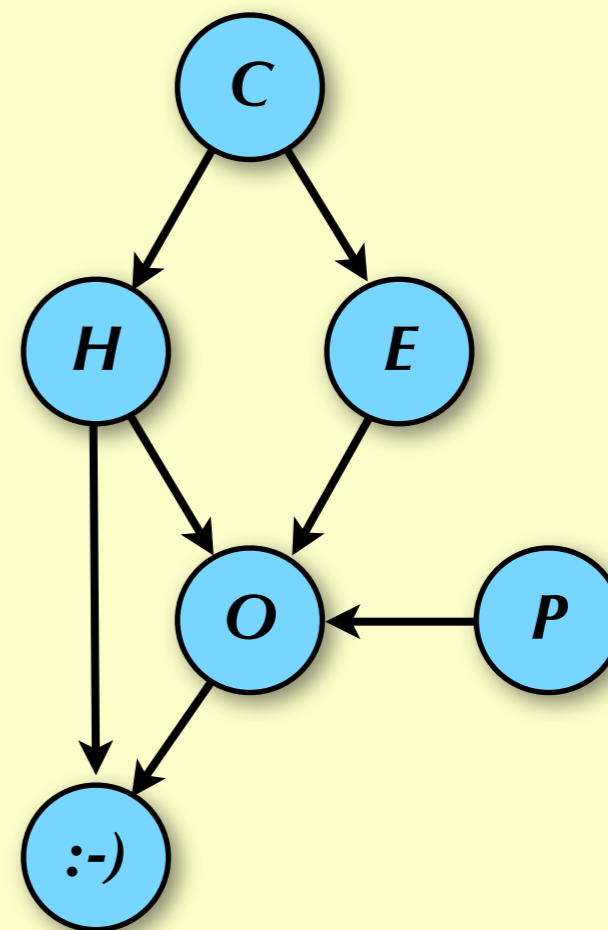


# Variable Elimination and Fill Edges



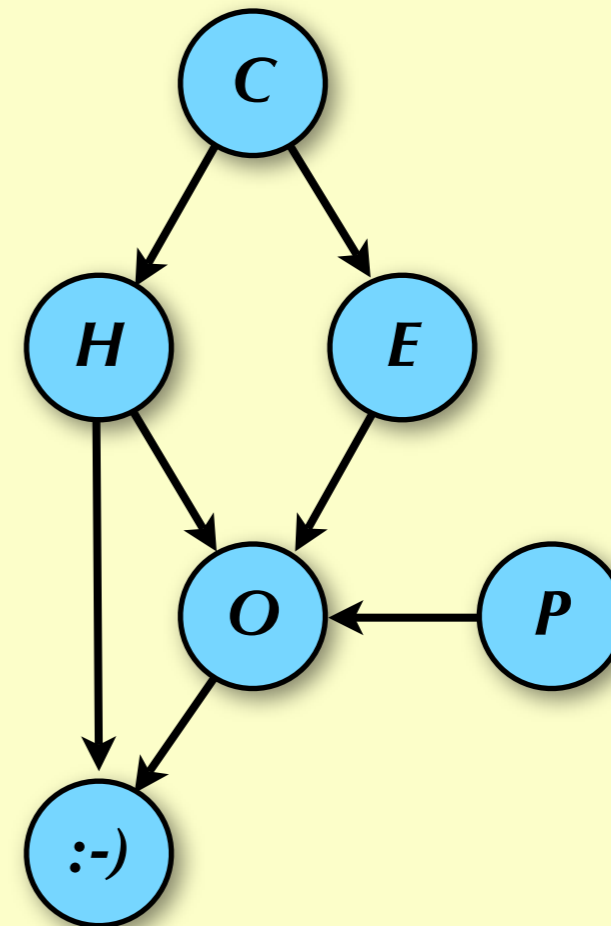
$$g(C, D, G, F) = \sum_e P(E = e | C, D) P(G | E = e, F)$$

# Recall The Happiness Problem



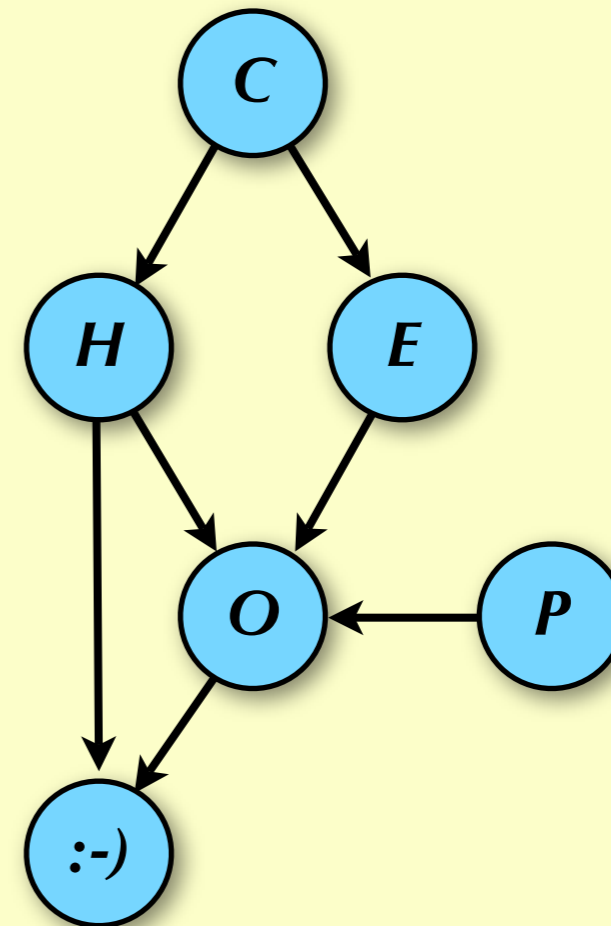
# Recall The Happiness Problem

- We eliminated:



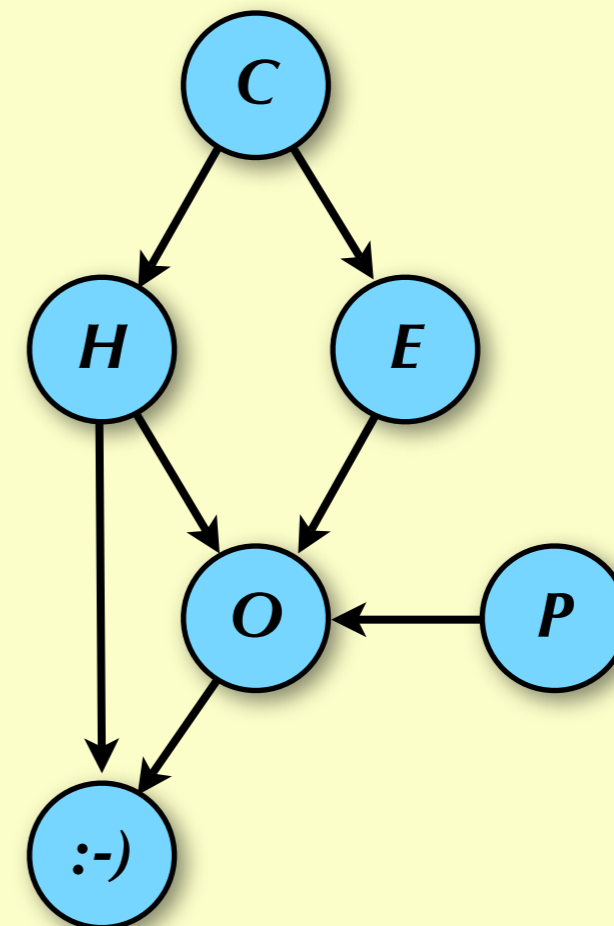
# Recall The Happiness Problem

- We eliminated:
  - P,E,O,H



# Recall The Happiness Problem

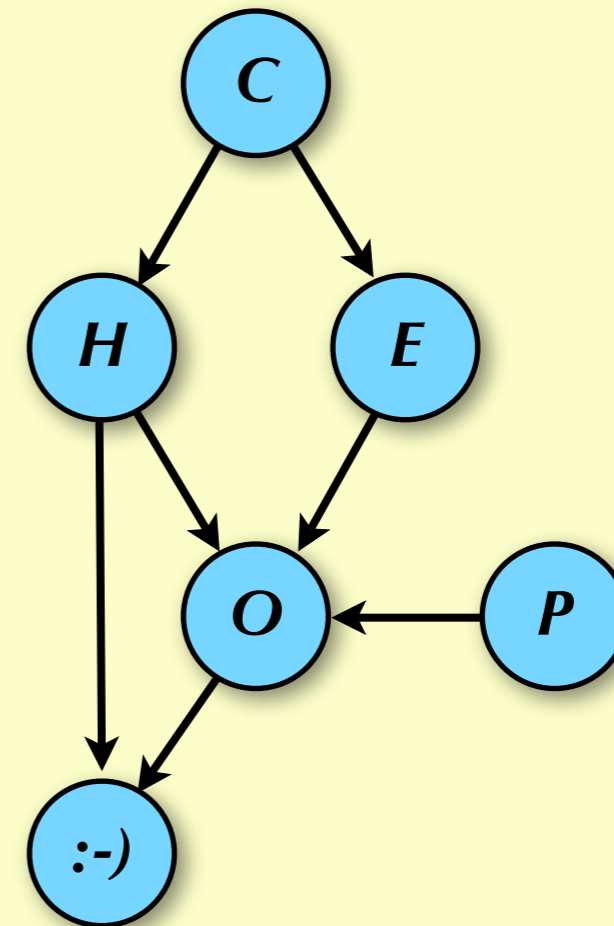
- We eliminated:
  - P,E,O,H
- Producing Functions





