Artificial Intelligence

15-381

April 5, 2007

Sequential Decision Problems & Markov Decision Processes

Recap of last lecture

- Reasoning over time
  - Markov Processes
  - Hidden Markov Models
  - modeling state transitions
  - probability of state sequences
  - inference of hidden states
  - forward and Viterbi algorithms

Prediction and Search in Probabilistic Worlds

Markov Systems, Markov Decision Processes, and Dynamic Programming

Note to other teachers and users of these slides. Andrew would be delighted if you found this source material useful in giving your own lectures. Feel free to use these slides verbatim, or to modify them to fit your own needs. PowerPoint originals are available. If you make use of a significant portion of these slides in your own lecture, please include this message, or the following link to the source repository of Andrew's tutorials: http://www.cs.cmu.edu/~awm/tutorials . Comments and corrections gratefully received.

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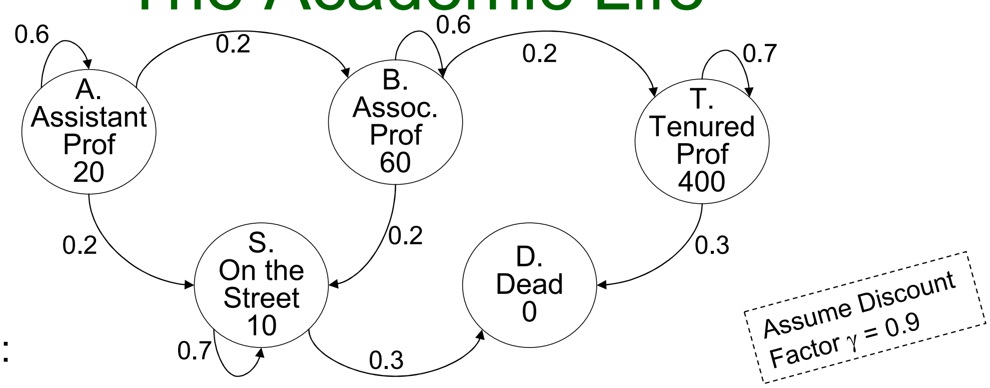
Thanks Andrew!

The Academic Life



Define:

J_A = Expected discounted future rewards starting in state A

J_B = Expected discounted future rewards starting in state B

J_T = " " " " " " " T

J_S = " " " " " " " S

J_D = " " " " " " " D

How do we compute J_A, J_B, J_T, J_S, J_D ?

Discounted Rewards

"A reward (payment) in the future is not worth quite as much as a reward now."

- Because of chance of obliteration
- Because of inflation

Example:

Being promised $10,000 next year is worth only 90% as much as receiving $10,000 right now.

Assuming payment *n* years in future is worth only $(0.9)^n$ of payment now, what is the AP's Future Discounted Sum of Rewards ?

Discount Factors

People in economics and probabilistic decision-making do this all the time.

The "Discounted sum of future rewards" using discount factor γ" is

(reward now) +

γ (reward in 1 time step) +

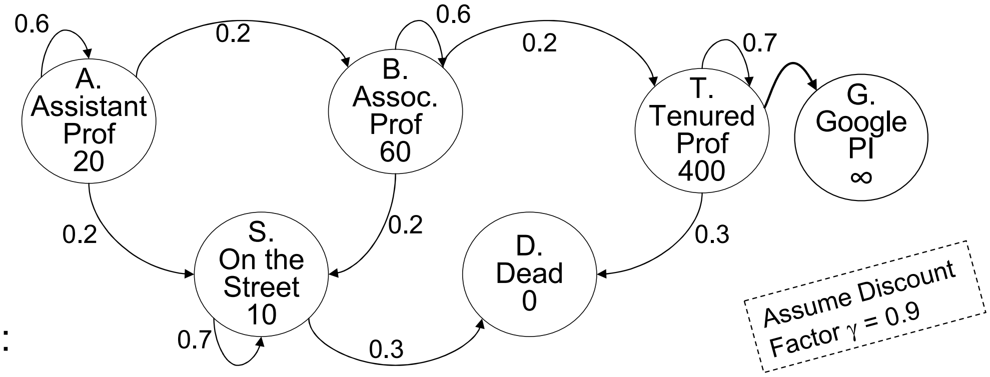
γ^2 (reward in 2 time steps) +

γ^3 (reward in 3 time steps) +

:

