| | |
|---|---|
| 10 Powerful Ideas | Bruce Magss-neto |
| QUIZ 3 REVIEW | April 26, 1998 |

**The easy stuff.**

This part is to test your ability to study material that has been labeled as likely to appear on exams. If you do not do well on this section, your study habits could use adjustment.

**1.** [5 points]
How many ways are there to seat $k$ children in $n > k$ chairs?

There are $\binom{n}{k}$ ways to choose which chairs will be filled, and then $k!$ ways to choose the permutation of the children to fill them, giving us a total of $k!\binom{n}{k} = \frac{n!}{(n-k)!}$ ways to seat the children by the product rule.

**2.** [5 points]
Write a closed-form expression for $\sum_{i=0}^{n} \binom{n}{i}$

This summation adds for each $i$ the number of ways to choose exactly $i$ bits to be 1 in an $n$-bit string. The total is the number of ways to choose any number of bits to be 1 in an $n$-bit string, which is the same as the number of $n$-bit strings, $2^n$.

**3.** [10 points]
What is the definition of independent events?

Two events $X$ and $Y$ are independent if and only if the joint probability is the product of their probabilities, or $Pr(X \cap Y) = Pr(X)Pr(Y)$. This is equivalent to saying that $Pr(X|Y) = Pr(X)$ **or** that $Pr(Y|X) = Pr(Y)$.

**4.** [10 points]
The *parity* of a bit vector is the mod-2 sum of the number of ones in the vector. How many bit-vectors are there of size $n$ that have even parity?

For each bit vector $X$ of length $n-1$, there is exactly one way to extend it to a bit vector of length $n$ that has even parity—by adding a 1 if $X$ has odd parity, or by adding a 0 if $X$ has even parity. Therefore there are as many even-parity $n$-bit vectors as there are unconstrained $(n-1)$-bit vectors, $2^{n-1}$.

## Reading Solutions.

This section tests your ability to understand solutions to previous problems by asking you to solve similar, slightly more complicated problems.

**5**. [15 points]
You are at point $(0, 0, 0)$ in three-dimensional space, and you need to get to point $(i, j, k)$. You can only make unit-length moves parallel to the three coordinate axes, and only moves that get you closer to your goal. How many different paths can you take?

No matter how you arrive at $(i, j, k)$, you must have taken $i + j + k$ steps. Of these steps, $i$ must be along one dimension, $j$ must be along another, and $k$ must be along the third. The question becomes how to distribute the three types of steps amongst the $i + j + k$ total steps. There are $\binom{i+j+k}{i}$ ways to choose which of the $i + j + k$ steps will be along the first dimension, $\binom{j+k}{j}$ ways to choose which of the $j + k$ remaining steps will be along the second dimension, and $\binom{k}{k} = 1$ way to choose which of the $k$ remaining steps will be along the third dimension. By the product rule, the total number of different paths is $\binom{i+j+k}{i}\binom{j+k}{j}$.

**6**. [15 points]
Consider a town with East-West streets numbered 1 through infinity, and North-South avenues numbered 1 through infinity. Suppose you are at 1st street and 1st avenue, and somewhat inebriated. You opt not to take a cab, and instead would like to *walk* to your home on $n$th street and $m$th avenue. You have forgotten where you live, however, and every minute you ponder the next movement to make. With probability $\frac{1}{2}$ you walk one street forward, and with probability $\frac{1}{2}$ you walk one avenue forward. You can only move in the direction of increasing streets or avenues, and if you miss your house you're doomed. If you happen to stumble upon the intersection $(n, m)$ by your house, you recognize it and proceed inside to safety. What is the probability that you make it?

You might remember a similar problem from the midterm. Any path that gets me to $(n, m)$ must require exactly $n + m - 2$ steps, $n-1$ of which must be in the "increasing street" direction and $m-1$ of which must be in the "increasing avenue" direction. There are therefore $\binom{n+m-2}{n-1}$ different paths to my house, since we need only choose which of these steps are in the "increasing street" direction. Each path of length $n + m - 2$ steps has probability $(0.5)^{n+m-2}$, since at each step we come to some decision that has probability 0.5. Treating each path as a separate event that leads us to the desired destination, the probability of getting to the desired intersection is the sum of the probabilities of each path, $\binom{n+m-2}{n-1}(0.5)^{n+m-2}$.

**Basic Techniques.**

This part will test your ability to apply techniques for which I have given explicit recipes. If you do not do well on this section, your study habits could use adjustment.