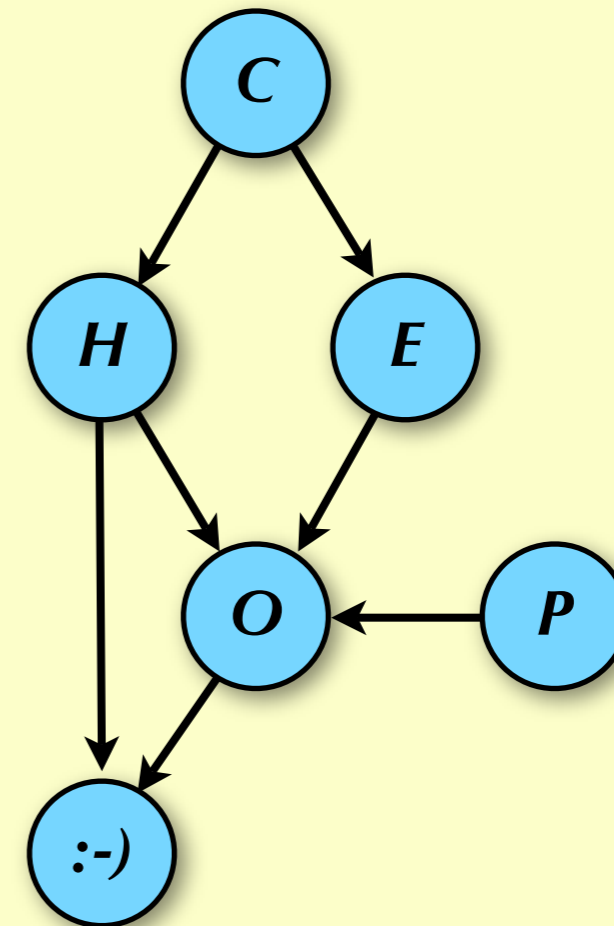
# Recall The Happiness Problem

- We eliminated:
  - P,E,O,H
- Producing Functions
  - $g_1(O,H,E)$



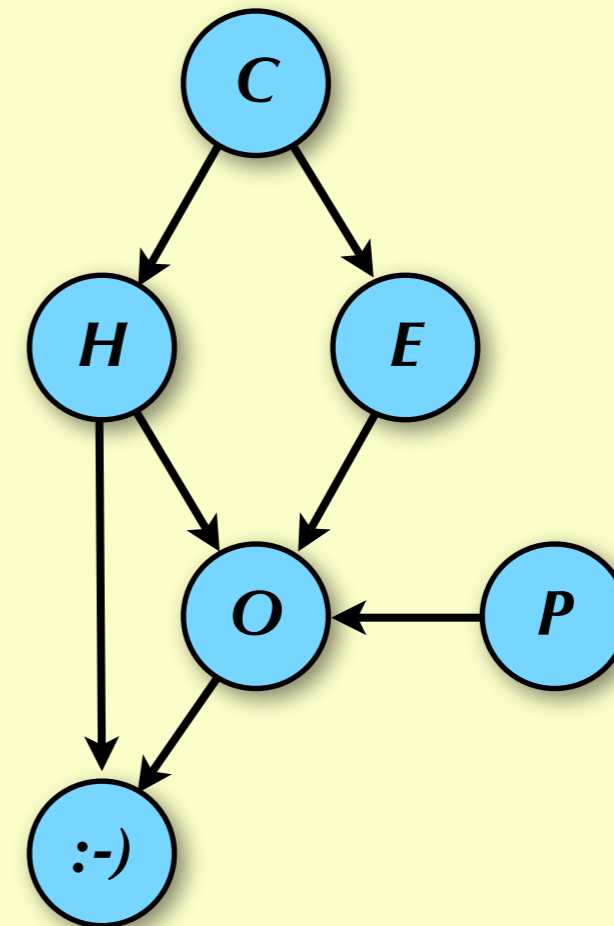
# Recall The Happiness Problem

- We eliminated:
  - P,E,O,H
- Producing Functions
  - $g_1(O,H,E)$
  - $g_2(O,H,C)$



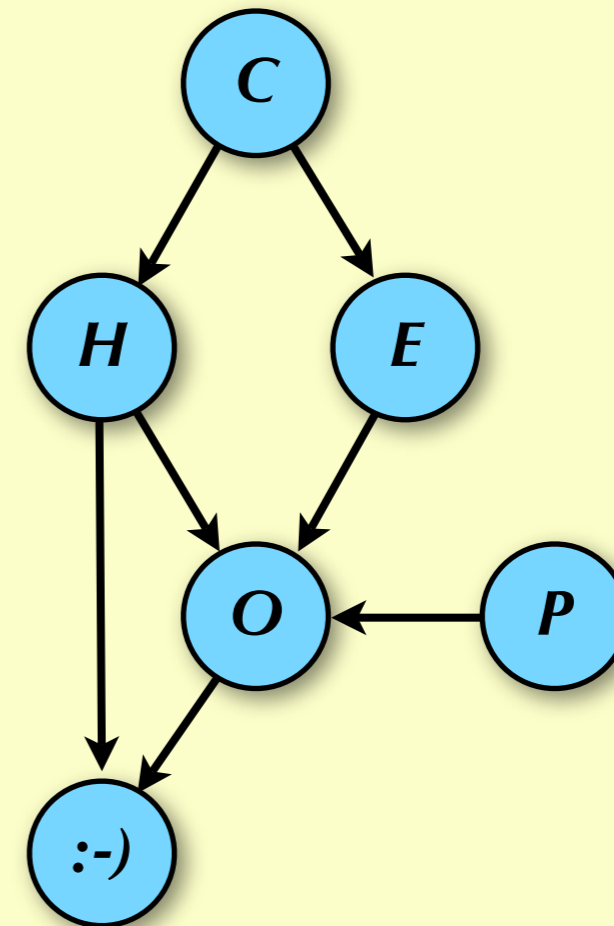
# Recall The Happiness Problem

- We eliminated:
  - P,E,O,H
- Producing Functions
  - $g_1(O,H,E)$
  - $g_2(O,H,C)$
  - $g_3(:-),H,C)$



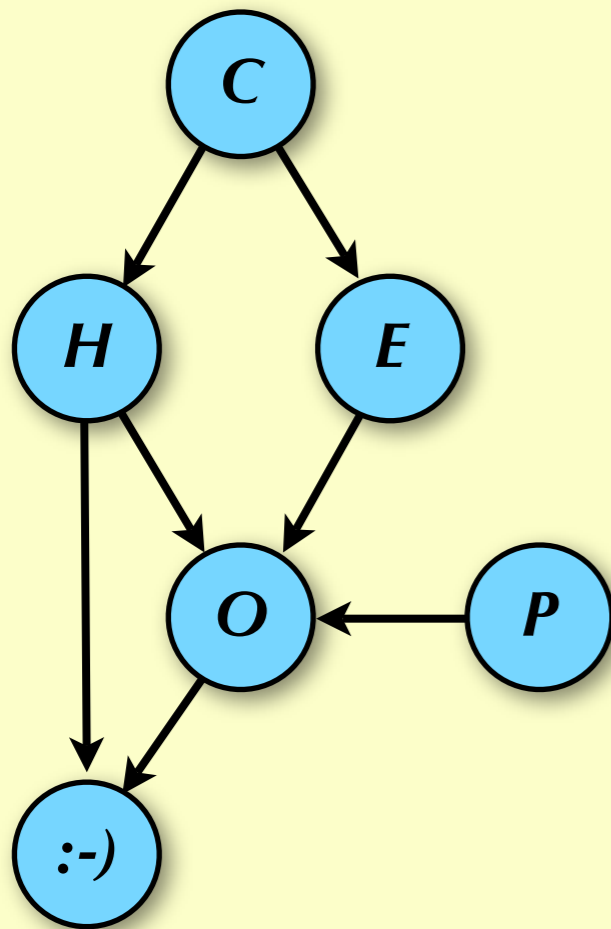
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- We eliminated:
  - $P, E, O, H$
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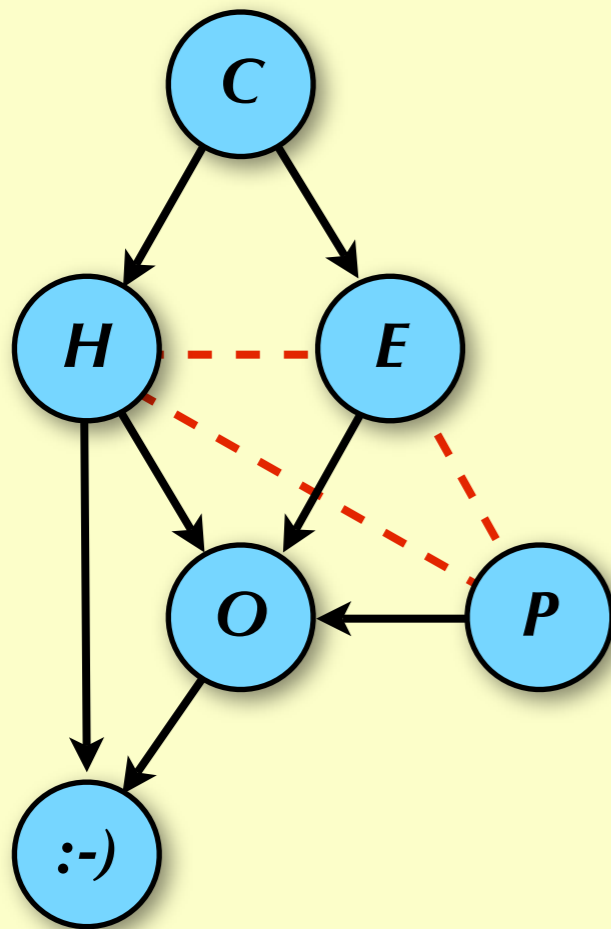
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- $P : g_1(O, H, E)$
- $E : g_2(O, H, C)$
- $O : g_3(:-, H, C)$
- $H : g_4(:-, C)$