: (infinite sum)

The Academic Life



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A Markov System with Rewards…

- Has a set of states $\{S_1\ S_2 \cdots S_N\}$
- Has a transition probability matrix

$$P = \begin{pmatrix} P_{11}\ P_{12} \cdots P_{1N} \\ P_{21} \\ \vdots \\ P_{N1} \quad \cdots \quad P_{NN} \end{pmatrix} \qquad P_{ij} = \text{Prob}(\text{Next} = S_j \mid \text{This} = S_i)$$

- Each state has a reward. $\{r_1\ r_2 \cdots r_N\}$
- There's a discount factor γ . $0 < \gamma < 1$

On Each Time Step …

0. Assume your state is S_i
1. You get given reward r_i
2. You randomly move to another state
   $P(\text{NextState} = S_j \mid \text{This} = S_i) = P_{ij}$
3. All future rewards are discounted by γ

Value Iteration: another way to solve a Markov System

Define

$J^1(S_i)$ = Expected discounted sum of rewards over the next 1 time step.

$J^2(S_i)$ = Expected discounted sum rewards during next 2 steps

$J^3(S_i)$ = Expected discounted sum rewards during next 3 steps

:

$J^k(S_i)$ = Expected discounted sum rewards during next *k* steps

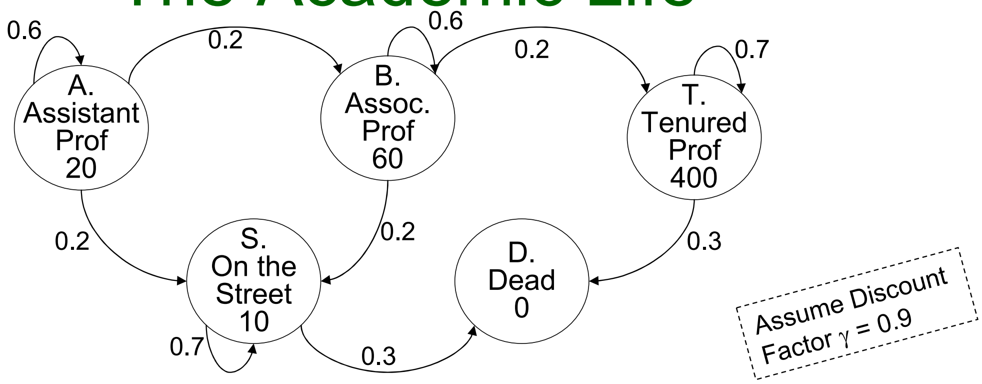
$J^1(S_i)$ = (what?)

$J^2(S_i)$ = (what?)

:

$J^{k+1}(S_i)$ = (what?)

The Academic Life



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$J^k(S_i)$ = Expected discounted sum rewards during next *k* steps

N = Number of states

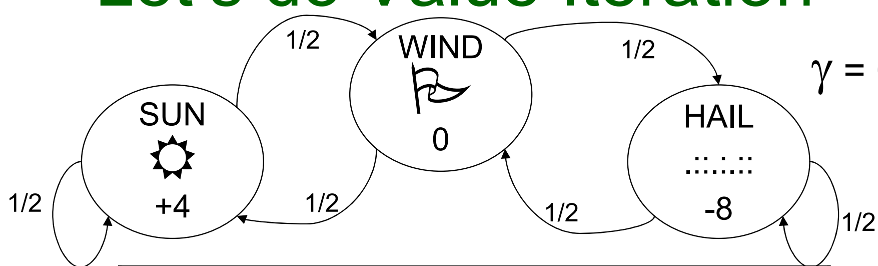
$J^1(S_i) = r_i$ (what?)

$$J^2(S_i) = r_i + \gamma \sum_{j=1}^{N} p_{ij} J^1(s_j)$$ (what?)

