Bits, Bytes, and Integers

15-213: Introduction to Computer Systems 2nd and 3rd Lectures, Jan 19 and Jan 24, 2012

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Today: Bits, Bytes, and Integers

- Representing information as bits
- Bit-level manipulations
- Integers
 - Representation: unsigned and signed
 - Conversion, casting
 - Expanding, truncating
 - Addition, negation, multiplication, shifting
 - Summary
- Representations in memory, pointers, strings

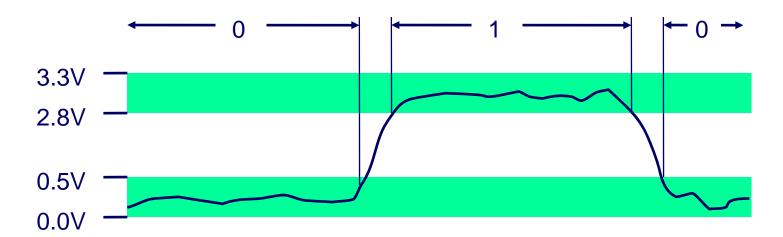
Binary Representations

Base 2 Number Representation

- Represent 15213₁₀ as 111011011011₂
- Represent 1.20₁₀ as 1.0011001100110011[0011]...₂
- Represent 1.5213 X 10⁴ as 1.1101101101101₂ X 2¹³

■ Electronic Implementation

- Easy to store with bistable elements
- Reliably transmitted on noisy and inaccurate wires



Encoding Byte Values

- Byte = 8 bits
 - Binary 000000002 to 1111111112
 - Decimal: 0₁₀ to 255₁₀
 - Hexadecimal 00₁₆ to FF₁₆
 - Base 16 number representation
 - Use characters '0' to '9' and 'A' to 'F'
 - Write FA1D37B₁₆ in C as
 - 0xFA1D37B
 - 0xfa1d37b

Hex Decimal Binary

0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
В	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Data Representations

C Data Type	Typical 32-bit	Intel IA32	x86-64
char	1	1	1
short	2	2	2
int	4	4	4
long	4	4	8
long long	8	8	8
float	4	4	4
double	8	8	8
long double	8	10/12	10/16
pointer	4	4	8

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Boolean Algebra

- Developed by George Boole in 19th Century
 - Algebraic representation of logic
 - Encode "True" as 1 and "False" as 0

And

Or

■ A&B = 1 when both A=1 and B=1

■ A | B = 1 when either A=1 or B=1

Not

Exclusive-Or (Xor)

~A = 1 when A=0

■ A^B = 1 when either A=1 or B=1, but not both

~	
0	1
1	0

General Boolean Algebras

- Operate on Bit Vectors
 - Operations applied bitwise

All of the Properties of Boolean Algebra Apply

Example: Representing & Manipulating Sets

Representation

- Width w bit vector represents subsets of {0, ..., w-1}
- $aj = 1 \text{ if } j \in A$
 - 01101001 { 0, 3, 5, 6 }
 - **76543210**
 - 01010101 { 0, 2, 4, 6 }
 - **76543210**

Operations

- &	Intersection	01000001	{ 0, 6 }
•	Union	01111101	{ 0, 2, 3, 4, 5, 6 }
■ ∧	Symmetric difference	00111100	{ 2, 3, 4, 5 }
■ ~	Complement	10101010	{ 1, 3, 5, 7 }

Bit-Level Operations in C

- Operations &, |, ~, ^ Available in C
 - Apply to any "integral" data type
 - long, int, short, char, unsigned
 - View arguments as bit vectors
 - Arguments applied bit-wise
- Examples (Char data type)
 - $\sim 0x41 \rightarrow 0xBE$
 - $\sim 01000001_2 \rightarrow 10111110_2$
 - $\sim 0x00 \rightarrow 0xFF$
 - $\sim 000000002 \rightarrow 1111111112$
 - $0x69 \& 0x55 \rightarrow 0x41$
 - $01101001_2 \& 01010101_2 \rightarrow 01000001_2$
 - $0x69 \mid 0x55 \rightarrow 0x7D$
 - $01101001_2 \mid 01010101_2 \rightarrow 011111101_2$

Contrast: Logic Operations in C

Contrast to Logical Operators

- **&**&, ||, !
 - View 0 as "False"
 - Anything nonzero as "True"
 - Always return 0 or 1
 - Early termination

Examples (char data type)

- $!0x41 \rightarrow 0x00$
- $!0x00 \rightarrow 0x01$
- $!!0x41 \rightarrow 0x01$
- $0x69 \&\& 0x55 \rightarrow 0x01$
- 0x69 || 0x55 → 0x01
- p && *p (avoids null pointer access)

Contrast: Logic Operations in C

Contrast to Logical Operators

- **&**&, ||, !
 - View 0 as "False"
 - Anything nonzero as "True"
 - Always return 0 or 1
 - Early termination

