Mathematical Foundations of Computer Science

Ten Powerful Ideas

and applications

Final Exam	Steven Rudich	May 9, 1997

Repeat After Me.

This part is to test your ability to study material and repeat it.

1. [2 points]
Give a formula for n choose k.

3. [3 points]
Give the definition of a tree.

3. [5 points]
Show that at any party with six people there are either three mutual acquaintances or three mutual strangers. (You may assume that any two people are either acquaintances or strangers.)

Basic Techniques.

This part will test your ability to reason with some of the basic techniques learned in the class.

4. [5 points]
Every node of a ternary tree has 3 children (except for the leaf nodes, which have no children). If a one-level ternary tree contains 1 node, how many nodes does an n-level ternary tree have? Your solution must be in closed form. I.e., no summation in the exression that gives the solution.

5. [5 points]
Prove that $1 + 2 + 3 + \cdots + n$ is divisible by n if and only if n is odd.

6. [10 points]
$f(0) = 2$, $f(1) = 1$, for $x > 1$, $f(x) = f(x-1) + 2f(x-2)$. Give a closed form solution for f.

A Moment's Thought!

This section tests your ability to think a little bit more insightfully. You must give complete explanations of your answers. **Do seven of the eight problems. Do not do all eight; if you do we will grade seven of our choice.**

7. [10 points]
Give a closed-form solution to

$$\sum_{k=0}^{n} \binom{n}{k} 3^k (-1)^{n-k}$$

8. [10 points]
Let $x_1, x_2, \ldots, x_m$ be any m integers. Prove that there must exist a pair whose difference is divisible by $m-1$.

9. [10 points]
What is the expected sum of a randomly chosen subset S of $\{1, 2, 3, \ldots, 20\}$? (In other words, S is equally likely to be any of the 2^{20} subsets. The sum of S is the sum of all its elements. The sum of the empty set is 0. What is the expectation of this sum?)

10. [10 points]
Count the number of ways to choose distinct subsets A_0, A_1, $A_2, \ldots, A_n$ of $\{ 1, 2, 3, \ldots, n \}$ such that $A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n$. ($A \subset B$ means that A is a subset of B.) Your solution should be inclosed form if possible.

11. [10 points]
Count the number of ways to choose subsets A_0, A_1, $A_2, \ldots, A_n$ of $\{ 1, 2, 3, \ldots, n \}$ such that $A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n$. The subsets are *not* required to be distinct. Your solution should be in closed form if possible.

12. [10 points]
Consider a dial that is equally likely to point to each of n regions numbered $1, 2, \ldots, n$. When we spin the dial three times what is the probability that the sum of the selected numbers is n?

13. [10 points]
Let F be the field of integers modulo 7. Find the degree two, univariate polynomial over F passing through the points $(1, 6)$, $(2, 4)$ and $(3, 4)$.

14. [10 points]
Find a simple, Eulerian graph with an even number of vertices and an odd number of edges. Recall that a simple graph has no self loops or multiple edges. An Eulerian graph is a graph with an Euler tour.

15. [10 points]
Prove that every graph with n vertices and $n-k$ edges has at least k components.