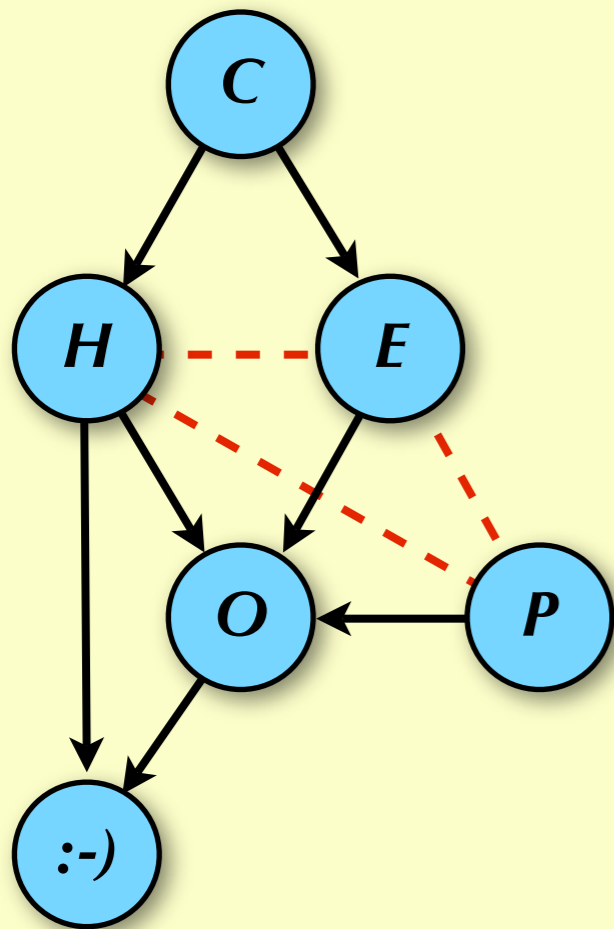
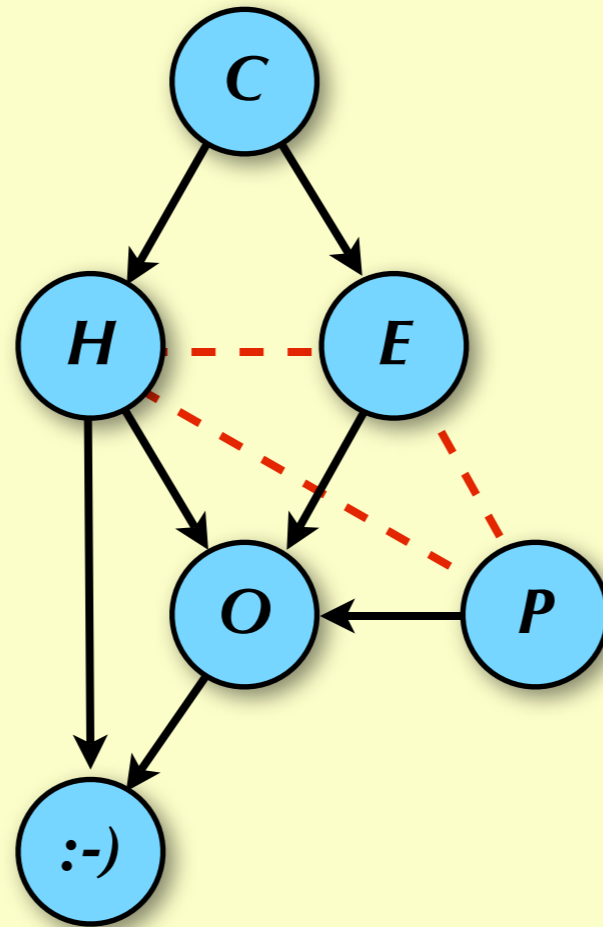
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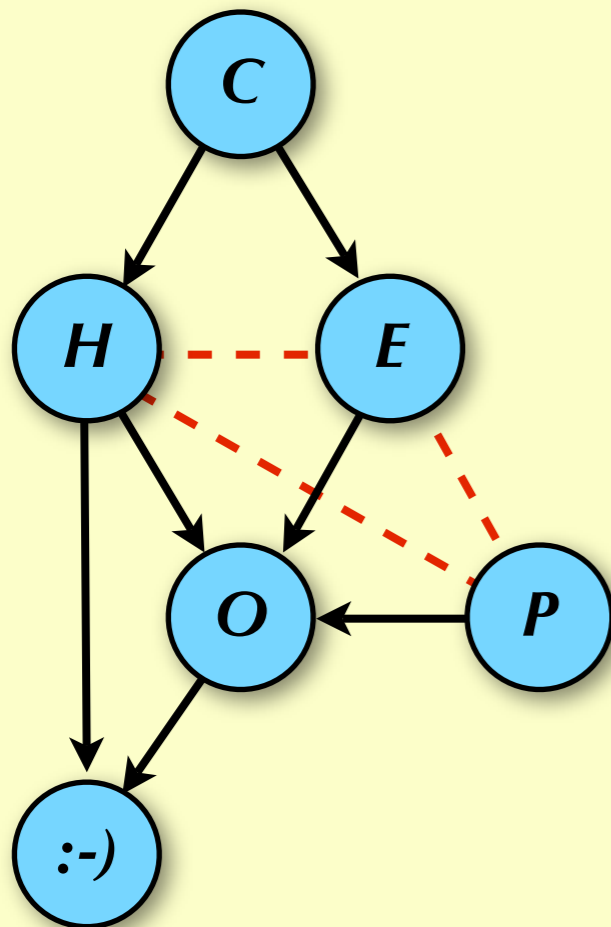
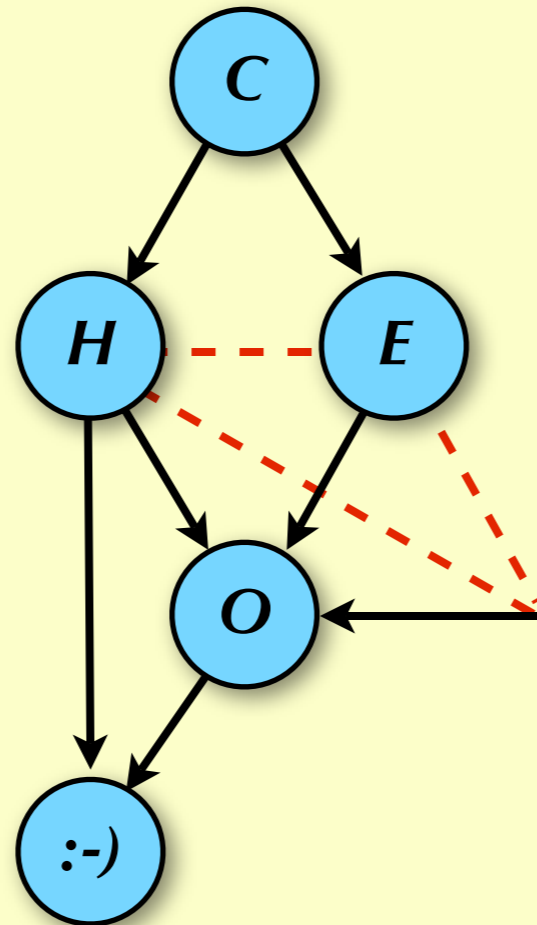
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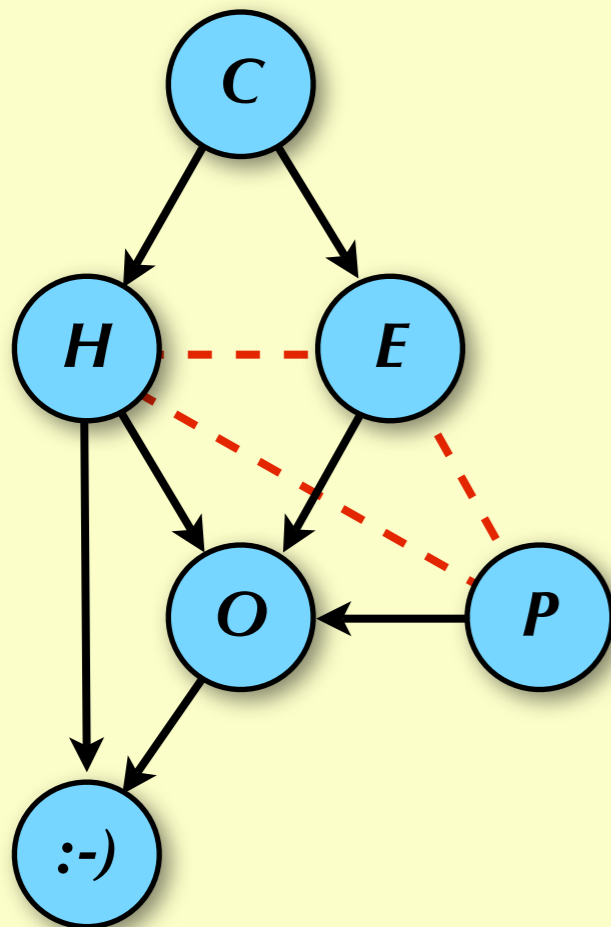
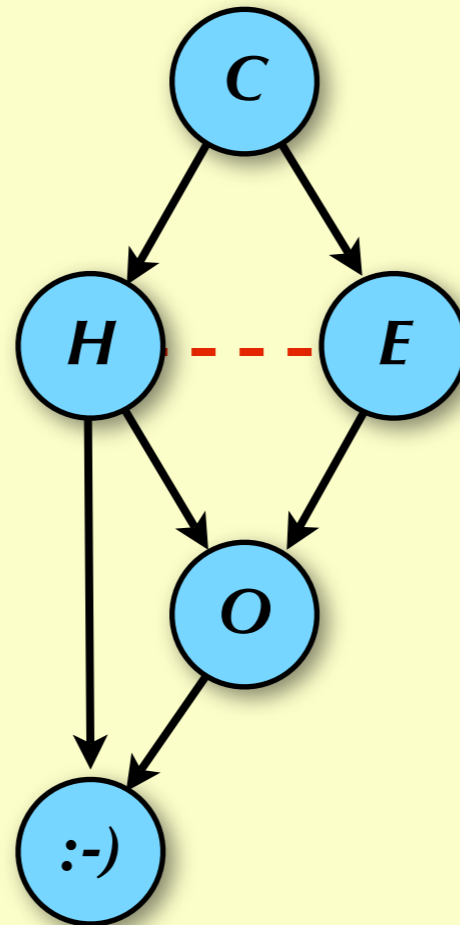
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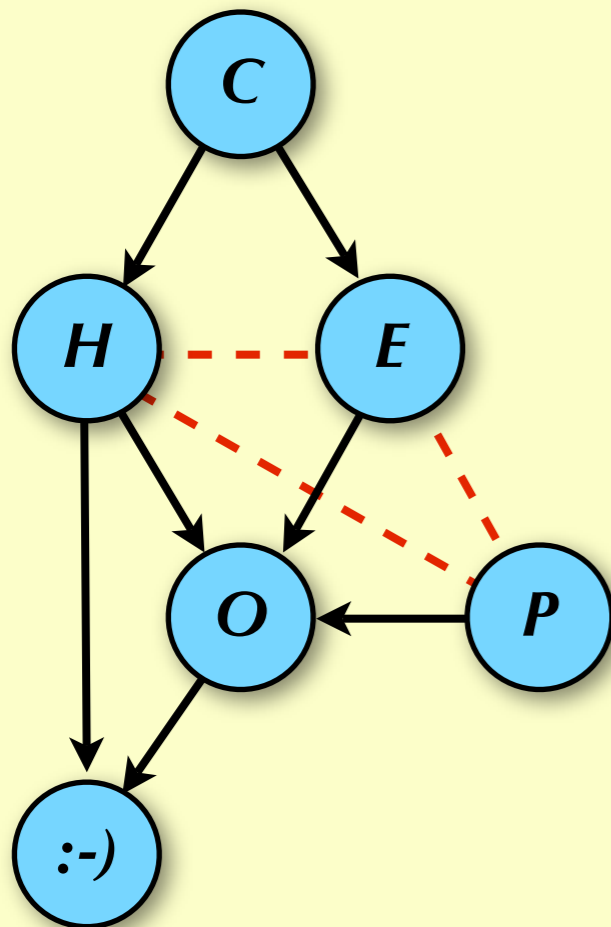