Examples (char data type)

- $!0x41 \rightarrow 0x00$
- $!0x00 \rightarrow 0x01$
- $!!0x41 \rightarrow 0x01$
- $0x69 \&\& 0x55 \rightarrow 0x01$
- 0x69 || 0x55 → 0x01
- p && *p (avoids null pointer access)

Watch out for && vs. & (and || vs. |)... one of the more common oopsies in C programming

Shift Operations

■ Left Shift: x << y

- Shift bit-vector x left y positions
 - Throw away extra bits on left
 - Fill with 0's on right

Right Shift: x >> y

- Shift bit-vector x right y positions
 - Throw away extra bits on right
- Logical shift
 - Fill with 0's on left
- Arithmetic shift
 - Replicate most significant bit on left

Hnd	ofina	d Ra	havior
Ullu	enne	u bei	llavioi

Shift amount < 0 or ≥ word size</p>

Argument x	01100010	
<< 3	00010 <i>000</i>	
Log. >> 2	00011000	
Arith. >> 2	00011000	

Argument x	10100010	
<< 3	00010 <i>000</i>	
Log. >> 2	00101000	
Arith. >> 2	<i>11</i> 101000	

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Encoding Integers

Unsigned

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

Two's Complement

$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

short int
$$x = 15213$$
;
short int $y = -15213$;

Sign Bit

C short 2 bytes long

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
У	-15213	C4 93	11000100 10010011

■ Sign Bit

- For 2's complement, most significant bit indicates sign
 - 0 for nonnegative
 - 1 for negative

Encoding Example (Cont.)

x = 15213: 00111011 01101101

y = -15213: 11000100 10010011

		·		
Weight	152	213	-152	213
1	1	1	1	1
2	0	0	1	2
4	1	4	0	0
8	1	8	0	0
16	0	0	1	16
32	1	32	0	0
64	1	64	0	0
128	0	0	1	128
256	1	256	0	0
512	1	512	0	0
1024	0	0	1	1024
2048	1	2048	0	0
4096	1	4096	0	0
8192	1	8192	0	0
16384	0	0	1	16384
-32768	0	0	1	-32768
C		45343		45242

Sum 15213 -15213

Numeric Ranges

Unsigned Values

•
$$UMax = 2^w - 1$$
111...1

■ Two's Complement Values

■
$$TMin = -2^{w-1}$$
100...0

■
$$TMax = 2^{w-1} - 1$$

011...1

Other Values

Minus 1111...1

Values for W = 16

	Decimal	Hex	Binary
UMax	65535	FF FF	11111111 11111111
TMax	32767	7F FF	01111111 11111111
TMin	-32768	80 00	10000000 000000000
-1	-1	FF FF	11111111 11111111
0	0	00 00	00000000 00000000

Values for Different Word Sizes

			W	
	8	16	32	64
UMax	255	65,535	4,294,967,295	18,446,744,073,709,551,615
TMax	127	32,767	2,147,483,647	9,223,372,036,854,775,807
TMin	-128	-32,768	-2,147,483,648	-9,223,372,036,854,775,808

Observations

- \blacksquare | TMin | = TMax + 1
 - Asymmetric range
- UMax = 2 * TMax + 1

C Programming

- #include #include *
- Declares constants, e.g.,
 - ULONG_MAX
 - LONG_MAX
 - LONG_MIN
- Values platform specific

Unsigned & Signed Numeric Values

Χ	B2U(<i>X</i>)	B2T(<i>X</i>)
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	-8
1001	9	- 7
1010	10	- 6
1011	11	- 5
1100	12	-4
1101	13	-3
1110	14	-2
1111	15	-1

Equivalence

Same encodings for nonnegative values

Uniqueness

- Every bit pattern represents unique integer value
- Each representable integer has unique bit encoding

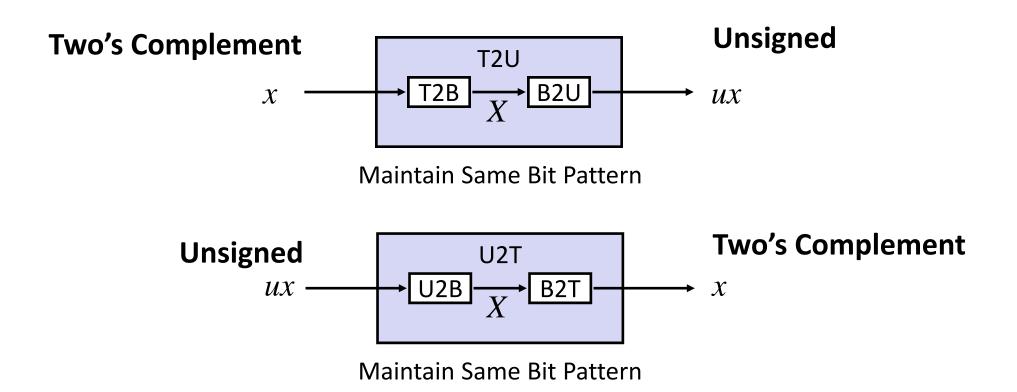
■ ⇒ Can Invert Mappings

- $U2B(x) = B2U^{-1}(x)$
 - Bit pattern for unsigned integer
- T2B(x) = B2T⁻¹(x)
 - Bit pattern for two's comp integer

