10 Powerful Ideas	Steven Rudich
QUIZ 2	April 16, 1997

NAME : Solution Set

ANDREW ID :

SECTION : A B C

INSTRUCTIONS. For the first 7 problems you do not need to give any reason or justification for your answer. For the last two problems you must give complete explanations. WRITE CLEARLY. WRITE YOUR NAME, ID, and SECTION above.

The easy stuff.

This part is to test your ability to study material that has been labeled as likely to appear on exams. If you do not do well on this section, your study habits could use adjustment.

1. [5 points]
How many ways are there to seat k children in $n > k$ chairs?

There are $\binom{n}{k}$ ways to choose which chairs will be filled, and then $k!$ ways to choose the permutation of the children to fill them, giving us a total of $k!\binom{n}{k} = \frac{n!}{(n-k)!}$ ways to seat the children by the product rule.

2. [5 points]
Write a closed-form expression for $\displaystyle\sum_{i=0}^{n} \binom{n}{i}$

This summation adds for each i the number of ways to choose exactly i bits to be 1 in an n-bit string. The total is the number of ways to choose any number of bits to be 1 in an n-bit string, which is the same as the number of n-bit strings, 2^n.

3. [10 points]
What is the definition of independent events?

Two events X and Y are independent if and only if the joint probability is the product of their probabilities, or $Pr(X \cap Y) = Pr(X)Pr(Y)$. This is equivalent to saying that $Pr(X|Y) = Pr(X)$ or that $Pr(Y|X) = Pr(Y)$.

4. [10 points]
The *parity* of a bit vector is the mod-2 sum of the number of ones in the vector. How many bit-vectors are there of size n that have even parity?

For each bit vector X of length $n-1$, there is exactly one way to extend it to a bit vector of length n that has even parity—by adding a 1 if X has odd parity, or by adding a 0 if X has even parity. Therefore there are as many even-parity n-bit vectors as there are unconstrained $(n-1)$-bit vectors, 2^{n-1}.

Reading Solutions.

This section tests your ability to understand solutions to previous problems by asking you to solve similar, slightly more complicated problems.

5. [15 points]
You are at point $(0,0,0)$ in three-dimensional space, and you need to get to point (i,j,k). You can only make unit-length moves parallel to the three coordinate axes, and only moves that get you closer to your goal. How many different paths can you take?

No matter how you arrive at (i,j,k), you must have taken $i+j+k$ steps. Of these steps, i must be along one dimension, j must be along another, and k must be along the third. The question becomes how to distribute the three types of steps amongst the $i+j+k$ total steps. There are $\binom{i+j+k}{i}$ ways to choose which of the $i+j+k$ steps will be along the first dimension, $\binom{j+k}{j}$ ways to choose which of the $j+k$ remaining steps will be along the second dimension, and $\binom{k}{k}=1$ way to choose which of the k remaining steps will be along the third dimension. By the product rule, the total number of different paths is $\binom{i+j+k}{i}\binom{j+k}{j}$.

6. [15 points]
Consider a town with East-West streets numbered 1 through infinity, and North-South avenues numbered 1 through infinity. Suppose you are at 1st street and 1st avenue, and somewhat inebriated. You opt not to take a cab, and instead would like to *walk* to your home on nth street and mth avenue. You have forgotten where you live, however, and every minute you ponder the next movement to make. With probability $\frac{1}{2}$ you walk one street forward, and with probability $\frac{1}{2}$ you walk one avenue forward. You can only move in the direction of increasing streets or avenues, and if you miss your house you're doomed. If you happen to stumble upon the intersection (n,m) by your house, you recognize it and proceed inside to safety. What is the probability that you make it?

You might remember a similar problem from the midterm. Any path that gets me to (n,m) must require exactly $n+m-2$ steps, $n-1$ of which must be in the "increasing street" direction and $m-1$ of which must be in the "increasing avenue" direction. There are therefore $\binom{n+m-2}{n-1}$ different paths to my house, since we need only choose which of these steps are in the "increasing street" direction. Each path of length $n+m-2$ steps has probability $(0.5)^{n+m-2}$, since at each step we come to some decision that has probability 0.5. Treating each path as a separate event that leads us to the desired destination, the probability of getting to the desired intersection is the sum of the probabilities of each path, $\binom{n+m-2}{n-1}(0.5)^{n+m-2}$.

Basic Techniques.

This part will test your ability to apply techniques for which I have given explicit recipes. If you do not do well on this section, your study habits could use adjustment.