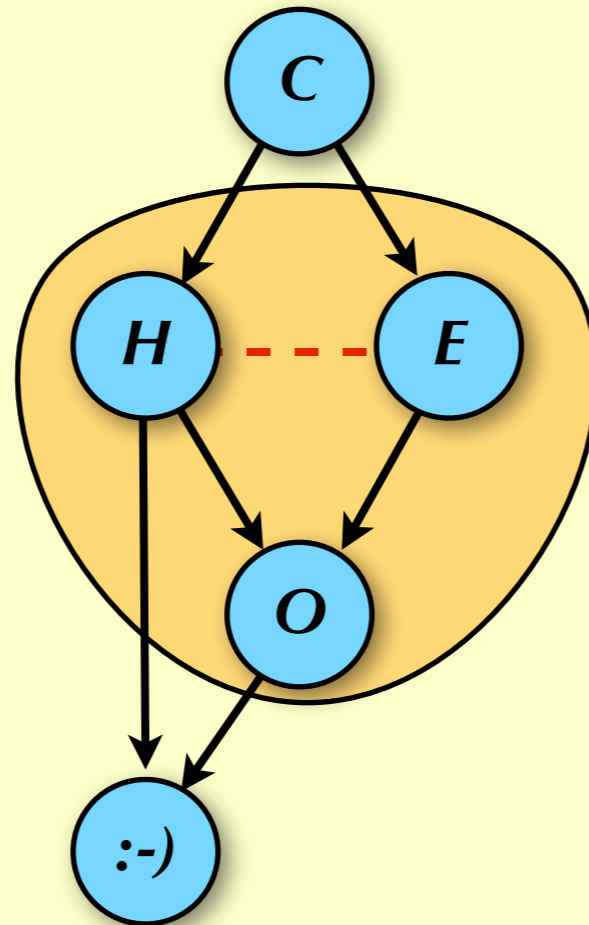
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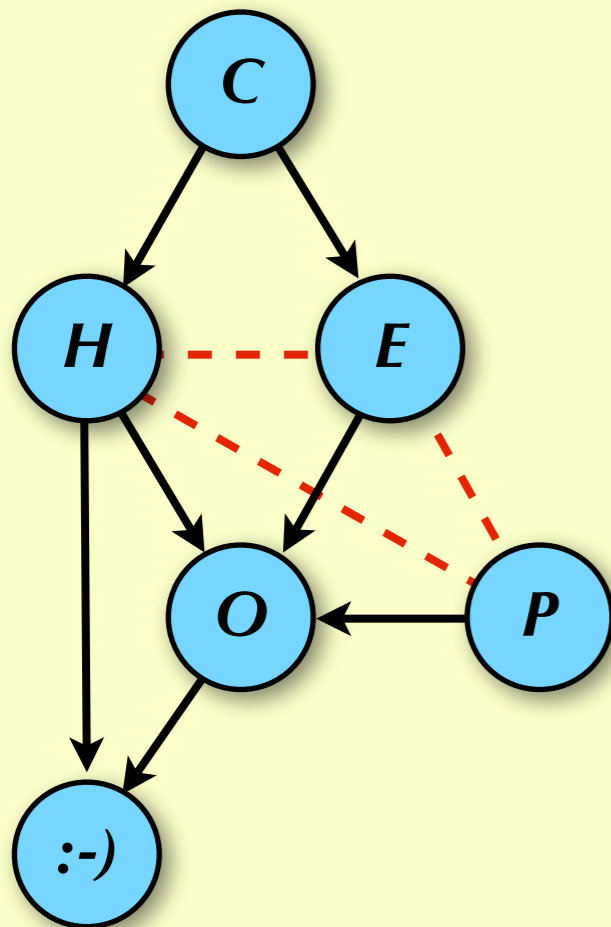
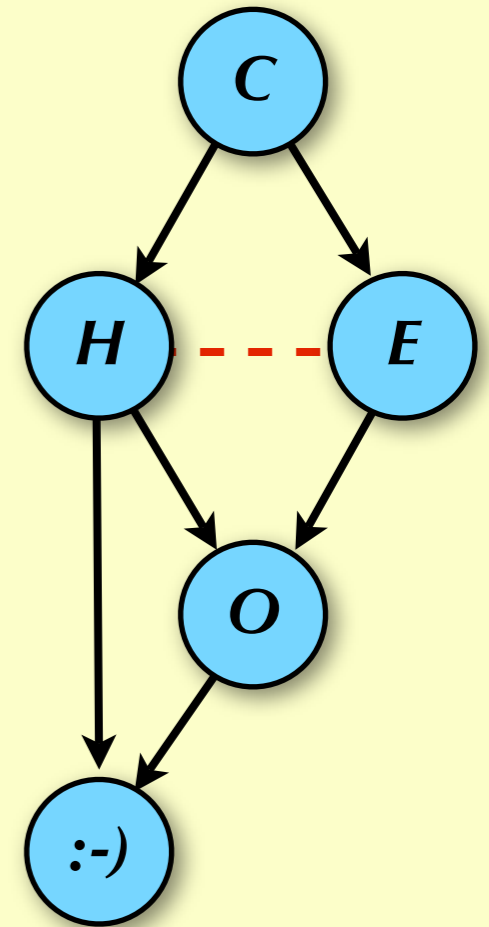
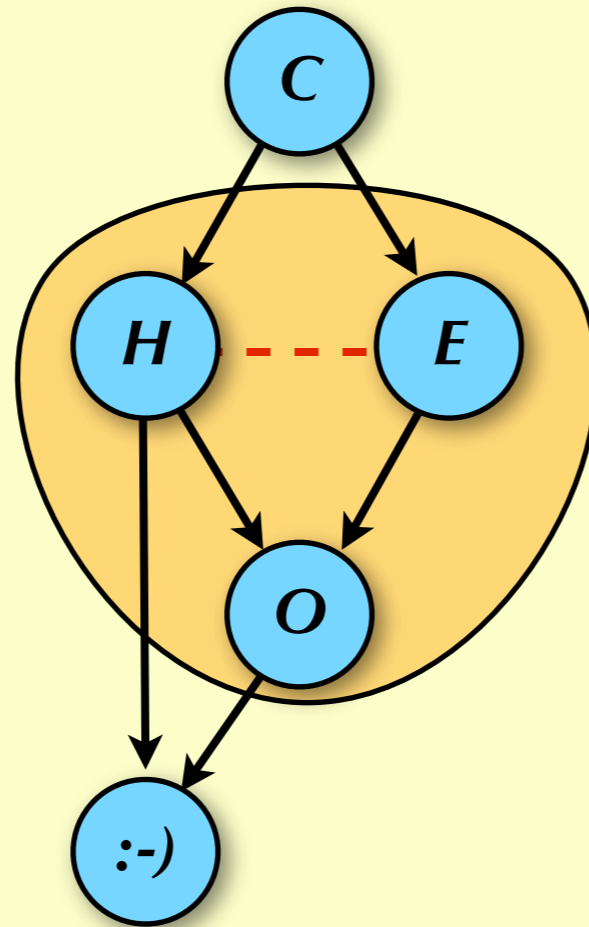
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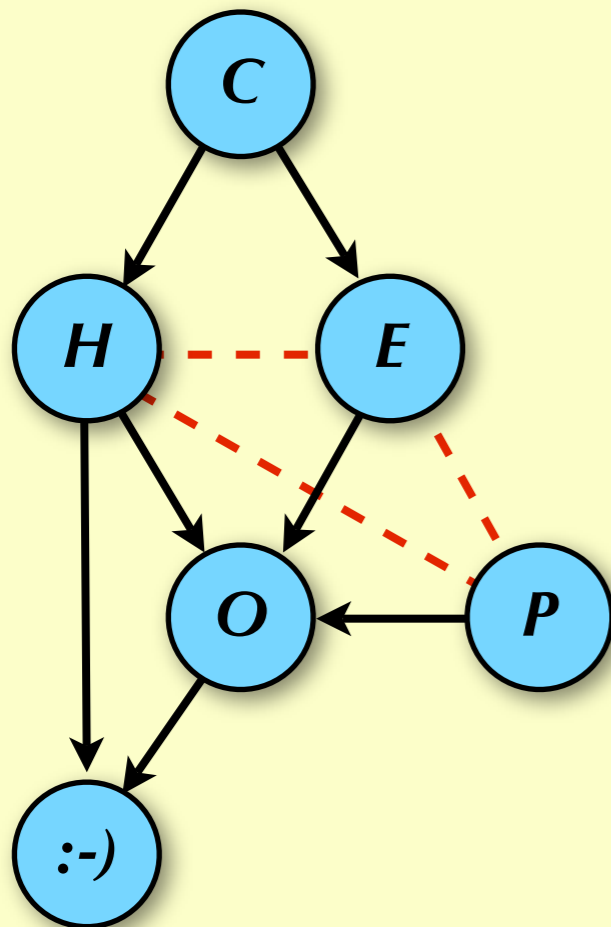
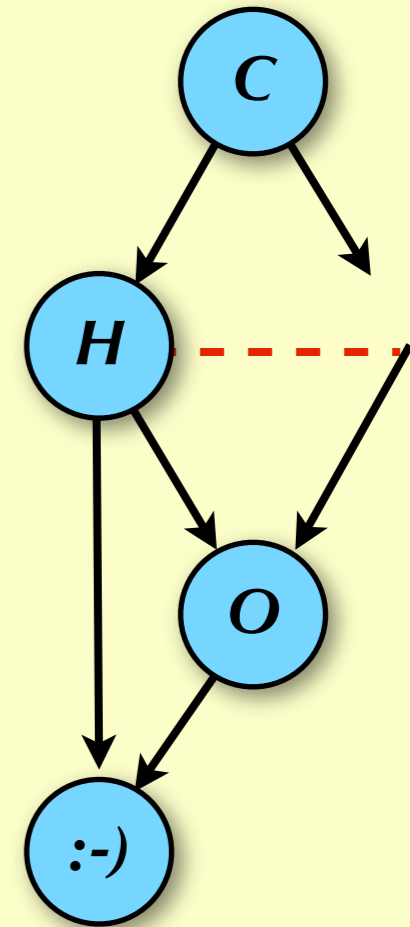
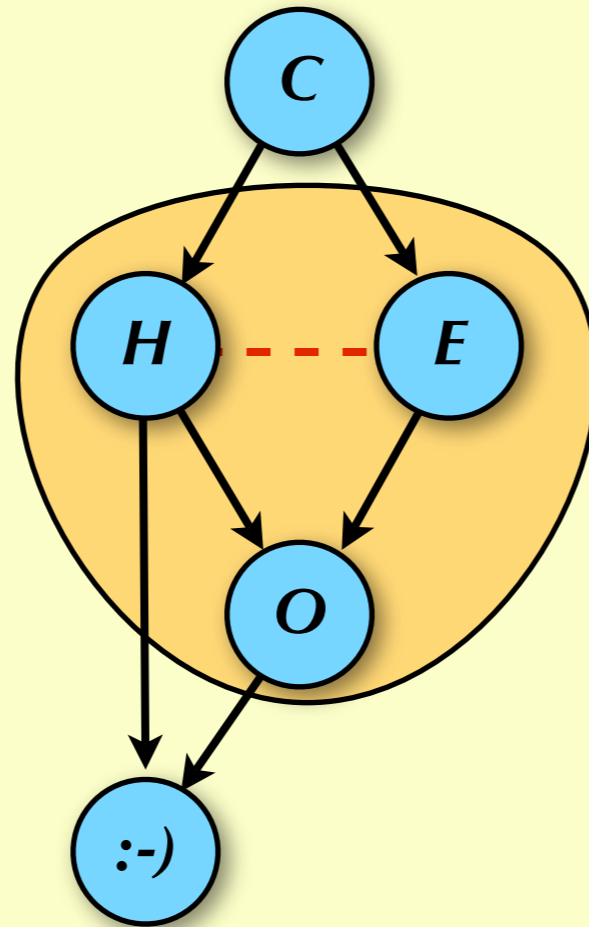