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Mapping Between Signed & Unsigned

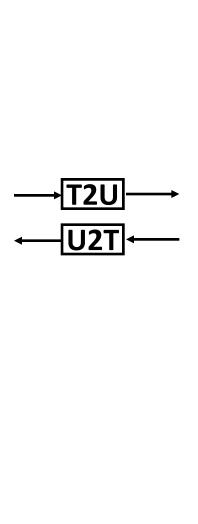


Mappings between unsigned and two's complement numbers:
 keep bit representations and reinterpret

Mapping Signed ↔ **Unsigned**

Bits
0000
0001
0010
0011
0100
0101
0110
0111
1000
1001
1010
1011
1100
1101
1110
1111

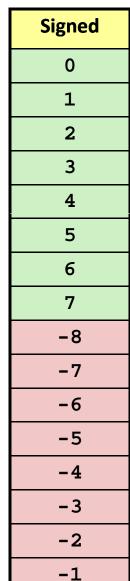
Signed
0
1
2
3
4
5
6
7
-8
-7
-6
- 5
-4
-3
-2
-1

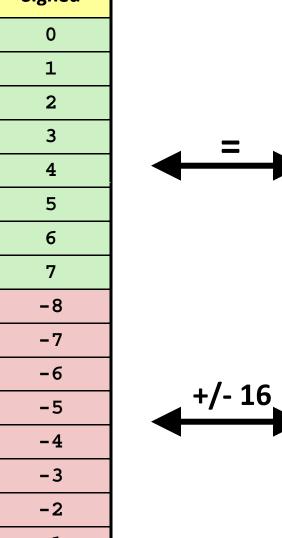


Unsigned	
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	

Mapping Signed ↔ **Unsigned**

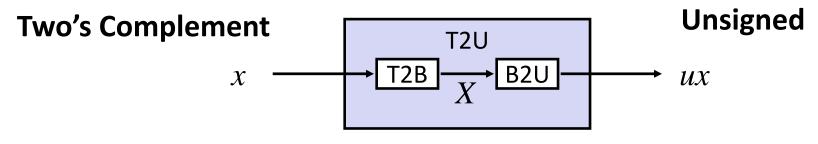
Bits
0000
0001
0010
0011
0100
0101
0110
0111
1000
1001
1010
1011
1100
1101
1110
1111



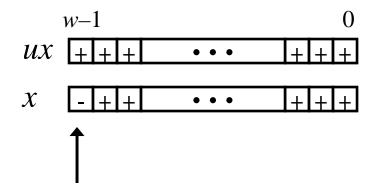


Unsigned
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15

Relation between Signed & Unsigned







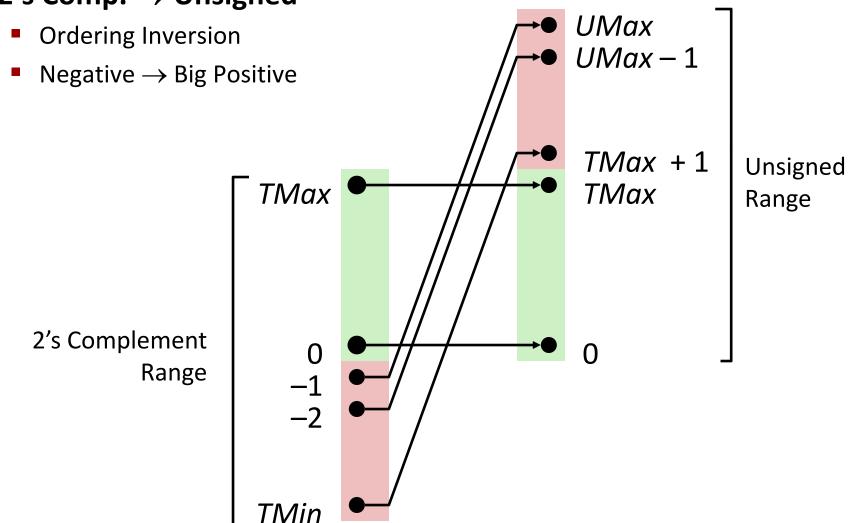
Large negative weight

becomes

Large positive weight

Conversion Visualized

2's Comp. → Unsigned



Signed vs. Unsigned in C

Constants

- By default are considered to be signed integers
- Unsigned if have "U" as suffix

```
OU, 4294967259U
```