:

$$J^{k+1}(S_i) = r_i + \gamma \sum_{j=1}^{N} p_{ij} J^k(s_j)$$ (what?)

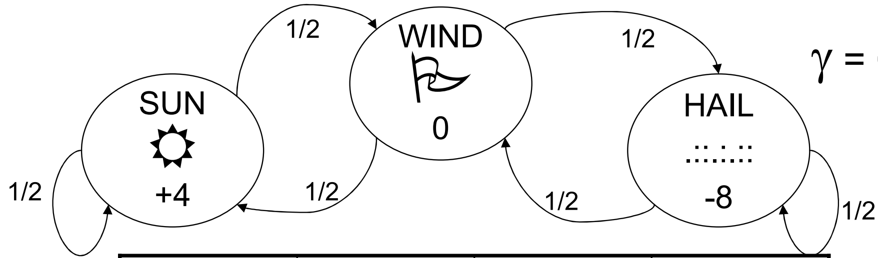
Let's do Value Iteration



$\gamma = 0.5$

k	J^k(SUN)	J^k(WIND)	J^k(HAIL)
1			
2			
3			
4			
5			

Let's do Value Iteration



$\gamma = 0.5$

k	J^k(SUN)	J^k(WIND)	J^k(HAIL)
1	4	0	-8
2	5	-1	-10
3	5	-1.25	-10.75
4	4.94	-1.44	-11
5	4.88	-1.52	-11.11

Value Iteration for solving Markov Systems

- Compute $J^1(S_i)$ for each *j*
- Compute $J^2(S_i)$ for each *j*

  :

- Compute $J^k(S_i)$ for each *j*

As $k \rightarrow \infty$ $J^k(S_i) \rightarrow J^*(S_i)$. Why?

When to stop? When

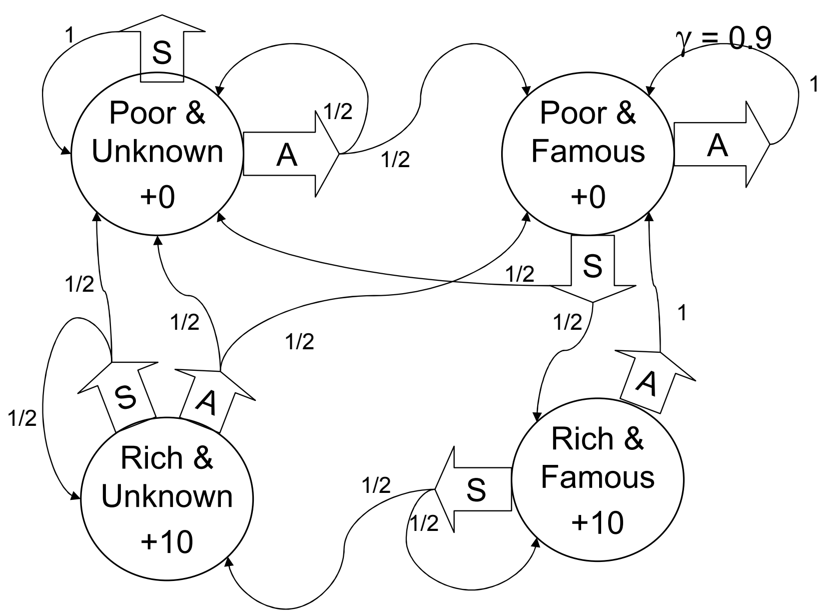
$$\max_i \left| J^{k+1}(S_i) - J^k(S_i) \right| < \xi$$

These are values, what about decisions?

A Markov Decision Process

You run a startup company.

In every state you must choose between Saving money or Advertising.



