10 Powerful Ideas	Bruce Maggs
Quiz 1 Review Exercises	Quiz 1: **Monday, February 23, 1998**

Solutions to these problems will be available on the course Web site by Saturday.

Dating

1. Show a set of bisexual dating preferences where there is no possible stable pairing among the people.

Induction

2. Show **by induction** that for all integer $n \geq 1$, $\phi^n = F_n\phi + F_{n-1}$, where ϕ is the golden ratio and F_n is the nth Fibonacci number.

3. Let's call a positive integer *purple* if it is the sum of consecutive positive integers all of which are less than itself. For example, $9 = 2 + 3 + 4$ is a purple number. Prove (**not** by induction!) that a positive integer is purple if and only if it is not a power of two. (Remember that the *if and only if* means that you have two statements to prove.)

4. A limo is two car lengths long and a car is a single car length long. Let $T(n)$ be the number of distinct possible ways of parking cars and limos along a curb that is n car lengths long. Order matters, but any two cars and any two limos are considered identical. If the curb is 4 car lengths long, then the distinct possible parking patterns are LL, LCC, CLC, CCL, and CCCC. Thus, $T(4) = 5$. Derive a recurrence relation for T that completely determines it, and justify your answer.

Growth Rates

5. Prove that big-O is transitive: $f(n) = O(g(n))$ and $g(n) = O(h(n))$ implies $f(n) = O(h(n))$.

6. Find the flaw in the following inductive proof:
   Prove: $n^2 = O(n)$
   Base case: for $n = 1$, $1^2 = O(1)$.
   Inductive step: Suppose $(n-1)^2 = O(n-1)$. Since $n^2 = O((n-1)^2)$, and $n-1 = O(n)$, then by transitivity of big-O we have $n^2 = O(n)$.

7. Show that $\log n! = \Theta(n \log n)$.

Recurrences

Find closed form solutions to the following recurrence relations.

8.

$$
\begin{aligned}
T(n) &= T(n-1) + 2T(n-2) \text{ for } n > 1 \\
T(1) &= 1 \\
T(0) &= 0
\end{aligned}
$$

9.

$$\begin{aligned} S(n) &= S(n-1) + n \text{ for } n > 0 \\ S(0) &= 1 \end{aligned}$$

10.

$$\begin{aligned} T(n) &= 2T(n-1) + 1 \text{ for } n > 0 \\ T(0) &= 0 \end{aligned}$$

11.

$$\begin{aligned} Q(n) &= \frac{1 + Q(n-1)}{Q(n-2)} \text{ for } n > 1 \\ Q(1) &= \beta \\ Q(0) &= \alpha \end{aligned}$$

12. Every node of a ternary tree has 3 children (except for the leaf nodes, which have no children). If a one-level ternary tree contains 1 node, give a closed form solution for the number of nodes in an n-level ternary tree have.

Groups

13. List all subgroups of $Z_4 = \{0, 1, 2, 3\}$ with the operation $+$ mod 4. Now list all subsets of $\{0, 1, 2, 3\}$. Does the null set (denoted $\emptyset$) belong to either list?

14. The group S_n, called the *symmetric group*, is the group of permutations of n elements, using composition as its operation. For instance, $\{1, 4, 3, 2\}$ is one member of S_4.
    (a) Assuming S_n is indeed a group, what would its identity element be?
    (b) What is $\mid S_n \mid$, the size of the group?
    (c) Prove that S_n is indeed a group for any n.
    (d) Does S_n have any subgroups?

15. Given any group $(G, \star)$ and any non-empty subset H of G such that $a, b \in H$ implies $a^{-1} \star b \in H$, show that $(H, \star)$ is a subgroup.

Counting

16. Give a closed form solution to $3 + 6 + 9 + \ldots + 3n$.

17. What is the formula for the power series expansion of $(x + y)^n$?

18. How many subsets can be formed from the elements of an n element set?

19. How many r-bit binary sequences have an odd number of 1s?

20. How many ways are there to rearrange the letters in the word EFFORTLESS?

21. Show that when $n \geq m$

$$\binom{m}{m} + \binom{m+1}{m} + \cdots + \binom{n}{m} = \binom{n+1}{m+1}$$

22. What is the coefficient of a^4b^6 in $(a^2+b)^8$?

23. How many integer solutions to the following equations?

$$X_1 + X_2 + X_3 + X_4 + X_5 = 50, \quad X_i \geq 0$$

24. An n-input, m-output boolean function is a function from $0,1^n$ to $0,1^m$. (Think of a circuit with n inputs and m outputs.) How many n-input, 1-output boolean functions are there? How many n-input, m-output boolean functions are there?

25. How many ways can three distinct numbers be chosen from the set $1, 2, \ldots, 100$ such that their sum is even?

26. How many ways are there to seat 5 children in a row of 12 chairs?

27. A *chord* is a line segment connecting two points on a circle. Suppose you are given n points on a circle such that no three of the chords connecting them intersect in the same point. Into how many line segments are the chords divided by their intersections?

28. How many ways can 3 exams be scheduled in a 5-day period such that no two exams are held the same day?

29. A palindrome is a word that reads the same forwards and backwards. How many seven-letter palindromes can be made from the English alphabet?

30. Show that the product of any k consecutive integers is divisible by $k!$

31. How many ways are there to place r red balls and w white balls into n distinct boxes such that each box contains at least one ball of each color?

32. Imagine a town with East-West streets numbered 1 through n, and North-South avenues numbered 1 through m. A taxi cab picks up a passenger at the corner of 1st street and 1st avenue. The passenger wishes to be delivered to nth street and mth avenue. It is quite clear that the passenger will be angry if the cab chooses a route longer than $(n-1)+(m-1)$ blocks, so we won't allow the cabby to take a route longer than this. In other words, the cabby must always be increasing his street number or his avenue number. Suppose that there is an accident at ith street and jth avenue. How many routes can the cabby take that avoid the intersection with the accident?

Toughies

33. Suppose there are $2n$ people in a circle. The first n people are "good guys," and the last n are "bad guys." Show that there is always an integer m (depending on n) such that, if we go around the circle executing every mth person, all the bad guys are executed first. (For example, when $n=3$, we can take $m=5$; when $n=4$ we can take $m=30$.)

More Practice

These books are all available in the Engineering & Science Library in Wean Hall.

1. *Elements of Discrete Mathematics*, by C.L. Liu.
2. *Discrete Mathematics*, by N.L. Biggs.

3. *Combinatorics: An Invitation*, by H.J. Straight.

4. *Concrete Mathematics: a foundation for computer science*, by R. Graham, D. Knuth, O. Patashnik.