Casting

Explicit casting between signed & unsigned same as U2T and T2U

```
int tx, ty;
unsigned ux, uy;
tx = (int) ux;
uy = (unsigned) ty;
```

Implicit casting also occurs via assignments and procedure calls

```
tx = ux;

uy = ty;
```

Casting Surprises

Expression Evaluation

- If there is a mix of unsigned and signed in single expression, signed values implicitly cast to unsigned
- Including comparison operations <, >, ==, <=, >=
- **Examples for** W = 32: **TMIN = -2,147,483,648**, **TMAX = 2,147,483,647**

■ Constant ₁	Constant ₂	Relation	Evaluation
0	0U	==	unsigned
-1	0	<	signed
-1	0U	>	unsigned
2147483647	-2147483647-1	>	signed
2147483647U	-2147483647-1	<	unsigned
-1	-2	>	signed
(unsigned)-1	-2	>	unsigned
2147483647	2147483648U	<	unsigned
2147483647	(int) 2147483648U	>	signed

Summary Casting Signed ↔ Unsigned: Basic Rules

- Bit pattern is maintained
- But reinterpreted
- Can have unexpected effects: adding or subtracting 2^w
- Expression containing signed and unsigned int
 - int is cast to unsigned!!

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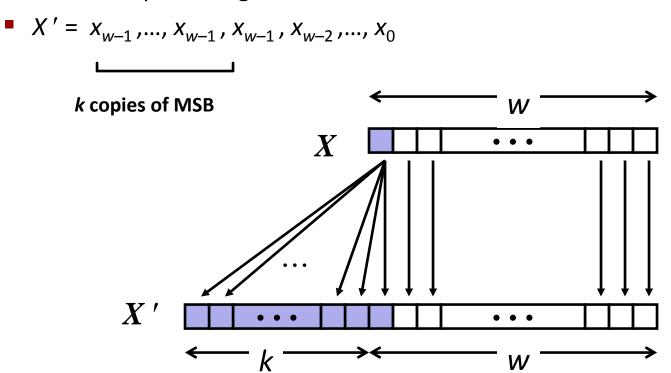
Sign Extension

■ Task:

- Given w-bit signed integer x
- Convert it to w+k-bit integer with same value

Rule:

Make k copies of sign bit:



Sign Extension Example

```
short int x = 15213;
int         ix = (int) x;
short int y = -15213;
int         iy = (int) y;
```

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
ix	15213	00 00 3B 6D	00000000 00000000 00111011 01101101
У	-15213	C4 93	11000100 10010011
iy	-15213	FF FF C4 93	11111111 11111111 11000100 10010011

- Converting from smaller to larger integer data type
- C automatically performs sign extension

Summary: Expanding, Truncating: Basic Rules

- Expanding (e.g., short int to int)
 - Unsigned: zeros added
 - Signed: sign extension
 - Both yield expected result
- Truncating (e.g., unsigned to unsigned short)
 - Unsigned/signed: bits are truncated
 - Result reinterpreted
 - Unsigned: mod operation
 - Signed: similar to mod
 - For small numbers yields expected behavour

Today: Bits, Bytes, and Integers

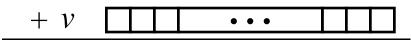
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Unsigned Addition

Operands: w bits

u •••

True Sum: w+1 bits



Discard Carry: w bits

$$UAdd_{w}(u, v)$$

u + v



Standard Addition Function

- Ignores carry output
- Implements Modular Arithmetic

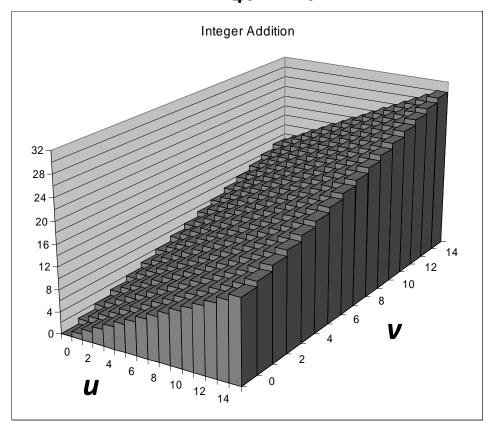
$$s = UAdd_w(u, v) = u + v \mod 2^w$$

Visualizing (Mathematical) Integer Addition

■ Integer Addition

- 4-bit integers u, v
- Compute true sum $Add_4(u, v)$
- Values increase linearly with u and v
- Forms planar surface

