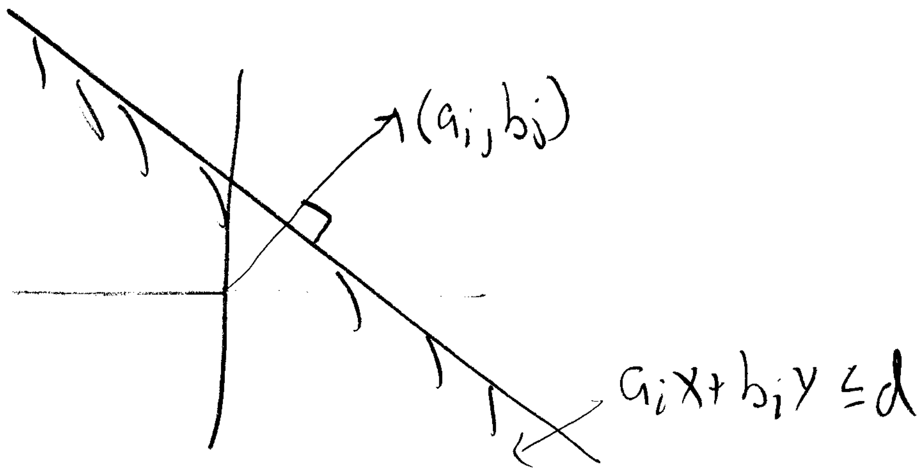
Geometric View

vector (a_i, b_i)

$a_i x + b_i y \le d \ge 0$ half plane normal to (a_i, b_i)

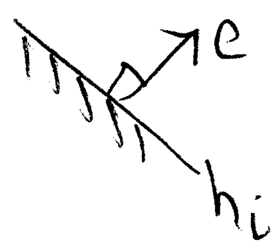


Input: Half-planes $\{h_1, \dots, h_n\}$ & vector $c \in \mathbb{R}^2$

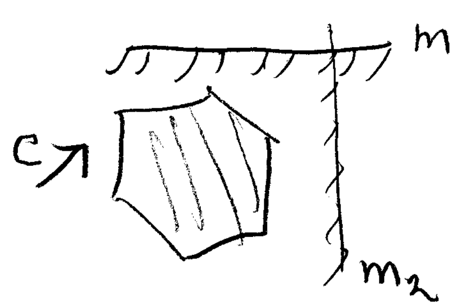
Goal: $x \in \bigcap_{i=1}^{n} h_i$ farthest in c direction.

Simplification:

1) No h_i normal to c ie



2) bounded feasible solutions
3) Given a "bding" box m_1 & m_2



1D-LP

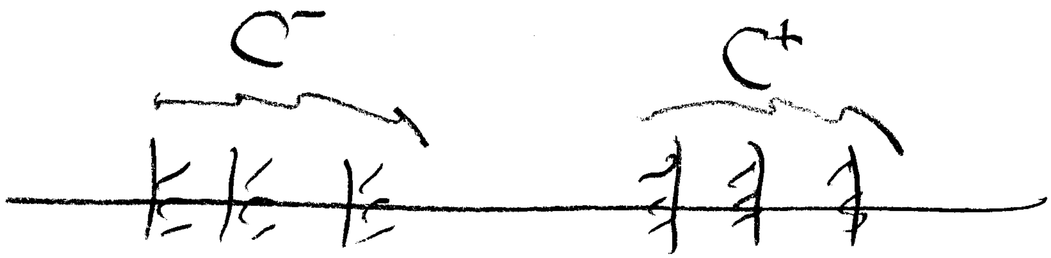
Input: Constraints $a_i x \le b_i$ $a_i \neq 0$

WLOG $a_i = \pm 1$

2 Types

$C^+ = \{ i \mid x \le b_i \}$

$C^- = \{ i \mid -x \le b_i \}$ i.e. $-b_i < x$



$\alpha = \max\{ -b_i \mid i \in C^- \}$

$\beta = \min\{ b_i \mid i \in C^+ \}$

Note: feasible iff $\alpha \le \beta$

If feasible return $= \begin{cases} \beta & \text{if } \operatorname{sign}(c) = 1 \\ \alpha & \text{o.w.} \end{cases}$

Thm 1D-LP is $O(n)$ time

Proceedure: 2D-LP$(m_1, m_2, h_1 \cdots h_n, c)$

1) $V_0 \leftarrow$ 2D-LP(m_1, m_2, c)

   ie $V_0 = CH(m_1) \cap CH(m_2)$

2) Randomly order $h_1, \cdots, h_n$

3) for $i = 1$ to n do

   if $V_{i-1} \in h_i$ then $V_i \leftarrow V_{i-1}$

   else (Make & solve 1D-LP prob)

   $L = CH(h_i)$

   $h_1' = L \cap h_1 \cdots L_{i-1}' = L \cap h_{i-1}$

   $c' = proj(c, L)$ note: $c' \neq 0$

   $V_i = $ 1D-LP$(h_1', \cdots h_{i-1}', c')$

   if V_i is "undef" report "no solution" &

   halt.