**7.** [20 points]
The Pittsburgh Pirates, by universal consensus the worst team in the major leagues, actually have a chance to make the playoffs in October. In order to do so, they'll need to win at least 90 of their remaining 150 games. Assuming the probability that the Pirates will win any single game is 40%, what's the probability that they'll make the playoffs? [A summation is ok as an answer]

We break all the possible winning scenarios into cases based on how many games the Pirates win in the remainder of the season, using the sum rule for disjoint events to sum the probabilities of each case. This leads to the following summation:

$\sum_{i=90}^{150} \binom{150}{i} (0.4)^i (0.6)^{150-i}$.

The first factor of each term in the summation is the number of win-loss sequences that can result if we enforce that the Pirates win $i$ of the 150 remaining games. The rest assigns a probability to each win-loss sequence having that constraint. The probability for a given sequence of wins and losses depends not on the order in which the wins and losses occur, but only on the total number of each, since the games are independent events; a win-loss sequence of length 150 with $i$ wins has probability $(0.4)^i(0.6)^{150-i}$ since each win in the sequence has probability 0.4 and each loss has probability $1 - 0.4 = 0.6$.

**A moment's thought.**

**8.** [20 points]
Let G be a simple graph, i.e., no self loops and at most one edge between any two nodes. Show that two different nodes of G must have the same degree, i.e., the same number of neighbors.

(Clarifications: We want you to show that *there exist* two distinct nodes in $G$ with the same degree. Also, limit the scope of the statement to graphs with more than 1 vertex.)

Suppose $G$ has $n$ nodes. Each node is connected to some number of other vertices (its *degree*), a value between 0 and $n-1$. There cannot exist a vertex with degree 0 *and* a vertex with degree $n-1$, because an $n-1$-degree vertex must be connected to everything, including the loner. So there are at most $n-1$ distinct values that the $n$ vertex degrees can take on. By the pigeonhole principle, there exists a value that is the degree for at least two different vertices.

## Graph theory

1. Call a vertex of a graph *odd* if its degree is odd, and *even* if its degree is even. Show that the number of odd vertices is even.

   This was covered in recitation. The basic idea is that the sum of the degree of each vertex is even. Split the vertices into two sets: those with even degree, and those with odd degree. The sum of the degree of the vertices with even degree is even, and the overall sum is even, therefore the sum of the degrees of the vertices with odd degree is even, which means that there must be an even number of them.

2. A mouse intends to eat a $3 \times 3 \times 3$ cube of cheese. Being tidy-minded, it begins at a corner and eats the whole of a $1 \times 1 \times 1$ cube before going on to an adjacent one. Can the mouse end in the center of the cube? Why or why not?

   No, the mouse cannot. Color the middle cube black, and also all of the $1 \times 1 \times 1$ cubes that reside in the middle of one of the edges of the overall cube. Color the remainer of the cubes white. The mouse must alternate eating white and black colored cubes. Assume the mouse can finish in the middle. We know that the mouse would have to end on a black cube if it ends in the middle, so pair each black cube eaten with the previous cube eaten (which must be white), so the mouse ate the same number of white cubes as black cubes (we know the mouse started on a white cube, since all of the corner cubes are white). However, there are 14 white cubes and only 13 black cubes, so at least one white cube was not consumed.

3. A *forest* is a graph containing no cycles, but not necessarily connected. Show that if $F = (V, E)$ is a forest with $c$ components, then

$$|E| = |V| - c$$

   Also covered in recitation. Basically, sum the edges and vertices for each compenent obtains the result (let $e_i$ and $v_i$ denote the number of edges and vertices in the i-th component):

$$\|E\| = \sum e_i = \sum (v_i - 1) = (\sum v_i) - c = \|V\| - c$$

4. The usual set of dominos may be represented by the symbol $[x|y]$ where $x$ and $y$ are members of the set $\{0, 1, 2, 3, 4, 5, 6\}$. The numbers $x$ and $y$ may be equal. Suppose the rules of the game require that the dominos be placed in a line so that adjacent dominos have matching numbers: $[x|y]$ is next to $[y|z]$, and so on. Note that the dominoes $[x|y]$ and $[y|x]$ are considered the same dominoe.

   (a) How many different dominoes are there? Explain your answer.

   There are $\binom{7}{2} + 7 = 28$ dominoes. There are $\binom{7}{2}$ dominoes with different $x$ and $y$, and 7 where $x = y$.

   (b) Show that it is possible to have a *complete game*, a game in which all the dominos are used. Hint: regard the dominos for which $x \neq y$ as the edges of the complete graph $K_7$.

   Consider $K_7$ labeled with $\{0, 1, \ldots, 6\}$. Now, each vertex has even degree, so there exists a Eulerian tour. This path is a sequence of edges. For each edge, map it to the dominoes with values equal to the labels on its end-points. Consider that sequence of dominoes, where the dominoe $[x|x]$ is added whenever $x$ appears first as a dominoe side. This sequence represents a complete game.