$Add_4(u, v)$

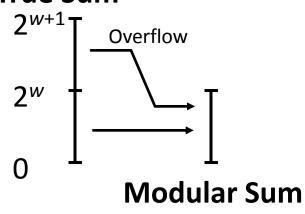


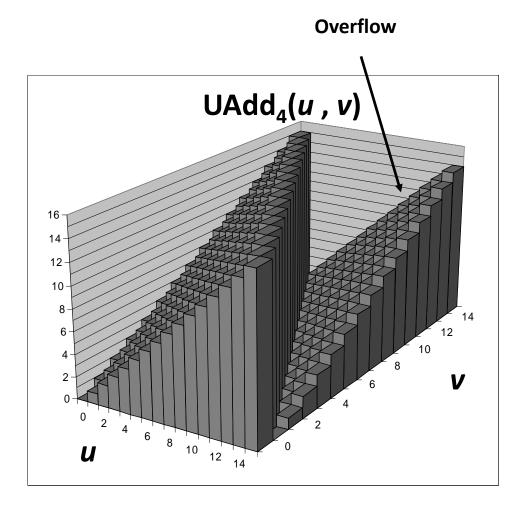
Visualizing Unsigned Addition

Wraps Around

- If true sum $\ge 2^w$
- At most once

True Sum





Two's Complement Addition

Discard Carry: w bits $TAdd_{w}(u, v)$

■ TAdd and UAdd have Identical Bit-Level Behavior

Signed vs. unsigned addition in C:

```
int s, t, u, v;
s = (int) ((unsigned) u + (unsigned) v);
t = u + v
```

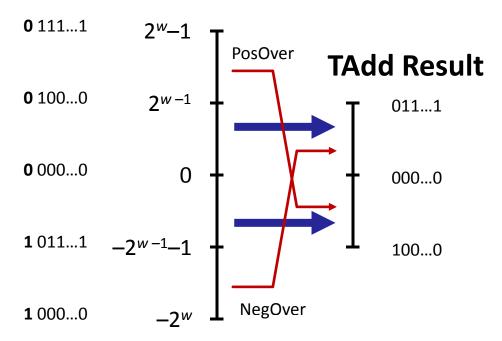
Will give s == t

TAdd Overflow

Functionality

- True sum requires w+1 bits
- Drop off MSB
- Treat remaining bits as 2's comp. integer

True Sum



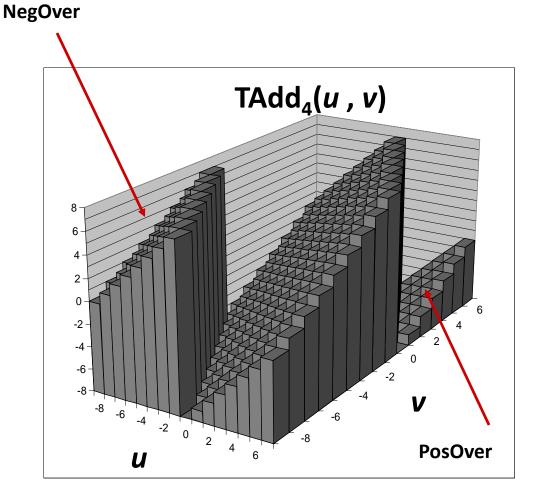
Visualizing 2's Complement Addition

Values

- 4-bit two's comp.
- Range from -8 to +7

Wraps Around

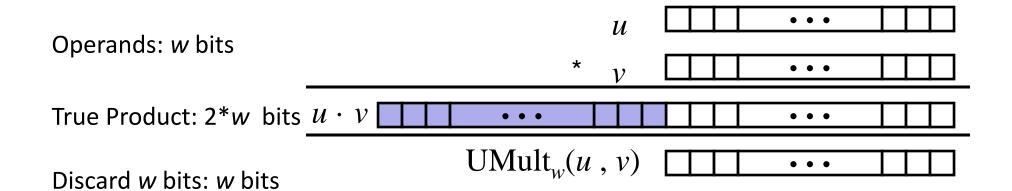
- If sum $\geq 2^{w-1}$
 - Becomes negative
 - At most once
- If sum $< -2^{w-1}$
 - Becomes positive
 - At most once



Multiplication

- Goal: Computing Product of w-bit numbers x, y
 - Either signed or unsigned
- But, exact results can be bigger than w bits
 - Unsigned: up to 2w bits
 - Result range: $0 \le x * y \le (2^w 1)^2 = 2^{2w} 2^{w+1} + 1$
 - Two's complement min (negative): Up to 2w-1 bits
 - Result range: $x * y \ge (-2^{w-1})*(2^{w-1}-1) = -2^{2w-2} + 2^{w-1}$
 - Two's complement max (positive): Up to 2w bits, but only for $(TMin_w)^2$
 - Result range: $x * y \le (-2^{w-1})^2 = 2^{2w-2}$
- So, maintaining exact results...
 - would need to keep expanding word size with each product computed
 - is done in software, if needed
 - e.g., by "arbitrary precision" arithmetic packages