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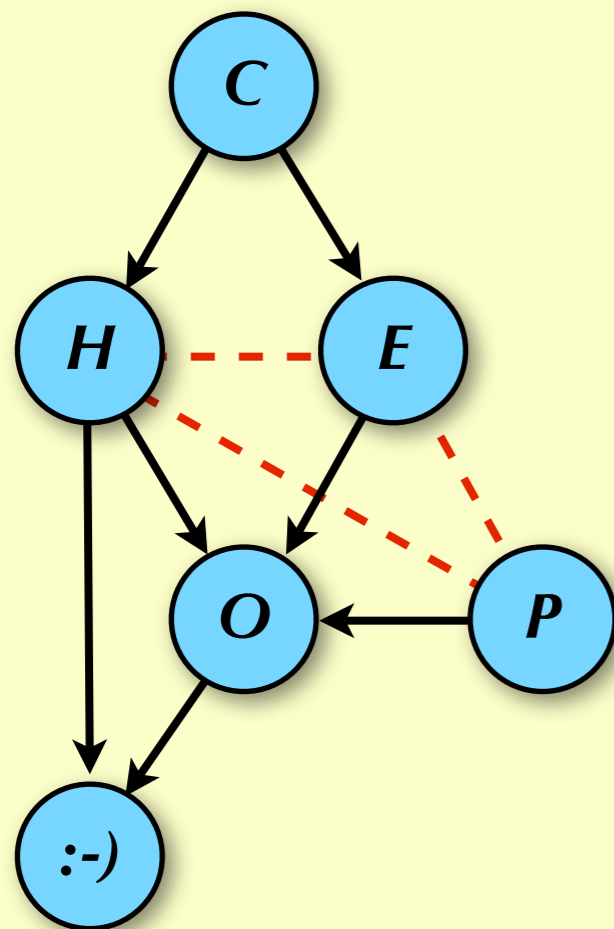
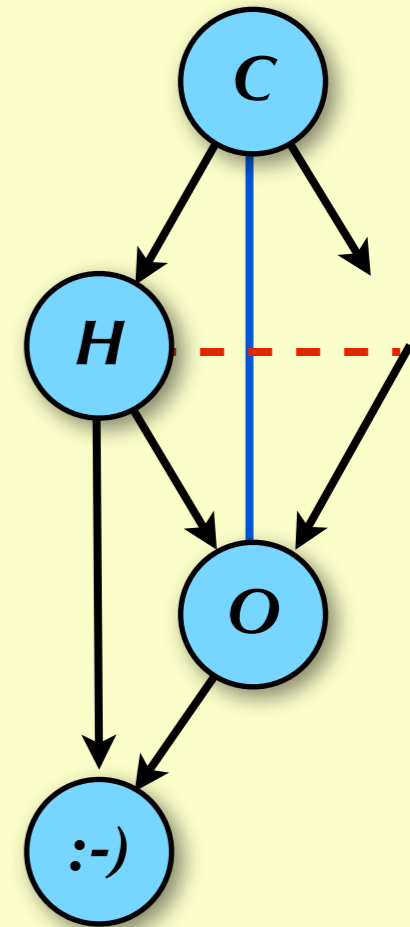
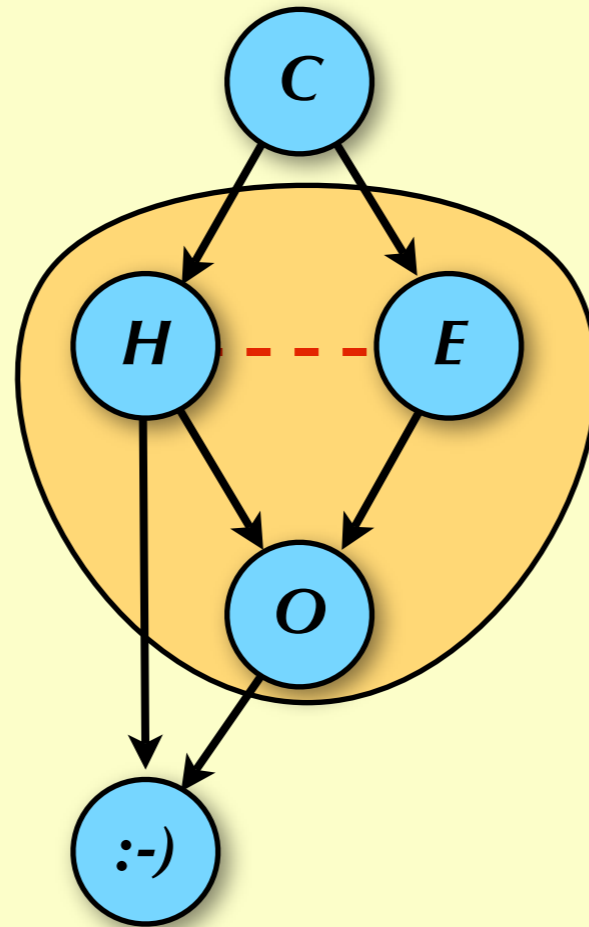
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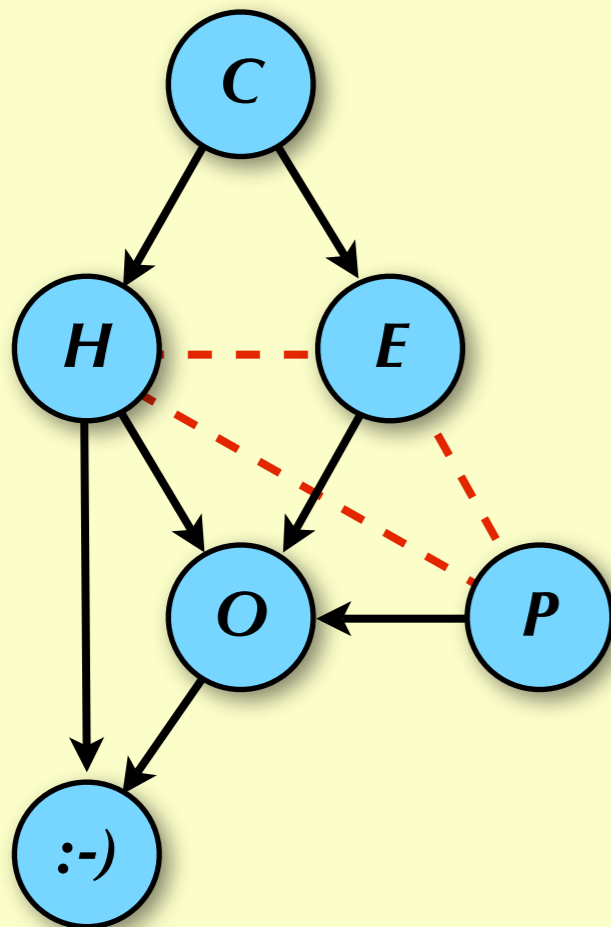
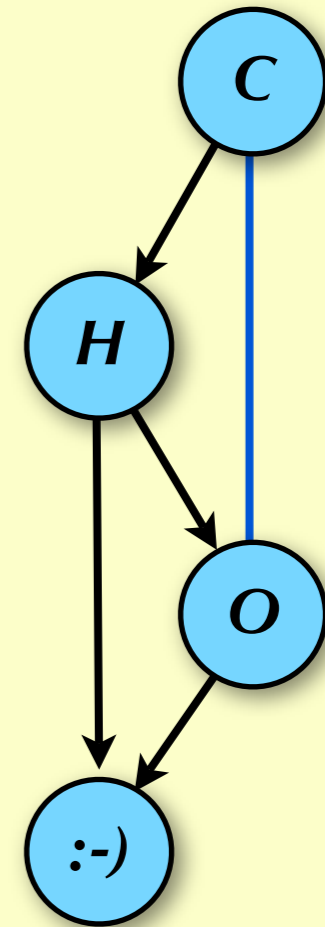
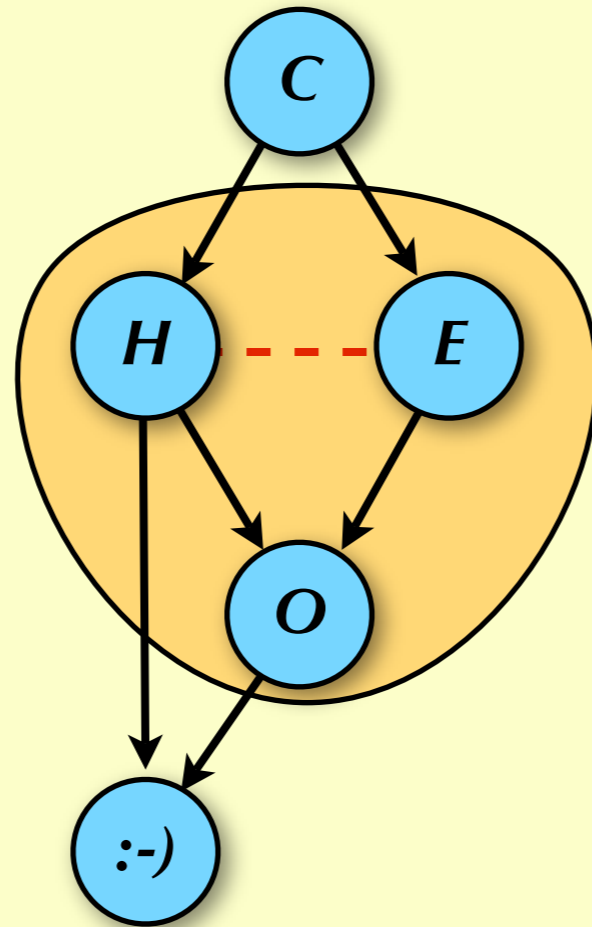
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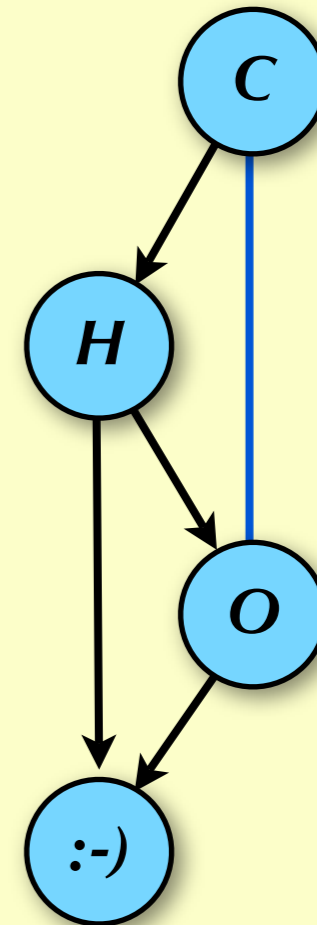