   (c) Calculate the number of complete games of dominos.

   wing the logic of the previous part, the placement of nondoubled dominos in a complete game corresponds to an Euler tour of $K_7$. How many Euler tours does $K_7$ have? First, choose a starting vertex (7 choices). Initially, you have 6 outgoing edges to choose from, but the next time you return to this vertex, you'll have only 4 unused edges, and the third time, only 2 edges. Thus there are $6 \cdot 4 \cdot 2$ ways to choose how you leave the starting vertex. For the other vertices the calculation is slightly different, because they are initially reached by an edge. Thus a non-starting vertex choose from 5 outgoing edges on its first visit, 3 on its second visit, and 1 on its third visit, giving $5 \cdot 3$ choices. Taking the product of these choices over all vertices[1] gives the number of Euler tours on $K_7$ as

$$7(6 \cdot 4 \cdot 2)(5 \cdot 3)^6$$

So we have this many ways to place the nondoubled dominos. To place each doubled domino somewhere among the nondoubled dominos, observe that every vertex occurs 3 times in the Euler tour, except the starting vertex which occurs 4 times. Each visit in the Euler tour corresponds to a place in the nondoubled dominos where a doubled domino can be inserted. Thus the doubled domino matching the starting vertex has 4 choices, and the other 6 doubled dominos have 3 choices, giving a grand total of

$$7(6 \cdot 4 \cdot 2)(5 \cdot 3)^6(4 \cdot 3^6) \approx 1.11 \times 10^{13} \text{ complete games}$$

5. Suppose you have a set of $n$ different (labeled) vertices.

   (a) How many complete graphs are there on $n$ vertices?

   one

   (b) How many paths are there in a complete graph?

   Recall that a path is a sequence of vertices (from one to $n$ vertices) with no cycles. Since order matters, for each path length $i$ ($1 \leq i \leq n$), we choose an $i$-permutation of the $n$ vertices, which can be done $\frac{n!}{(n-i)!}$. Summing over all path lengths gives

   $$\frac{\sum_{i=1}^{n} n!}{(n-i)!}$$

   (c) How many simple cycles are there in a complete graph? (Recall that a simple cycle is a path that starts and ends at the same vertex.)

   The answer is $\frac{(n-1)!}{2}$. Recall that the number of rotation-invariant ways to seat $n$ people around a circular table is $(n-1)!$. Cycle on $n$ different vertices are like seatings of $n$ people around a table, but with the additional symmetry of reflection—that is to say, it doesn't matter whether the nodes are connected in clockwise or counter-clockwise order, since the edges are undirected. Hence we've overcounted by a factor of two.

   (d) How many graphs are there on $n$ vertices?

   $2^{\binom{n}{2}}$. For each possible edge, we choose whether it's in the graph or not.

6. Let $G$ be a planar graph with $e$ edges and $f$ faces, such that every face has at least $k$ sides.

   (a) Show that $f \leq 2e/k$.

   Suppose you walk around each face counting edges. Since each edge butts against exactly two faces, you count exactly $2e$ edges. However, since every face has at least $k$ sides, you will count at least $fk$ edges. Thus $2e \geq fk$, which leads immediately to the result.

---

[1]You may question this application of the Product Rule – doesn't the set of vertex $i$'s remaining permitted edges depend on the choices that other vertices have already made? So how can we blithely multiply vertex $i$'s choices by vertex $j$'s choices, if they seem dependent? Yes, the set of permissible remaining edges may vary depending other vertices' choices, but the *size* of the set is always the same! So we're safe.

(b) Use this fact to show that $K_{3,3}$ is nonplanar.

Suppose you could draw $K_{3,3}$ in the plane. Every face has a cycle (the edges surrounding it). However, it's easy to see that every cycle in $K_{3,3}$ has at least 4 edges – cycles must have at least 3 edges in any case, but you need an even-length cycle to get back to the half of the graph you started in. Thus every face of the putatively planar $K_{3,3}$ must have at least 4 sides. Since $e = 9$, we must have $fleq4$. According to Euler's formula $n - e + f = 2$, however, we must have $f = 5$. So $K_{3,3}$ is nonplanar, by contradiction.

## Counting and Probability

7. Among the set of digits $\{1, 2, 3, 4, 5\}$, first one digit is chosen uniformly at random, and then another. What's the probability that an odd digit is picked the second time?

   $3/5$, by linearity of expectation.