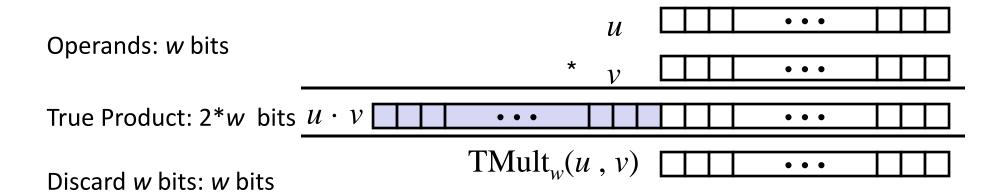
Unsigned Multiplication in C



- Standard Multiplication Function
 - Ignores high order w bits
- Implements Modular Arithmetic

$$UMult_{w}(u, v) = u \cdot v \mod 2^{w}$$

Signed Multiplication in C



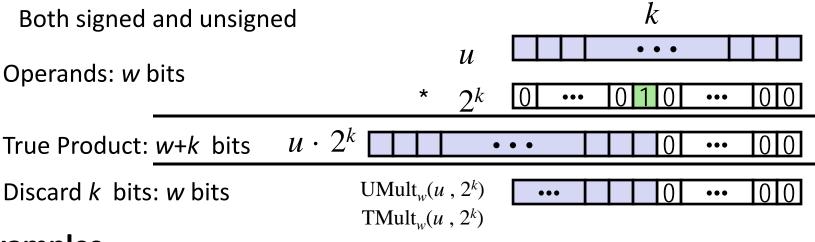
Standard Multiplication Function

- Ignores high order w bits
- Some of which are different for signed vs. unsigned multiplication
- Lower bits are the same

Power-of-2 Multiply with Shift

Operation

- $\mathbf{u} << \mathbf{k}$ gives $\mathbf{u} * \mathbf{2}^k$

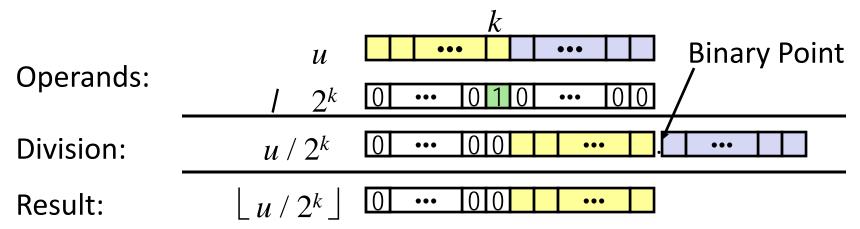


Examples

- u << 3
- u << 5 u << 3</p> * 24
- Most machines shift and add faster than multiply
 - Compiler generates this code automatically

Unsigned Power-of-2 Divide with Shift

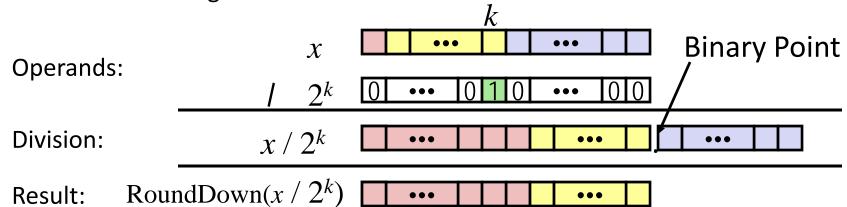
- Quotient of Unsigned by Power of 2
 - $\mathbf{u} \gg \mathbf{k}$ gives $\left[\mathbf{u} / \mathbf{2}^{k} \right]$
 - Uses logical shift



	Division	Computed	Hex	Binary		
x	15213	15213	3B 6D	00111011 01101101		
x >> 1	7606.5	7606	1D B6	00011101 10110110		
x >> 4	950.8125	950	03 B6	00000011 10110110		
x >> 8	59.4257813	59	00 3B	00000000 00111011		

Signed Power-of-2 Divide with Shift

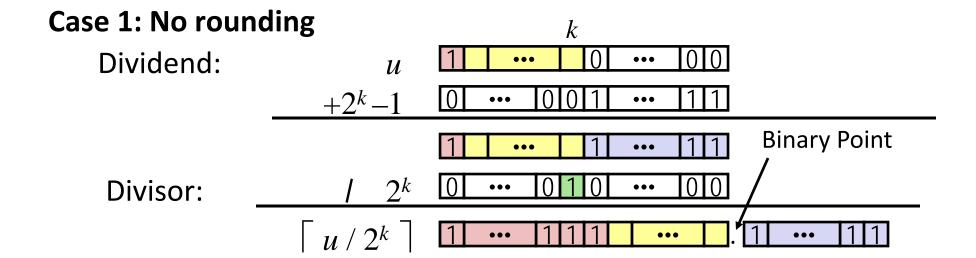
- Quotient of Signed by Power of 2
 - $\mathbf{x} \gg \mathbf{k}$ gives $\left[\mathbf{x} / \mathbf{2}^{k} \right]$
 - Uses arithmetic shift
 - Rounds wrong direction when **u** < 0