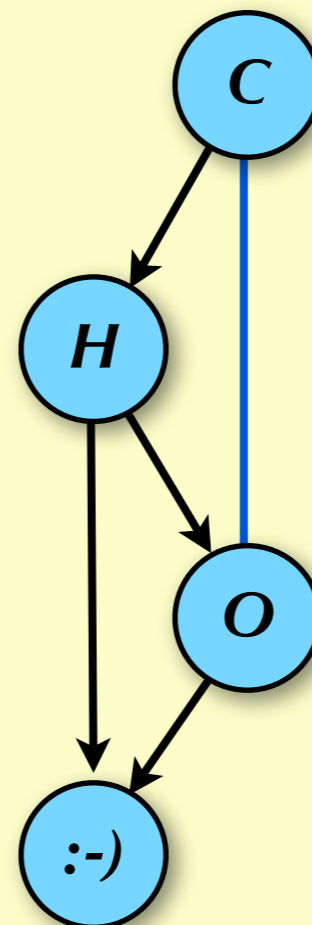
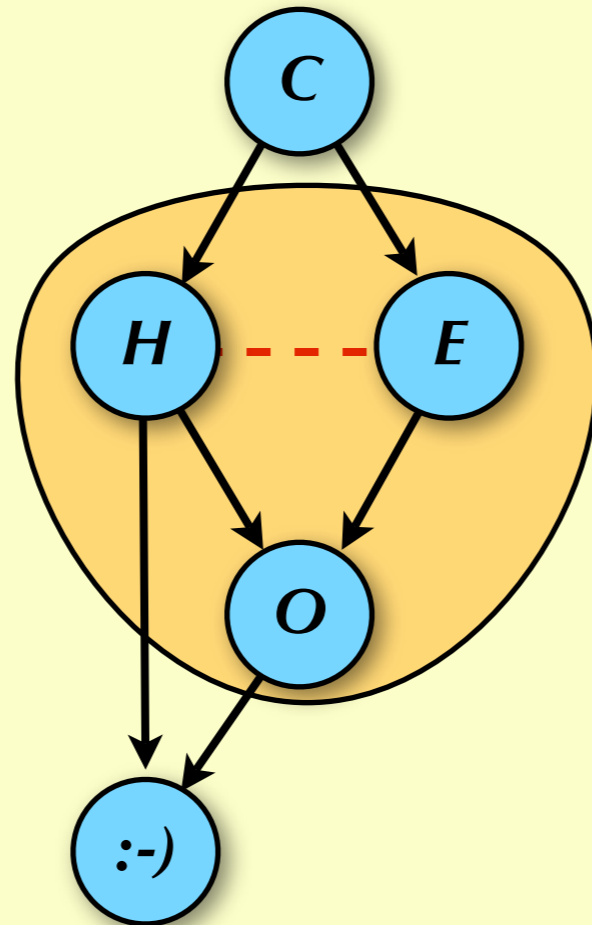
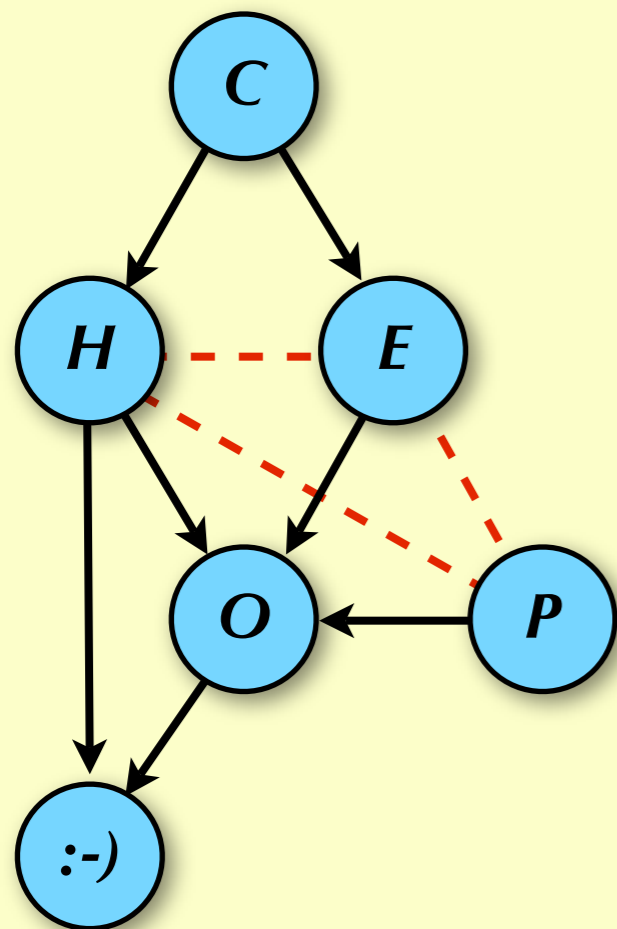
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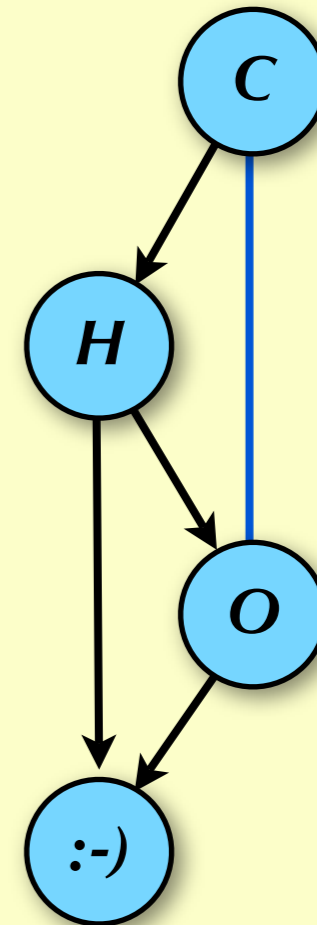
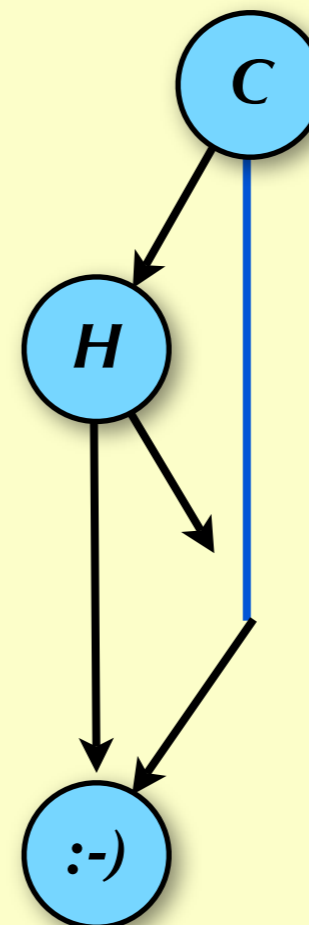
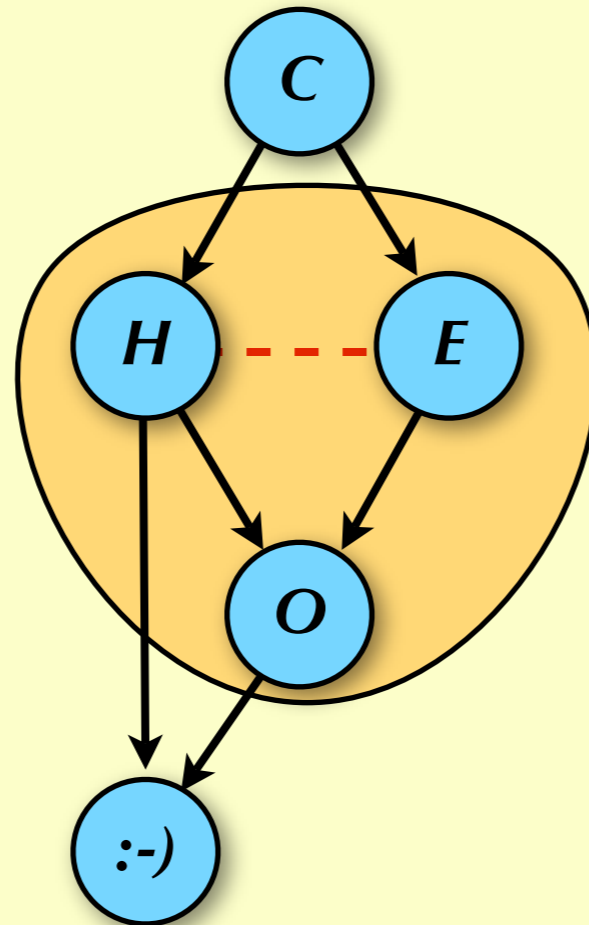
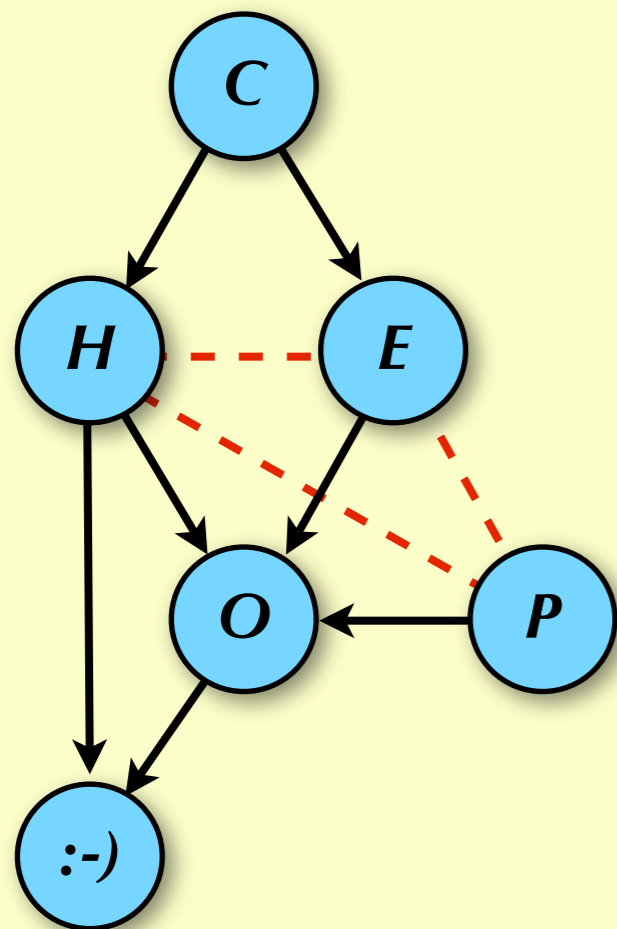
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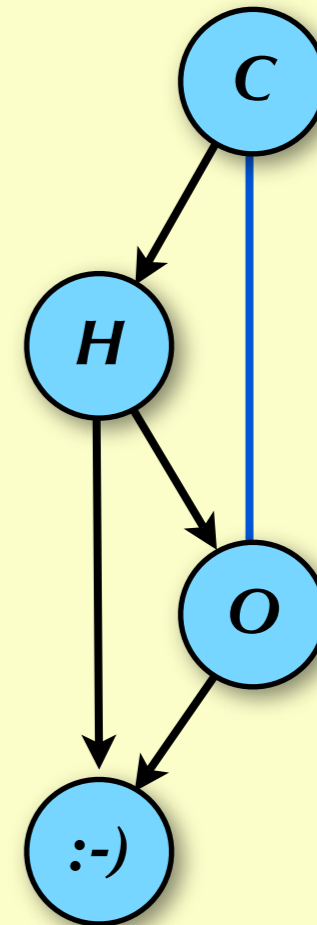