Correctness:

Claim: At time (*) $V_i = LP(h_1, \ldots h_i, c)$

pf by induct

base OK

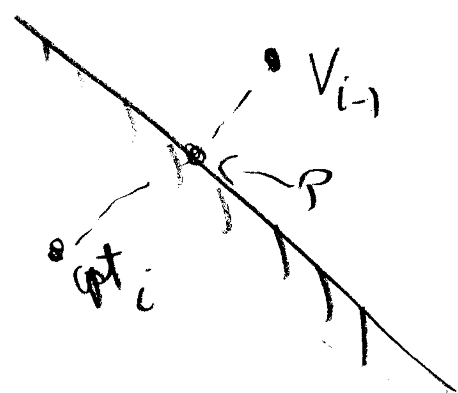
assume V_{i-1} is correct

case: $V_{i-1} \in h_i$ then $V_{i-1} \in$ feasible region $\Rightarrow$ opt

case: $V_{i-1} \notin h_i$

<u>Claim</u> $V_i \in CH(h_i) = L$

picture proof



$c^T V_{i-1} > c^T Opt_i$

$\Rightarrow c^T p \geq c^T Opt_i$

& p feasible

$\therefore p = Opt_i$

Timing

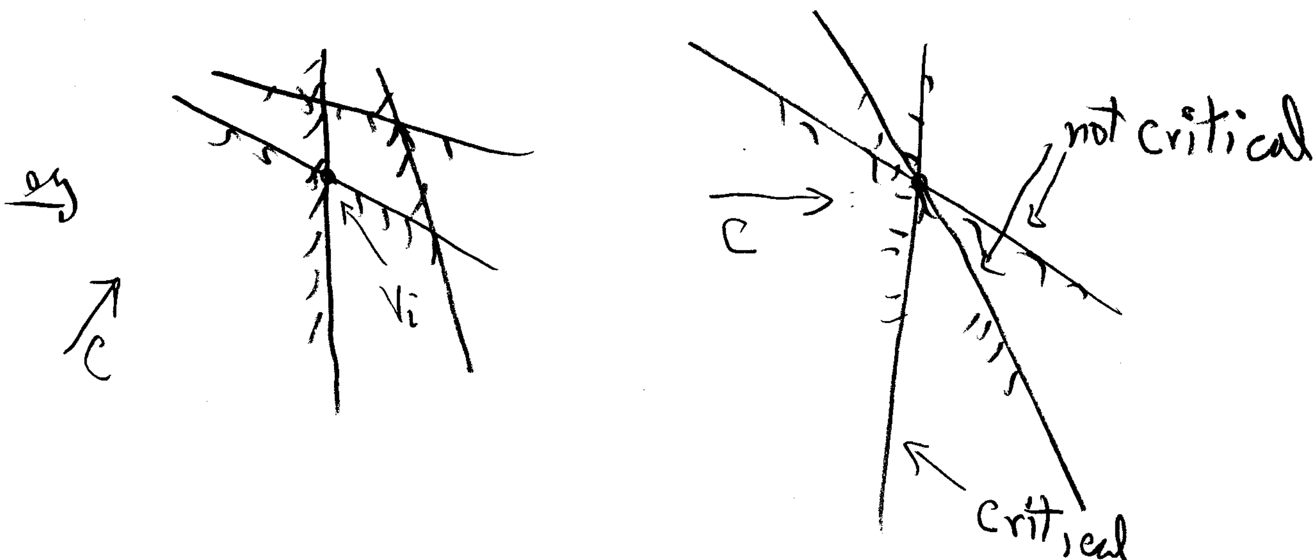
Claim 2D-LP is $O(n)$ expected time.

We use backward analysis.

Suppose we remove h_j from $h_1 \cdots h_i$.

Def h_j critical if removing it changes opt solution

eg



Note At most 2 critical constraints

$$\text{cost}(h_j) = \begin{cases} k & \text{if } h_j \text{ not critical} \\ k \cdot i & \text{o.w.} \end{cases}$$

constant k.

Worst case exactly 2 critical

$$E_i = \left((2)\,ki + (i-2)\,k\right)/i \leq \frac{3ki}{i} = 3k$$

Total Expected cost $\sum_{i=1}^{n} 3k = O(n)$