	Division	Computed	Hex	Binary		
У	-15213	-15213	C4 93	11000100 10010011		
y >> 1	-7606.5	-7607	E2 49	1 1100010 01001001		
y >> 4	-950.8125	-951	FC 49	1111 1100 01001001		
y >> 8	-59.4257813	-60	FF C4	1111111 11000100		

Correct Power-of-2 Divide

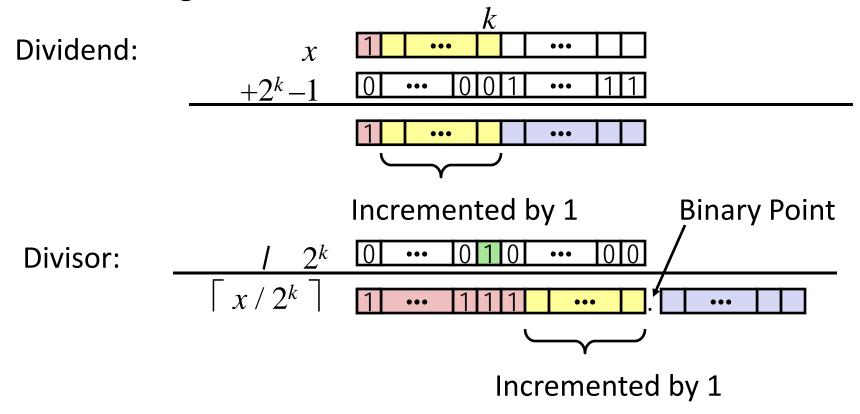
- Quotient of Negative Number by Power of 2
 - Want $\lceil \mathbf{x} / \mathbf{2}^k \rceil$ (Round Toward 0)
 - Compute as $\lfloor (x+2^k-1)/2^k \rfloor$
 - $\ln C: (x + (1 << k)-1) >> k$
 - Biases dividend toward 0



Biasing has no effect

Correct Power-of-2 Divide (Cont.)

Case 2: Rounding



Biasing adds 1 to final result

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Arithmetic: Basic Rules

Addition:

- Unsigned/signed: Normal addition followed by truncate, same operation on bit level
- Unsigned: addition mod 2^w
 - Mathematical addition + possible subtraction of 2^w
- Signed: modified addition mod 2^w (result in proper range)
 - Mathematical addition + possible addition or subtraction of 2^w

Multiplication:

- Unsigned/signed: Normal multiplication followed by truncate, same operation on bit level
- Unsigned: multiplication mod 2^w
- Signed: modified multiplication mod 2^w (result in proper range)

Why Should I Use Unsigned?

- Don't Use Just Because Number Nonnegative
 - Easy to make mistakes

```
unsigned i;
for (i = cnt-2; i >= 0; i--)
a[i] += a[i+1];
```

Can be very subtle

```
#define DELTA sizeof(int)
int i;
for (i = CNT; i-DELTA >= 0; i-= DELTA)
```

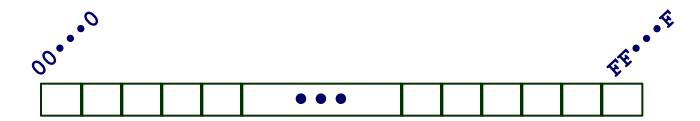
Do Use When Performing Modular Arithmetic

- Multiprecision arithmetic
- Do Use When Using Bits to Represent Sets
 - Logical right shift, no sign extension

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Byte-Oriented Memory Organization



Programs refer to data by address

- Conceptually, envision it as a very large array of bytes
 - In reality, it's not, but can think of it that way
- An address is like an index into that array
 - and, a pointer variable stores an address

■ Note: system provides private address spaces to each "process"

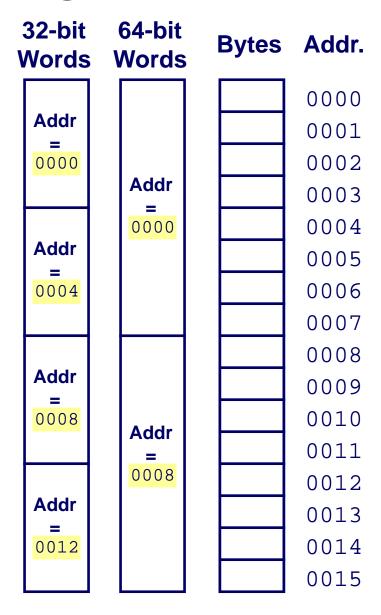
- Think of a process as a program being executed
- So, a program can clobber its own data, but not that of others

Machine Words

- Any given computer has a "Word Size"
 - Nominal size of integer-valued data
 - and of addresses
 - Most current machines use 32 bits (4 bytes) as word size
 - Limits addresses to 4GB (2³² bytes)
 - Becoming too small for memory-intensive applications
 - leading to emergence of computers with 64-bit word size
 - Machines still support multiple data formats
 - Fractions or multiples of word size
 - Always integral number of bytes

Word-Oriented Memory Organization

- Addresses Specify Byte Locations
 - Address of first byte in word
 - Addresses of successive words differ by 4 (32-bit) or 8 (64-bit)



For other data representations too ...