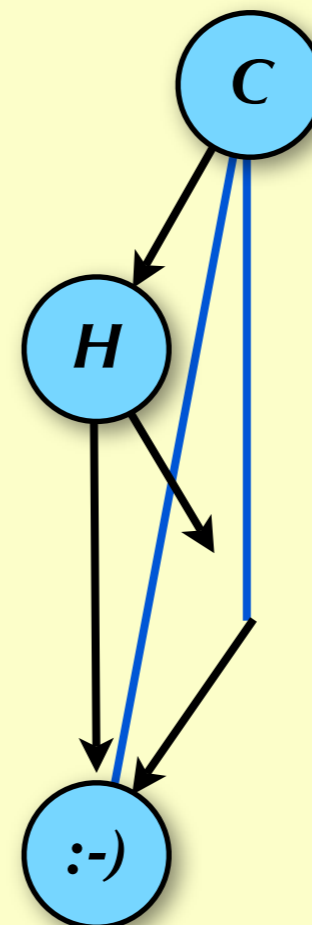
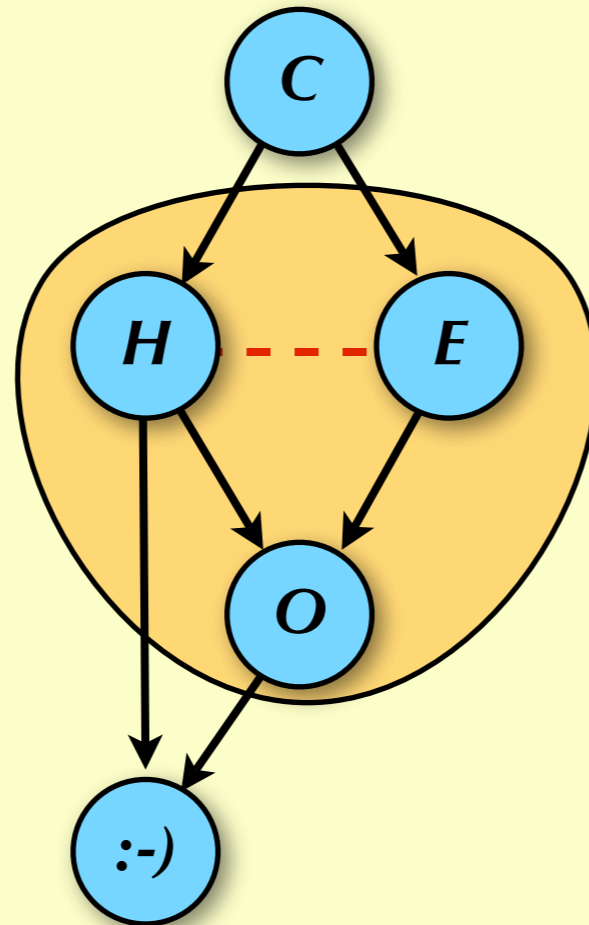
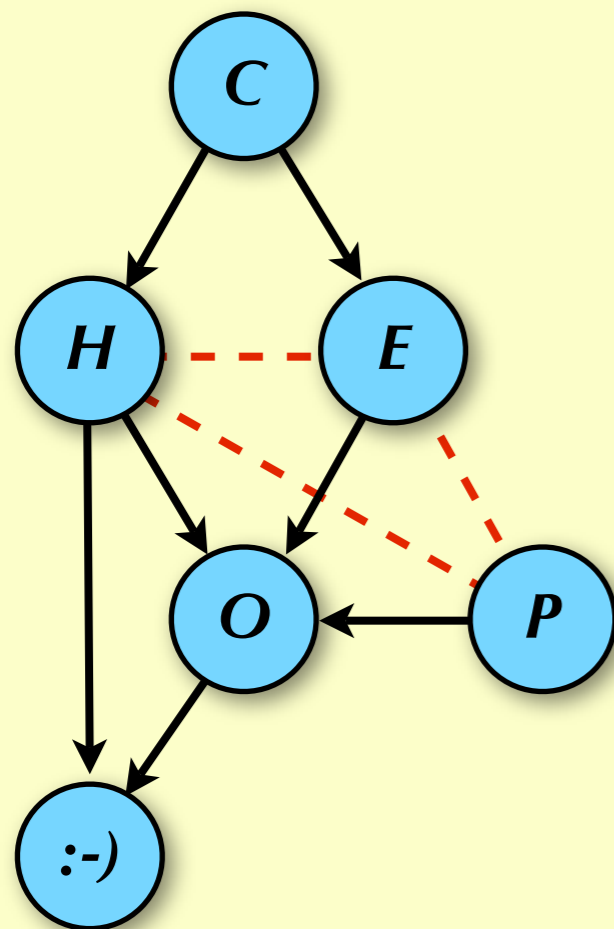
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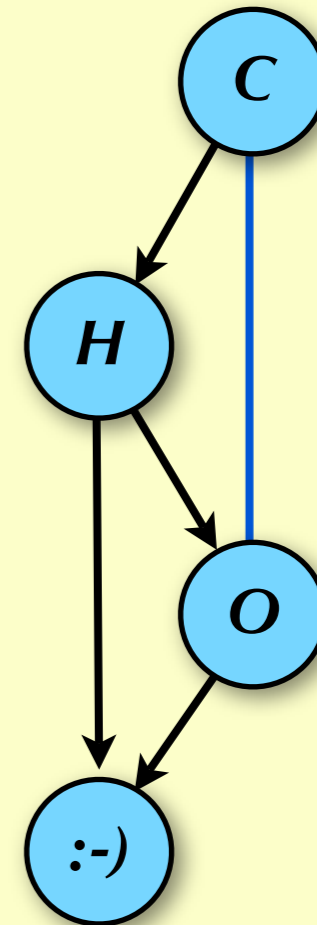
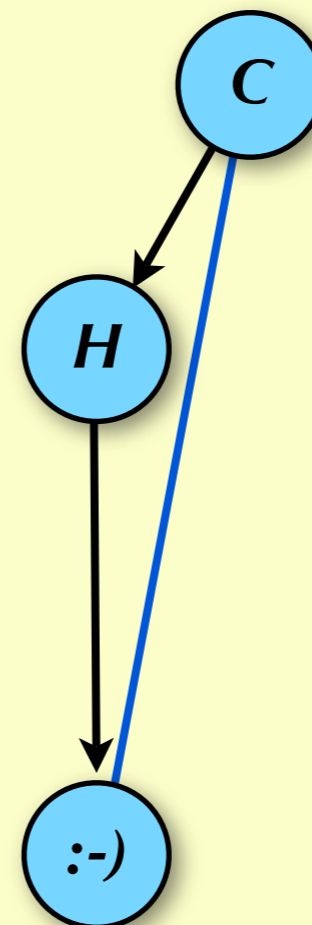
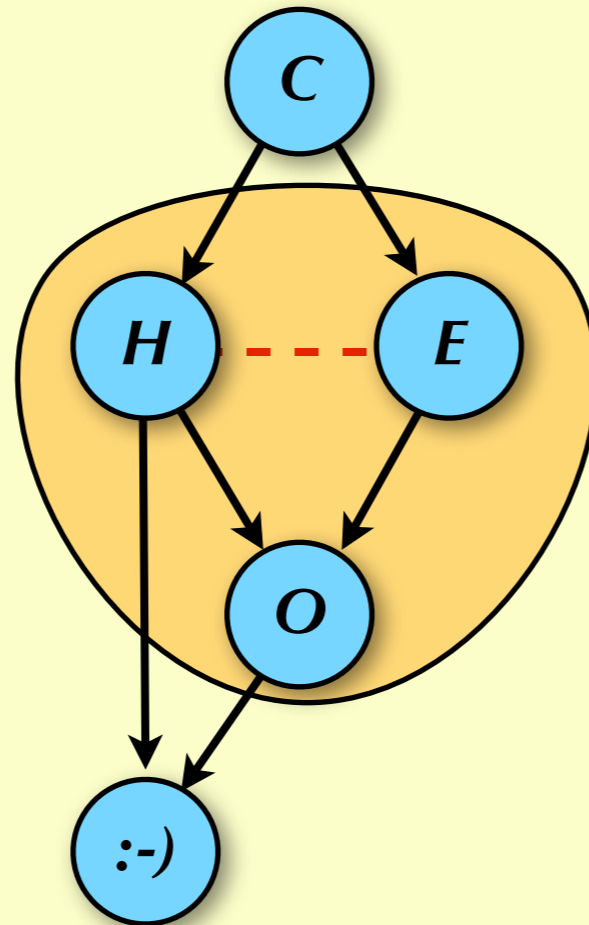
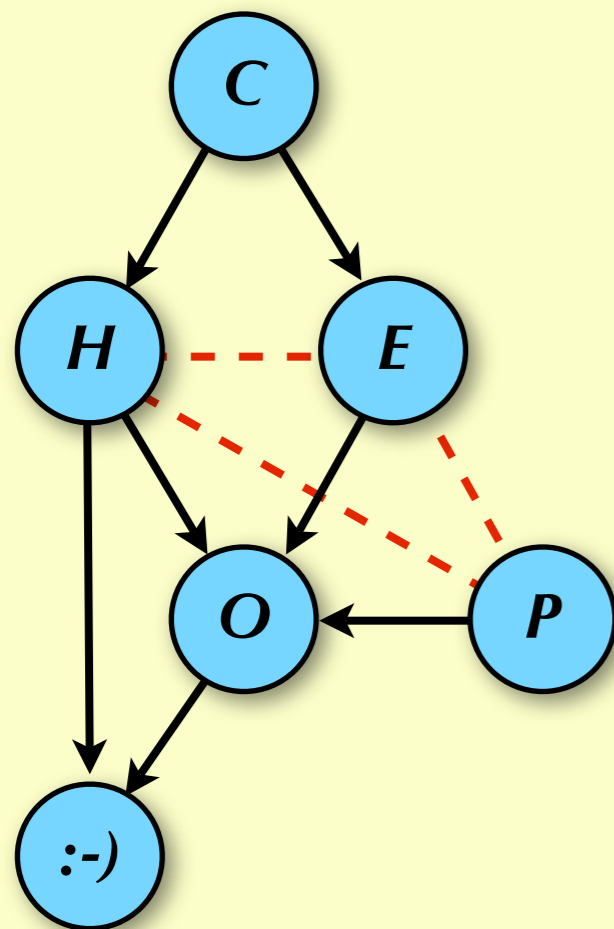
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# Summary

- Covered basic concepts of Variable Elimination
  - Worked through an example
  - Observed correspondence to graph operations
- Remember
  - There are other inference algorithms
  - Graphical models are a generic framework for modeling probability
    - Exploit structure (conditional independence) to manage representation and computational complexity
- Don't forget Midway milestone due Monday