7. [20 points]

The Pittsburgh Pirates, by universal consensus the worst team in the major leagues, actually have a chance to make the playoffs in October. In order to do so, they'll need to win at least 90 of their remaining 150 games. Assuming the probability that the Pirates will win any single game is 40%, what's the probability that they'll make the playoffs? [A summation is ok as an answer]

We break all the possible winning scenarios into cases based on how many games the Pirates win in the remainder of the season, using the sum rule for disjoint events to sum the probabilities of each case. This leads to the following summation:

$\sum_{i=90}^{150} \binom{150}{i} (0.4)^i (0.6)^{150-i}$.

The first factor of each term in the summation is the number of win-loss sequences that can result if we enforce that the Pirates win i of the **150** remaining games. The rest assigns a probability to each win-loss sequence having that constraint. The probability for a given sequence of wins and losses depends not on the order in which the wins and losses occur, but only on the total number of each, since the games are independent events; a win-loss sequence of length **150** with i wins has probability $(0.4)^i(0.6)^{150-i}$ since each win in the sequence has probability 0.4 and each loss has probability $1 - 0.4 = 0.6$.

A moment's thought.

8. [20 points]
Let G be a simple graph, i.e., no self loops and at most one edge between any two nodes. Show that two different nodes of G must have the same degree, i.e., the same number of neighbors.

(Clarifications: We want you to show that *there exist* two distinct nodes in G with the same degree. Also, limit the scope of the statement to graphs with more than 1 vertex.)

Suppose G has n nodes. Each node is connected to some number of other vertices (its *degree*), a value between 0 and $n-1$. There cannot exist a vertex with degree 0 *and* a vertex with degree $n-1$, because an $n-1$-degree vertex must be connected to everything, including the loner. So there are at most $n-1$ distinct values that the n vertex degrees can take on. By the pigeonhole principle, there exists a value that is the degree for at least two different vertices.

EXTRA CREDIT

Warning: this is a trick question.

9. [20 points]
Let $\Sigma = \{a, b\}$. Let S be the following set:

$\{XY \mid X \in \Sigma^*, Y \in \Sigma^*,$ and the number of a's in **X** equals the number of b's in **Y**$\}$

Prove or disprove that S is regular, i.e., can be accepted by some finite automaton.

S is regular, because it is precisely Σ^*, the set of all finite strings over $\Sigma = \{a, b\}$. The finite automaton consisting of one state (starting and accepting) with self-loops on both symbols a and b would accept this language.

To prove this, we must show that any string Z is in S, *i.e.* is of the form $\{XY \mid X \in \Sigma^*, Y \in \Sigma^*,$ and the number of a's in **X** equals the number of b's in **Y**$\}$. Equivalently, we must show that for every string Z there is a place where we can split the string in two, such that the left piece has as many a's as the right piece has b's.

We will prove this by induction on the length n of an arbitrary string Z. There is one string of length zero, the null string, and trivially we can split this into two concatenated null strings that satisfy the desired constraints. Suppose the property holds for all strings of length n, and consider an arbitrary string Z of length $n+1$. Z can be written CXY, where $C \in \{a, b\}$ and the remainder is a string of length n, which by the inductive hypothesis can be split into strings X and Y with as many a's in X as there are b's in Y. If $C = b$, then the strings $X' = CX$ and $Y' = Y$ satisfy the desired splitting constraint, and so $Z = X'Y' \in S$. If $C = a$, then consider the character x appearing before Y in $Z = CXY$. (x is well-defined, since even if X is null, C cannot be. Let $W = CX - x$, *i.e.* the truncation of CX by one character.) Regardless of x, xY has as many b's as W has a's, and so we can write $X' = W$ and $Y' = xY$ to satisfy the desired splitting constraint, again showing that $Z = X'Y' \in S$.

An alternate, and perhaps simpler way of proving that $Z \in S$ for all Z resembles the "necklace" extra credit from homework 4. Consider an arbitrary string Z. Define $f(i)$, where i is an integer index of a possible splitting point in Z, to be the number of b's after index i *minus* the number of a's before index i. $f(0)$ is the number of b's in the string, and $f(|Z|)$ is negative the number of a's in the string. Since $f(i+1)$ and $f(i)$ differ by at most one for all i, and since f goes from a positive number to a negative number in scanning Z, there must be some i' such that $f(i') = 0$. At splitting point i' we satisfy the desired constraint, showing that $Z \in S$.