C Data Type	Typical 32-bit	Intel IA32	x86-64	
char	1	1	1	
short	2	2	2	
int	4	4	4	
long	4	4	8	
long long	8	8	8	
float	4	4	4	
double	8	8	8	
long double	8	10/12	10/16	
pointer	4	4	8	

Byte Ordering

- So, how are the bytes within a multi-byte word ordered in memory?
- Conventions
 - Big Endian: Sun, PPC Mac, Internet
 - Least significant byte has highest address
 - Little Endian: x86
 - Least significant byte has lowest address

Byte Ordering Example

Example

- Variable x has 4-byte value of 0x01234567
- Address given by &x is 0x100

Big Endian		0x100	0x101	0x102	0x103		
			01	23	45	67	
Little Endian		0x100	0x101	0x102	0x103		
			67	45	23	01	

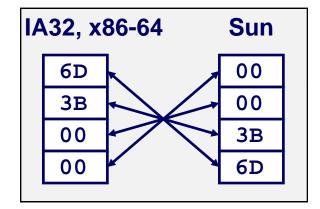
Representing Integers

Decimal: 15213

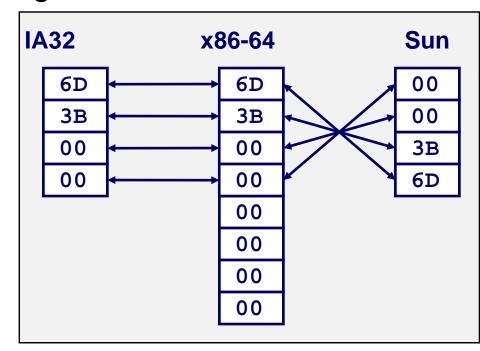
Binary: 0011 1011 0110 1101

Hex: 3 B 6 D

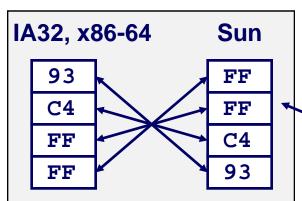
int A = 15213;



long int C = 15213;



int B = -15213;



Two's complement representation

Examining Data Representations

- Code to Print Byte Representation of Data
 - Casting pointer to unsigned char * allows treatment as a byte array

```
typedef unsigned char *pointer;

void show_bytes(pointer start, int len){
  int i;
  for (i = 0; i < len; i++)
     printf("%p\t0x%.2x\n",start+i, start[i]);
  printf("\n");
}</pre>
```

Printf directives:

%p: Print pointer

%x: Print Hexadecimal

show_bytes Execution Example

```
int a = 15213;
printf("int a = 15213;\n");
show_bytes((pointer) &a, sizeof(int));
```

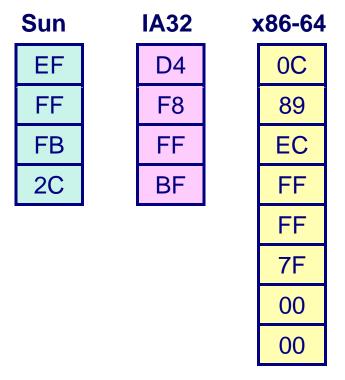
Result (Linux):

```
int a = 15213;
0x11ffffcb8 0x6d
0x11ffffcb9 0x3b
0x11ffffcba 0x00
0x11ffffcbb 0x00
```

Representing Pointers

int
$$B = -15213;$$

int *P = &B



Different compilers & machines assign different locations to objects

Representing Strings

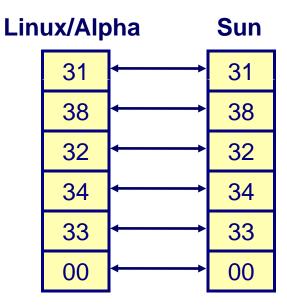
char S[6] = "18243";

Strings in C

- Represented by array of characters
- Each character encoded in ASCII format
 - Standard 7-bit encoding of character set
 - Character "0" has code 0x30
 - Digit i has code 0x30+i
- String should be null-terminated
 - Final character = 0

Compatibility

Byte ordering not an issue



Integer C Puzzles

Initialization

•
$$x * x >= 0$$

•
$$x > 0 \&\& y > 0$$

•
$$x >= 0$$

•
$$(x|-x)>>31==-1$$

•
$$ux >> 3 == ux/8$$

•
$$x >> 3 == x/8$$

•
$$x & (x-1) != 0$$

$$\Rightarrow$$
 ((x*2) < 0)

$$\Rightarrow$$
 (x<<30) < 0

$$\Rightarrow x + y > 0$$

$$\Rightarrow$$
 -x <= 0

$$\Rightarrow$$
 -x >= 0