8. Say you draw uniformly at random from the elements of the group $S_n$, the group of permutations on $n$ elements. What's the probability that in the permutation you elect, the $k$th element will be in the $i$th position?

$$\frac{(n-1)!}{n!} = 1/n$$

9. What's the probability that among $k$ random digits, 0 doesn't appear?

   $9^k/10^k$

10. $n$ balls are placed randomly into $n$ cells. What's the probability that exactly one cell remains empty?

$$\frac{(n-1)^n}{n^n}$$

11. A box contains 90 good and 10 defective screws. What's the probability that if 10 screws are used, none will be defective?

$$\frac{\binom{90}{10}}{\binom{100}{10}}$$

12. Eight eligible bachelors and seven beautiful models have purchased seats in a 15-seat row of a theatre. What's the expected number of adjacent seats ticketed for marriageable couples?

    Introduce

$$x_i = \begin{cases} 1 & \text{if seat } i-1 \text{ and } i \text{ are occupied by people of opposite sex} \\ 0 & \text{otherwise} \end{cases}$$

What we're after is $E[x_2 + x_3 + \ldots x_{15}]$. By linearity of expectation, this is $E[x_2] + E[x_3] + \ldots E[x_15]$. Now let's consider $E[x_j]$ for some $j$. Well,

$$E[x_j] = 1 \cdot p(x_j = 1) + 0 \cdot p(x_j = 0) = p(x_j = 1)$$

We still need to compute $p(x_j = 1)$. The probability that seat $j-1$ has a male in it is $8/15$. Given this, the probability that the $j$th seat has a female in it is $7/14$. So the probability of male-female in these two seats is $(8/15)(7/14)$. By a similar argument, the probability of female-male in these two seats is $(7/15)(8/14)$. Adding these together gives $8/15$, the probability that seat $j-1$ and $j$ are occupied by people of opposite sex. This holds for all $j$.

So we have $E[x_j] = 8/15$, and the answer is then $E[x_2] + E[x_3] + \ldots E[x_15] = 14(8/15)$.

13. What is $213729^{1018} \pmod{20}$?

    By the modular exponentiation rule we learned in Lecture 29, we know that

$$x^{\varphi(n)} = 1 \pmod{n}$$

    Recall that $\varphi(n)$, "the Euler phi function," is the number of integers less than $n$ that are relatively prime to $n$. For instance, $\varphi(4) = 2$ (only 1 and 3 are relatively prime to 4), and $\varphi(5) = 4$ (since all integers less than a prime are relatively prime to it). Furthermore, we showed in Lecture 29 that if $p$ and $q$ are relatively prime, then $\varphi(pq) = \varphi(p)\varphi(q)$. Since 4 and 5 are relatively prime, $\varphi(20) = \varphi(4)\varphi(5) = 8$.

    So we can simplify the problem tremendously:

$$\begin{aligned} 213729^{1018} \pmod{20} &= (213729 \bmod 20)^{1018 \bmod 8} \pmod{20} \\ &= 9^2 \pmod{20} \\ &= 1 \end{aligned}$$

## A Moment's Thought

14. What is the expected number of simple cycles in a random graph drawn on $n$ labeled vertices with edge probability $p$?

    A simple cycle can have between 3 vertices and $n$ vertices. To choose a cycle, we first choose its size $i$; then a selection of $i$ vertices to involve in the cycle; and then a way of cyclically connecting the chosen vertices (see the derivation for the number of simple cycles on $n$ labeled nodes, above.) Each possible size-$i$ cycle exists with probability $p^i$. The desired expectation is therefore

$$\sum_{i=3}^{n} \binom{n}{i} \frac{(i-1)!}{2} p^i$$

15. Goornks reproduce asexually on staircase steps. On every step it hits, a Goornk (of either type) produces a *weak* goornk, which promptly hops up to the next step, and a *strong* goornk, which hops up two steps at once, bypassing the step in between. How many goornks will eventually arrive at the top of a staircase with $k$ steps if a single goornk reproduces at the bottom?

    We can place the set of goornks that arrive at the top step in one-to-one correspondence with the set of dot/dash sequences of length $k$ or fewer units, where dots are 1 unit long and dashes are 2 units long. This is because each goornk has a distinct *lineage*, describable by a family-tree path of weak goornks and strong goornks originating at the bottom of the staircase. The sequence "–..-", for example, means "the strong goornk produced from the weak goornk produced from the weak goornk produced from the strong goornk produced from the strong goornk produced from the Mother of All Goornks." (This goornk might was born on the 8th step and hopped his way up to the top from there.) All such sequences are distinct, viable lineages and therefore represent distinct goornks that arrive at the top.

To count the set of dot/dash sequences of length $k$ or fewer units (where length is defined as above), we break the set into cases based on the length of the sequence and the number of dashes in the sequence. For a fixed length $i$, there are $\sum_{j=0}^{i/2} \binom{(j)+(i-2j)}{j} = \sum_{j=0}^{i/2} \binom{i-j}{j}$ sequences of length $i$, so the total count is the following double sum:

$$\sum_{i=0}^{k} \sum_{j=0}^{i/2} \binom{i-j}{j}$$