Markov Decision Processes

An MDP has…

- A set of states $\{s_1 \cdots S_N\}$
- A set of actions $\{a_1 \cdots a_M\}$
- A set of rewards $\{r_1 \cdots r_N\}$ (one for each state)
- A transition probability function

$$P_{ij}^{k} = \text{Prob}\left(\text{Next} = j \,\middle|\, \text{This} = i \text{ and I use action } k\right)$$

On each step:

0. Call current state S_i
1. Receive reward r_i
2. Choose action $\in \{a_1 \cdots a_M\}$
3. If you choose action a_k you'll move to state S_j with probability P_{ij}^{k}
4. All future rewards are discounted by γ

Modeling Environments with Markov Models

Types of Markov Models

State	Passive	Active
Fully Observable	Markov Model	MDP
Hidden State	HMM	POMDP

Advanced topic. We won't cover these in detail.

- MDP
  - tractable to solve
  - relatively easy to specify
  - assumes perfect knowledge of state
- POMDP
  - Treats all sources of uncertainty (acting, sensing, environment) in a uniform framework
  - Allows for taking actions that gain information
  - Difficult to specify all the conditional probabilities
  - Almost always infeasible to solve optimally

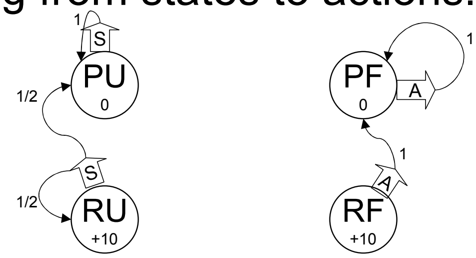
A Policy

A policy is a mapping from states to actions.

Examples

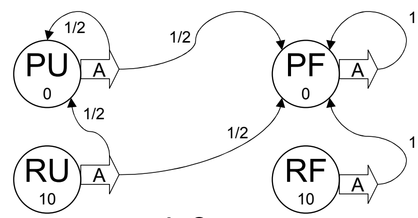
Policy Number 1:

STATE →	ACTION
PU	S
PF	A
RU	S
RF	A



Policy Number 2:

STATE →	ACTION
PU	A
PF	A
RU	A
RF	A



- How many possible policies in our example?
- Which of the above two policies is best?
- How do you compute the optimal policy?

Interesting Fact

For every M.D.P. there exists an optimal policy.

It's a policy such that for every possible start state there is no better option than to follow the policy.

(Not proved in this lecture)

Computing the Optimal Policy

Idea One:

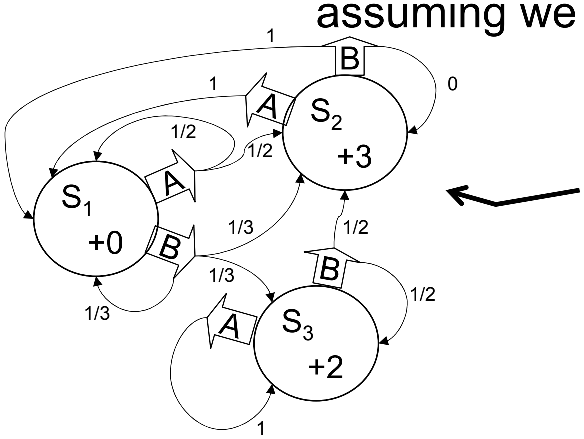
Run through all possible policies.

Select the best.

What's the problem ??

Optimal Value Function

Define $J^*(S_i)$ = Expected Discounted Future Rewards, starting from state S_i, assuming we use the optimal policy



Question

What (by inspection) is an optimal policy for that MDP?

(assume $\gamma = 0.9$)

What is $J^*(S_1)$?
What is $J^*(S_2)$?
What is $J^*(S_3)$?

Computing the Optimal Value Function with Value Iteration

Define

$J^k(S_i)$ = Maximum possible expected sum of discounted rewards I can get if I start at state S_i and I live for *k* time steps.

