10 Powerful Ideas	Bruce Maggs-neto
QUIZ 3 REVIEW	April 23, 1998

The easy stuff.

This part is to test your ability to study material that has been labeled as likely to appear on exams. If you do not do well on this section, your study habits could use adjustment.

1. [5 points]
How many ways are there to seat k children in $n > k$ chairs?

2. [5 points]
Write a closed-form expression for $\sum_{i=0}^{n} \binom{n}{i}$

3. [10 points]
What is the definition of independent events?

4. [10 points]
The *parity* of a bit vector is the mod-2 sum of the number of ones in the vector. How many bit-vectors are there of size n that have even parity?

Reading Solutions.

This section tests your ability to understand solutions to previous problems by asking you to solve similar, slightly more complicated problems.

5. [15 points]
You are at point $(0, 0, 0)$ in three-dimensional space, and you need to get to point (i, j, k). You can only make unit-length moves parallel to the three coordinate axes, and only moves that get you closer to your goal. How many different paths can you take?

6. [15 points]
Consider a town with East-West streets numbered 1 through infinity, and North-South avenues numbered 1 through infinity. Suppose you are at 1st street and 1st avenue, and somewhat inebriated. You opt not to take a cab, and instead would like to *walk* to your home on nth street and mth avenue. You have forgotten where you live, however, and every minute you ponder the next movement to make. With probability $\frac{1}{2}$ you walk one street forward, and with probability $\frac{1}{2}$ you walk one avenue forward. You can only move in the direction of increasing streets or avenues, and if you miss your house you're doomed. If you happen to stumble upon the intersection (n, m) by your house, you recognize it and proceed inside to safety. What is the probability that you make it?

Basic Techniques.

This part will test your ability to apply techniques for which I have given explicit recipes. If you do not do well on this section, your study habits could use adjustment.

7. [20 points]
The Pittsburgh Pirates, by universal consensus the worst team in the major leagues, actually have a chance to make the playoffs in October. In order to do so, they'll need to win at least 90 of their remaining 150 games. Assuming the probability that the Pirates will win any single game is 40%, what's the probability that they'll make the playoffs? [A summation is ok as an answer]

A moment's thought.

8. [20 points]
Let G be a simple graph, i.e., no self loops and at most one edge between any two nodes. Show that two different nodes of G must have the same degree, i.e., the same number of neighbors.

(Clarifications: We want you to show that *there exist* two distinct nodes in G with the same degree. Also, limit the scope of the statement to graphs with more than 1 vertex.)

Graph theory

1. Call a vertex of a graph *odd* if its degree is odd, and *even* if its degree is even. Show that the number of odd vertices is even.

2. A mouse intends to eat a $3 \times 3 \times 3$ cube of cheese. Being tidy-minded, it begins at a corner and eats the whole of a $1 \times 1 \times 1$ cube before going on to an adjacent one. Can the mouse end in the center of the cube? Why or why not?

3. A *forest* is a graph containing no cycles, but not necessarily connected. Show that if $F = (V, E)$ is a forest with c components, then
$$|E| = |V| - c$$

4. The usual set of dominos may be represented by the symbol $[x|y]$ where x and y are members of the set $\{0, 1, 2, 3, 4, 5, 6\}$. The numbers x and y may be equal. Suppose the rules of the game require that the dominos be placed in a line so that adjacent dominos have matching numbers: $[x|y]$ is next to $[y|z]$, and so on.

   (a) How many different dominoes are there? Explain your answer.

   (b) Show that it is possible to have a *complete game*, a game in which all the dominos are used. Hint: regard the dominos for which $x \neq y$ as the edges of the complete graph K_7.

   (c) Calculate the number of complete games of dominos.

5. Suppose you have a set of n different (labeled) vertices.

   (a) How many complete graphs are there on n vertices?

   (b) How many paths are there in a complete graph?

   (c) How many simple cycles are there in a complete graph? (Recall that a simple cycle is a path that starts and ends at the same vertex.)

   (d) How many graphs are there on n vertices?

6. Let G be a planar graph with e edges and f faces, such that every face has at least k sides.

   (a) Show that $f \leq 2e/k$.

   (b) Use this fact to show that $K_{3,3}$ is nonplanar.

Counting and Probability

7. Among the set of digits $\{1, 2, 3, 4, 5\}$, first one digit is chosen uniformly at random, and then another. What's the probability that an odd digit is picked the second time?

8. Say you draw uniformly at random from the elements of the group S_n, the group of permutations on n elements. What's the probability that in the permutation you elect, the kth element will be in the ith position?

9. What's the probability that among k random digits, 0 doesn't appear?

10. n balls are placed randomly into n cells. What's the probability that exactly one cell remains empty?

11. A box contains 90 good and 10 defective screws. What's the probability that if 10 screws are used, none will be defective?

12. Eight eligible bachelors and seven beautiful models have purchased seats in a 15-seat row of a theatre. What's the expected number of adjacent seats ticketed for marriageable couples?

13. What is $213729^{1018} \pmod{20}$?

A Moment's Thought

14. What is the expected number of simple cycles in a random graph drawn on n labeled vertices with edge probability p?

15. Goornks reproduce asexually on staircase steps. On every step it hits, a Goornk (of either type) produces a *weak* goornk, which promptly hops up to the next step, and a *strong* goornk, which hops up two steps at once, bypassing the step in between. How many goornks will eventually arrive at the top of a staircase with k steps if a single weak goornk reproduces at the bottom?