10 Powerful Ideas	Bruce Maggs
Final Review Exercises	Final Exam: **Monday, May 4, 1998**
	1pm-4pm, DH2210

Quickies

1.
$$\sum_{i=0}^{n} 2^i =$$

2.
$$\sum_{i=0}^{n} x^i =$$

3.
$$\sum_{i=0}^{n} i =$$

4.
$$\sum_{i=0}^{n} \binom{n}{i} =$$

5.
$$\binom{n}{i-1} + \binom{n}{i} =$$

6. What is the coefficient of $a^2b^3c^5$ in $(a+b+c)^{10}$?

Induction

7. Solve the following recurrences:

(a)
$$\begin{aligned} T(n) &= 6T(n-1) - 8T(n-2) \\ T(0) &= 0 \\ T(0) &= 1 \end{aligned}$$

(b)
$$\begin{aligned} T(n) &= 2T(n/2) + n^2 \\ T(1) &= 1 \end{aligned}$$

8. Prove by induction that $\phi^n = F_n\phi + F_{n-1}$ for $n \geq 1$.

Groups and Fields

9. State the four properties that define a group.

10. List all the subgroups of $(\mathcal{Z}_p, +)$, where p is prime.

11. Prove that the set of n-bit bitstrings is a group under the operation bitwise XOR.

12. Prove that the set of n-bit bitstrings with even parity is a subgroup of the group in the previous problem.

13. Show that the set of complex numbers $a + bi$ is a field.

Counting and Probability

14. The numbers 1447, 1005, and 1231 have something in common: each is a 4-digit number beginning with 1 that has exactly two identical digits. How many such numbers are there?

15. What is the largest 2-digit prime factor of the integer $n = \binom{200}{100}$?

16. Letters in Morse code are formed by a succession of dots and dashes with repetitions allowed. How many different letters could be represented by 10 or fewer dots and dashes?

17. n balls are placed randomly into n cells. What's the probability that exactly one cell remains empty?

18. A fair coin is flipped n times. What's the probability that all the heads occur at the end of the sequence?

19. A box contains 90 good and 10 defective screws. What's the probability that if 10 screws are used, none will be defective?

20. What is
$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} \cdots$$
and why?

21. What is $(x + y + z)^m$ as a sum of terms?

22. True or false:
    (a) $E[AB] = E[A]E[B]$ if and only if A and B are independent.
    (b) $E[A + B] = E[A] + E[B]$ if and only if A and B are independent.
    (c) $E[logA] = logE[A]$ for all random variables A.

23. What is the probability of rolling two pair (such as 5 5 3 3 6) on the first roll in Yahtzee?

24. What is the probability of getting two pair (such as Q Q A A 3) in poker?

25. If you choose a k-bit number by randomly picking bits, what is the probability of getting a power of 2?

26. If you pick a k-bit number at random, what is the probability of picking a prime?

27. What is $90^{1584} \pmod{11}$? (Hint: use Fermat's Little Theorem)

28. Give two proofs for the identity

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

    One proof should be algebraic (manipulating symbols), the other combinatoric (showing a correspondence between two sets).

29. If you pick two socks at random from a drawer with 5 gray socks and 4 black socks, what is the probability that the two socks match?

30. What is the probability that, out of n people chosen at random, at least two have the same birthday?

31. How many people must you choose at random in order to ensure that at least *ten* people have the same birthday (ignore leap years)?

Infinities, Undecidability, Intractability, Languages

32. Give an example of each of the following, or explain why it don't exist.

    (a) a countably infinite field
    (b) a finite group
    (c) an uncountable regular language
    (d) a one-to-one onto function from $\mathcal{N}$ to $\mathcal{R}$.

Graph Theory

33. How many different Hamiltonian cycles are possible in a labeled complete graph on n vertices?

34. How many trees can be made with 10 labeled nodes and 10 edges?

A Moment's Thought

35. For $\{1, 2, 3, \ldots, n\}$ and each of its nonempty subsets a unique *alternating sum* is defined as follows: Arrange the numbers in the subset in decreasing order and then, beginning with the largest, alternately add and subtract successive numbers. (For example, the alternating sum for $\{1, 2, 4, 6, 9\}$ is $9 - 6 + 4 - 2 + 1 = 6$ and for $\{5\}$ it is simply 5.) Find the sum of all such alternating sums for $n = 7$.

36. Imagine constructing a graph as follows. Draw 10 nodes. Then for each pair of nodes, draw an edge between them with probability p. What is the expected number of nodes with degree exactly two?