Note that $J^1(S_i) = r_i$

Let's compute $J^k(S_i)$ for the startup example

k	$J^k(PU)$	$J^k(PF)$	$J^k(RU)$	$J^k(RF)$
1				
2				
3				
4				

Let's compute $J^k(S_i)$ for the startup example

k	$J^k(PU)$	$J^k(PF)$	$J^k(RU)$	$J^k(RF)$
1	0	0	10	10
2	0	4.5	14.5	19
3	2.03	8.55	16.53	25.08
4	4.76	12.20	18.35	28.72

Bellman's Equation

$$J^{n+1}(S_i) = \max_k \left[r_i + \gamma \sum_{j=1}^{N} P_{ij}^k J^n(S_j) \right]$$

Value Iteration for solving MDPs

- Compute $J^1(S_i)$ for all *i*
- Compute $J^2(S_i)$ for all *i*
- :
- Compute $J^n(S_i)$ for all *i*

…..until converged

$$\left[\text{converged when } \max_i \left| J^{n+1}(S_i) - J^n(S_i) \right| \langle \xi \right]$$

…Also known as

Dynamic Programming

Finding the Optimal Policy

1. Compute $J^*(S_i)$ for all i using Value Iteration (a.k.a. Dynamic Programming)
2. Define the best action in state S_i as

$$\arg\max_k \left[r_i + \gamma \sum_j P_{ij}^k J^*(S_j) \right]$$

(Why?)

Applications of MDPs

This extends the search algorithms of your first lectures to the case of probabilistic next states.

Many important problems are MDPs….

… Robot path planning
… Travel route planning
… Elevator scheduling
… Bank customer retention
… Autonomous aircraft navigation
… Manufacturing processes
… Network switching & routing

Policy Iteration

Another way to compute optimal policies

Write $\pi(S_i)$ = action selected in the i'th state. Then π is a policy.

Write π^t = t'th policy on t'th iteration

Algorithm:

π° = Any randomly chosen policy

$\forall i$ compute $J^{\circ}(S_i)$ = Long term reward starting at S_i using π°

$$\pi_1(S_i) = \arg\max_a \left[r_i + \gamma \sum_j P_{ij}^a J^{\circ}(S_j) \right]$$

J_1 = ….

$\pi_2(S_i)$ = ….

… Keep computing π^1 , π^2 , π^3 …. until $\pi^k = \pi^{k+1}$. You now have an optimal policy.

Policy Iteration & Value Iteration: Which is best ???

It depends.

Lots of actions? Choose Policy Iteration
Already got a fair policy? Policy Iteration
Few actions, acyclic? Value Iteration

Best of Both Worlds:

Modified Policy Iteration [Puterman]

…a simple mix of value iteration and policy iteration

3rd Approach

Linear Programming

Dealing with large numbers of states

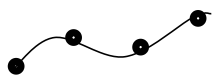
Don't use a Table…

STATE	VALUE
s_1	
S_2	
:	
$S_{15122189}$	

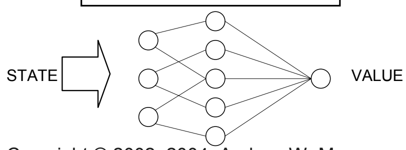
use…

(Generalizers)

Splines

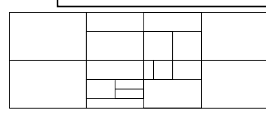


A Function Approximator



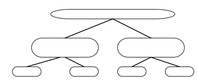
(Hierarchies)

Variable Resolution

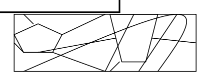


[Munos 1999]

Multi Resolution



Memory Based



What You Should Know

- Definition of a Markov System with Discounted rewards
- How to solve it with Matrix Inversion
- How (and why) to solve it with Value Iteration
- Definition of an MDP, and value iteration to solve an MDP
- Policy iteration
- Great respect for the way this formalism generalizes the deterministic searching of the start of the class
- But awareness of what has been sacrificed.